Intermittency in turbulence and singular solutions: the case of incompressible 3D flows at large Reynolds number

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Outline + one remark

The Euler-Leray equations for a self-similar singular solutions of an inviscid incompressible fluid are derived from the Euler equations. The similarity exponents take into account Kelvin's theorem of conservation of circulation or energy conservation (if energy is finite)

1) What are Euler-Leray equations + a strategy for an explicit (analytical) solution.

2) Amazing agreement between predictions of Euler-Leray with intermittency observed by Yves Gagne in Modane wind tunnel 1998.

Remark: Other examples of dissipation of kinetic energy in singular (or quasi singular) sets: shock waves in compressible fluids, white caps of gravity waves.

Challenge (+ work in progress): put localized (space and time) dissipation in a coherent framework using statistical methods.

Derivation of Leray's equations.1

In 1934 Jean Leray ("Essai sur le mouvement d'un fluide visqueux emplissant l'espace", Acta Math. **63** (1934) p. 193 - 248) published a paper on the equations for an incompressible fluid in 3D. He introduced many ideas, among them the notion of weak solution and also what problem should be solved to show the existence (or not) of a solution singular after a finite time starting from smooth initial data.

Leray assumed a solution of Navier-Stokes 3D blowing-up in finite time at a point, following self-similar evolution for reasonably constrained smooth initial data. Unknown yet if this is correct, either for Euler and/or NS.

Derivation of Leray's equations.2

Euler equations (inviscid, incompressible):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$

and

$$\nabla \cdot \mathbf{u} = 0,$$

Leray looked (with viscosity added, Navier-Stokes equations) to self-similar solutions of the type:

$$\mathbf{u}(\mathbf{r}, t) = (t^* - t)^{-\alpha} \mathbf{U}(\mathbf{r}(t^* - t)^{-\beta}),$$

where t^* is the time of the singularity (set to zero), where α and β are positive exponents to be found and where $\mathbf{U}(.)$ is to be derived by solving Euler or NS equations.
That such a velocity field is a solution of Euler or NS equations implies $1 = \alpha + \beta$. The conservation of circulation in Euler equations implies $0 = \alpha - \beta$, and therefore $\alpha = \beta = 1/2$. If one imposes instead that the energy in the collapsing domain is conserved, one must satisfy the constraint $-2\alpha + 3\beta = 0$, which yields $\alpha = 3/5$ and $\beta = 2/5$, the Sedov-Taylor exponents.

Derivation of Leray's equations.3

No set of singularity exponents can satisfy both constraints of energy conservation and of constant circulation on convected closed curves. $\alpha = \beta = 1/2$ if there are smooth curves invariant under Leray stretching.
Otherwise one has to take the Sedov-Taylor scaling, assuming that
1) the collapsing solution has finite energy,
2) no closed curve is carried inside the singular domain while keeping finite length and remaining smooth.

Derivation of Leray's equations.4

Introduce boldface letters such that $\mathbf{R} = \mathbf{r}(-t)^{-\beta}$. The Euler equations become the Euler-Leray equations for $\mathbf{U}(\mathbf{R})$:

$$-(\alpha\ \mathbf{U} + \beta\ \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P,$$

and

$$\nabla \cdot \mathbf{U} = 0$$

A general time dependence can be kept besides the one due to the rescaling of the velocity and distances by defining as new time variable $\tau = -\ln(t^* - t)$. This maps the dynamical equation into

$$\frac{\partial \mathbf{U}}{\partial \tau} - (\alpha \mathbf{U} + \beta \mathbf{R} \cdot \nabla \mathbf{U}) + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla P,$$

$$\nabla \cdot \mathbf{U} = 0$$

A set equivalent to the original Euler equations.

A singularity of the "pure" self-similar type (without dependence with respect to τ) must decay at large distances in such a way that it becomes independent on time. Otherwise it would depend singularly on time everywhere and so not be a point-wise singularity. Moreover the relevant solution(s) of Euler-Leray is a smooth function of **R**. Otherwise it makes a singular solution at any time, not at a single time (this eliminates solutions of the type of Landau submerged jet).

The first constraint (solution independent on time at large distances) is satisfied if $\mathbf{U} \sim 1/R$ at R large. Returning to the initial space-time dependence one gets (with $\alpha = \beta$) $u \sim (-t)^{-1/2}/r(-t)^{-1/2} \sim 1/r$ with no time dependence.

At $t = 0$ (time of singularity) the velocity field of the singular solution is exactly like $1/r$ times a function of the angle to satisfy incompressibility (a property perhaps checkable by PIV).

Explicit solution of Euler-Leray equations: an outline

Sketch of solution of the full Euler-Leray equations in axisymmetric geometry with swirl and dependence on τ:

1) Starts from a localized solution of steady localized Euler equation by solving Bragg-Hawthorne equations. Because this has finite energy one takes Sedov-Taylor exponents.

2) Because steady Euler equations are invariant under arbitrary dilation of amplitude and argument (being homogeneous of order 2 and invariant under dilation of coordinates) one can assume that the solution of Bragg-Hawthorne has very large amplitude.

3) This makes the (linear) streaming term added by Leray arbitrarily small compared to the leading order term which is quadratic.

4) Solving Euler-Leray by perturbation one meets two solvability conditions because of the two dilation symmetries of the steady Euler equations. They can be satisfied by adding two small oscillations with arbitrary amplitudes. Each oscillation generates a Stokes drift at quadratic order. Adding the two contributions to this drift one can meet the two solvability conditions.

One obvious motivation for working on Euler-Leray singularities is their possible connection with the (loosely defined) phenomenon of intermittency in high Reynolds number flows. This raises several questions:

1. What is specific to Leray singularities compared to other schema for intermittency?

2. What would be specific of an Euler-Leray singularity in a time record of large Reynolds number flow?

Is it possible to "observe" Euler-Leray singularities?.2

Point 1 : If intermittency is caused by Leray-like singularities, they should yield strong positive correlation between singularities of the velocity and of the acceleration. This is what is observed. Compared to the scaling prediction derived from Kolmogorov-like exponents this (positive) correlation is a strong indication of the occurrence of singularities near large fluctuations. Moreover Kolmogorov theory extended to dissipative scales excludes exponents of the singularity of the velocity fluctuations vs distance which is less than $1/3$: otherwise dissipation is divergent everywhere in space, clearly impossible.
The only way out is to have dissipation events at random points in space and time in the limit of large Reynolds number, instead of being spread continuously in space and time.

Euler-Leray singularities and intermittency.1

Kolmogorov K41 theory It is based upon the idea that turbulent fluctuations at very large Reynolds number (where the effect of viscosity is formally small) depend on the energy dissipated in the turbulent flow per unit of mass and of time.
Kolmogorov theory is successful for predicting the spectrum of velocity fluctuations (Kolmogorov-Obukhov spectrum $k^{-5/3}$) but is contradicted by intermittency. Because of it the fluctuations fail to satisfy the relationship predicted by Kolmogorov between the velocity fluctuation and the distance between two points of measurement.
Using the scaling law with ϵ, one finds that the velocity correlation $v(R+r) - v(R)$ is of order $(\epsilon r)^{1/3}$ when the distance r is in the (wide) range between the largest scales and the length scale short enough to make the viscosity relevant. This law predicts that, as r gets smaller and smaller, the amplitude of the velocity fluctuation decreases, not what is observed. The exponent of velocity vs distance cannot be less than $1/3$: otherwise diverging dissipation at small distances, dissipation is not spread uniformly in space.

Euler-Leray singularities and intermittency.3

We have very long and high quality records of velocity fluctuations in high-speed wind tunnel of Modane in the French Alps, obtained by hot-wire anemometry (Yves Gagne et al. 1998), and all sorts of correlations can be studied.

Suppose the large bursts of velocity are due to Euler-Leray singularities. It means that $u(r,t)$ scales like $(-t)^{-\alpha}$ as t tends to zero (0 taken arbitrarily as the instant of the singularity). The acceleration γ (time derivative of Eulerian u) is of order of $(-t)^{-(2\beta+\alpha)}$ as t tends to zero. Therefore near the singularity both the velocity and the acceleration diverge, this last one the most strongly and in this large burst u^3 is of order γ if conservation of circulation is taken:

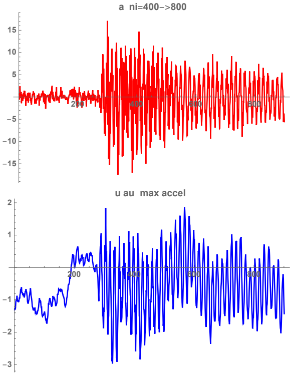
$$u^3 \sim \Gamma\gamma$$

The multiplicative constant is of the order of a "typical" value of the circulation. With the Sedov-Taylor exponents, on has instead:

$$u^8 \sim E\gamma^3$$

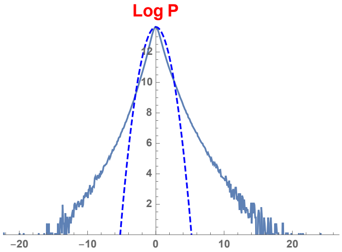
where E is the energy inside the collapsing domain.

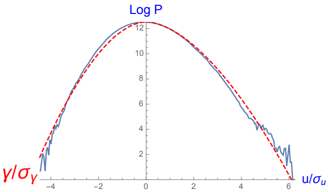
burst from Modane 2014; $\gamma(t)$ (red); $u(t)$ (blue)



$\gamma/g = 56000$; (Maximum ratio $\gamma/g = 10^6$ for Modane-2014 ; and $\gamma/g = 6000$ for Modane-1998) g acceleration of gravity.

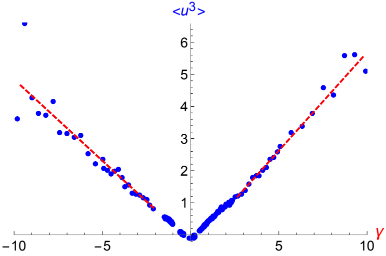
Gaussian Statistics?

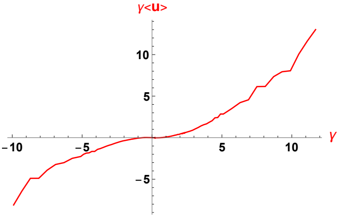




Non Gaussian acceleration ; velocity $\sim$ Gaussian

Scaling relations : $u^3 = \Gamma\gamma$ or $u\gamma \sim \epsilon$?

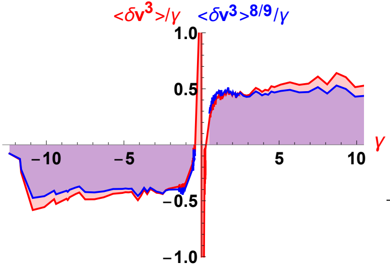


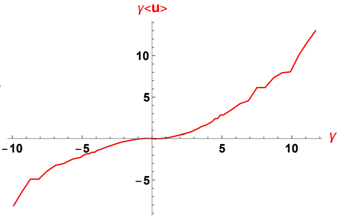


Scalings Leray/circulation: $u^3 = \Gamma\gamma$; $u\gamma \sim \epsilon$ invalid

Scaling Kolmogorov

Circulation scaling vs Sedov-Taylor scaling vs Kolmogorov scaling





Scalings circulation (left red): $u^3 \sim \Gamma\gamma$
scaling Sedov-Taylor (left blue) $u^8 \sim E\gamma^3$;
Scaling Kolmogorov (right) $u\gamma \sim \epsilon$ on the right
Notice: Taylor frozen turbulence does not apply because the large velocity fluctuations are noticeably larger than the mean velocity.