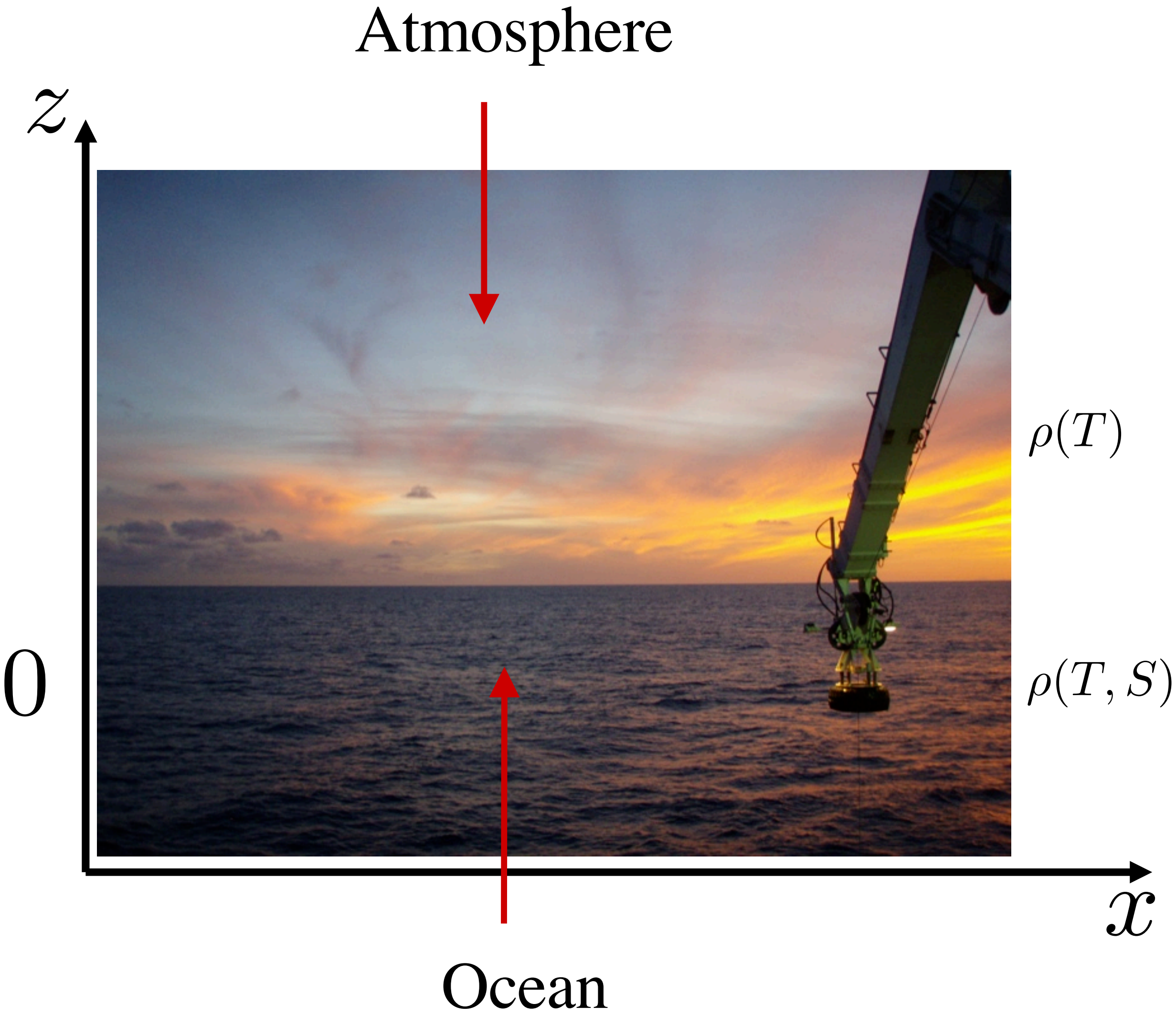
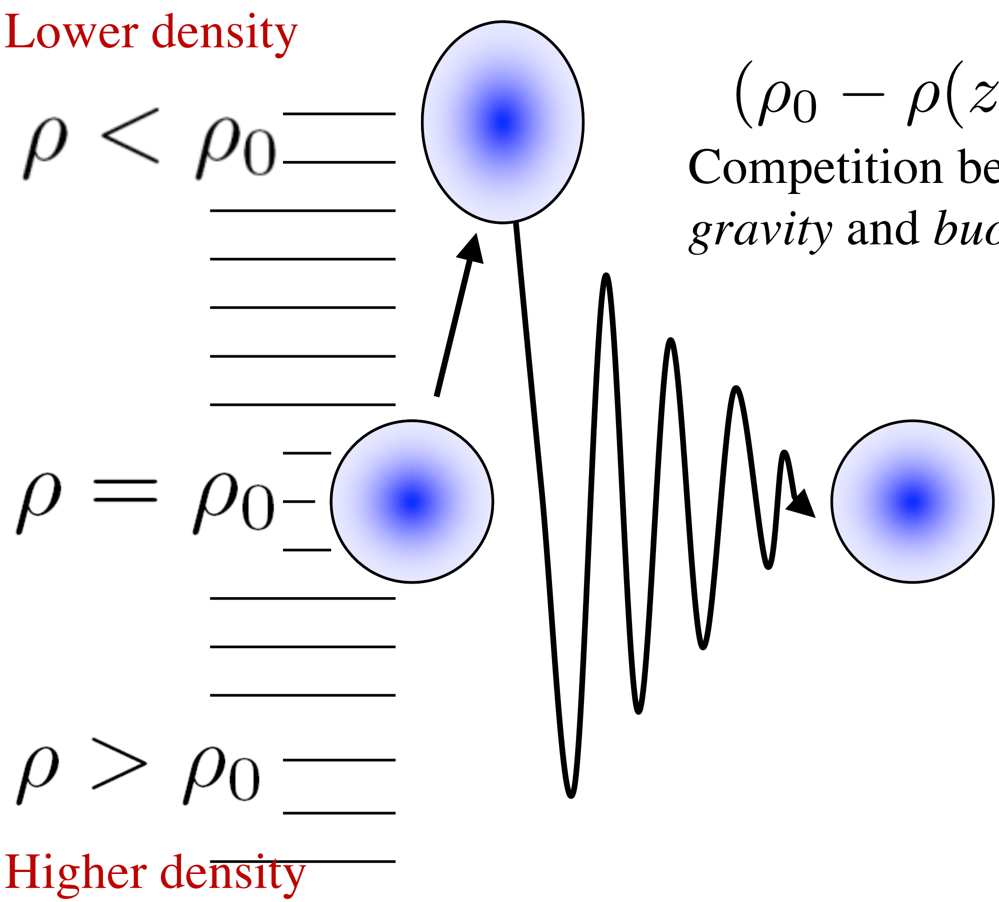
Stratified Fluids

Density $\rho(z)$

Stability $$\frac{\mathrm{d}\rho(z)}{\mathrm{d}z} < 0$$

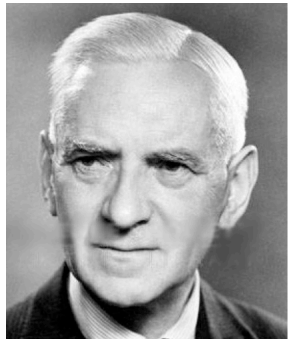


Brunt-Väisälä frequency



$(\rho_0 - \rho(z))\,\vec{g}$

Competition between *gravity* and *buoyancy*

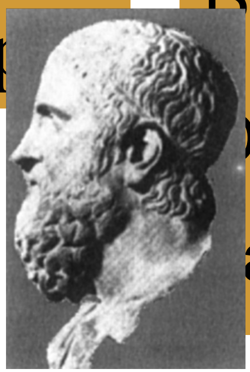


Brunt

Väisälä

$$N^2 = -\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}$$

Examp



eriod

cean → 30 min

aboratory → 10 s

- Slow oscillations
- Wave propagation

Basic equations

Incompressible flow

$$\nabla \cdot \boldsymbol{u} = 0,$$

Navier-Stokes Eq.

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_{\mathrm{ref}}} \nabla p + b \boldsymbol{e}_z + \nu \nabla^2 \boldsymbol{u}$$

Mass conservation

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + u_z N^2 = 0.$$

$N(z) = \left(-g\left(\partial_z \rho_0\right) / \rho_{\mathrm{ref}}\right)^{1/2}$

$b = g\left(\rho_0(z) - \rho\right) / \rho_{\mathrm{ref}}$

Restricting to 2D and introducing the streamfunction

$$\boldsymbol{u} = \left(\partial_z \psi, 0, -\partial_x \psi\right)$$

one gets

$$\partial_t \nabla^2 \psi + J\left(\nabla^2 \psi, \psi\right) = -\partial_x b + \nu \nabla^4 \psi.$$

$$\partial_t b + J(b, \psi) - N^2 \partial_x \psi = 0.$$

where $J(\psi, b) = \partial_x \psi \, \partial_z b - \partial_x b \, \partial_z \psi$

One can finally combine above equations in

$$\underbrace{\partial_{tt} \nabla^2 \psi + N^2 \partial_{xx} \psi}_{\text{Linear terms}} = \underbrace{\nu \nabla^4 \partial_t \psi}_{\text{Viscosity}} + \underbrace{\partial_t J\left(\psi, \nabla^2 \psi\right) + \partial_x J(b, \psi)}_{\text{Nonlinear terms}}$$

Linear terms — Viscosity — Nonlinear terms

Unusual wave equation: Linear Approximation

$$\nabla^2 \psi_{tt} + N^2 \psi_{xx} = 0$$

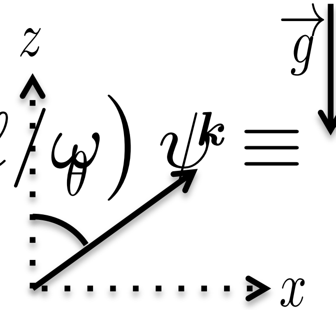
Different from the D'Alembert's equation

Plane wave solution

$$\psi = \psi_0 \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \longrightarrow \omega = \pm N \frac{\ell}{k} = \pm N \sin\theta$$

with $\boldsymbol{k} = (\ell, 0, m)$ and $k = |\boldsymbol{k}| = (\ell^2 + m^2)^{1/2}$

$$\partial_t b = N^2 \partial_x \psi \longrightarrow b = -\left(N^2 \ell / \omega\right) \psi \equiv \mathcal{P}\psi$$



Unusual wave equation: Nonlinear

ψ : The streamfunction b : The buyoancy

$$\nabla^2\psi_{tt} + N^2\psi_{xx} = \boxed{\partial_t J(\psi, \nabla^2\psi) + \partial_x J(b,\psi)}$$

Plane wave solution

with $\boldsymbol{k} = (\ell, 0, m)$ and $k = |\boldsymbol{k}| = (\ell^2 + m^2)^{1/2}$

$$\psi = \psi_0\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \longrightarrow J(\psi, \nabla^2\psi) = 0$$

$J(\psi, -k^2\psi) = 0$

$$b = -\left(N^2\ell/\omega\right)\psi \longrightarrow J(\psi, b) = 0$$

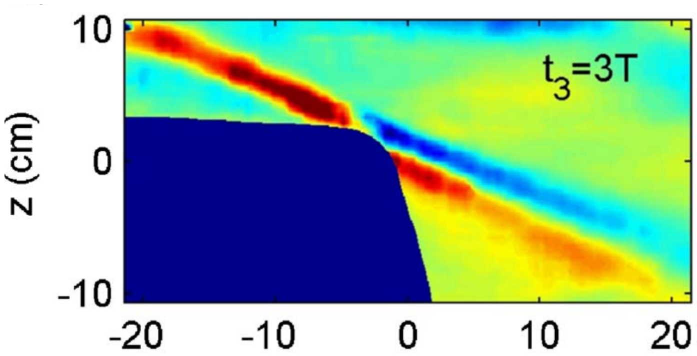
$J(\psi, \mathcal{P}\psi) = 0.$

$$\boxed{\partial_{tt}\nabla^2\psi + N^2\partial_{xx}\psi} = \boxed{\partial_t J(\psi, \nabla^2\psi) + \partial_x J(b,\psi)} = 0$$

- *Plane waves* are solutions of the *nonlinear* equation !
- *Uniform beams*, regardless of their transversal profile, are also *exact NL solutions.*

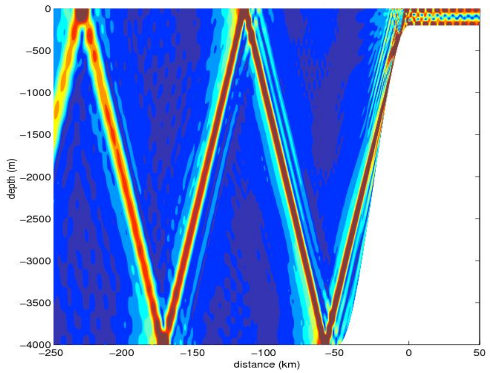
Tidal flow over topography

Experiments



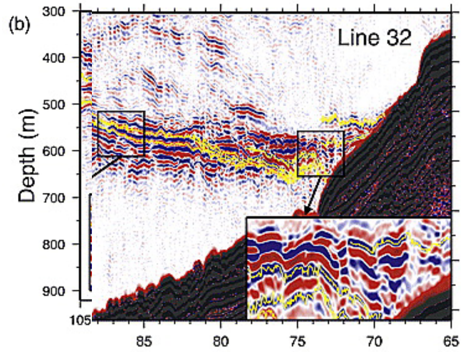
Gostiaux & Dauxois, PoF 2007

Numerical Simulations

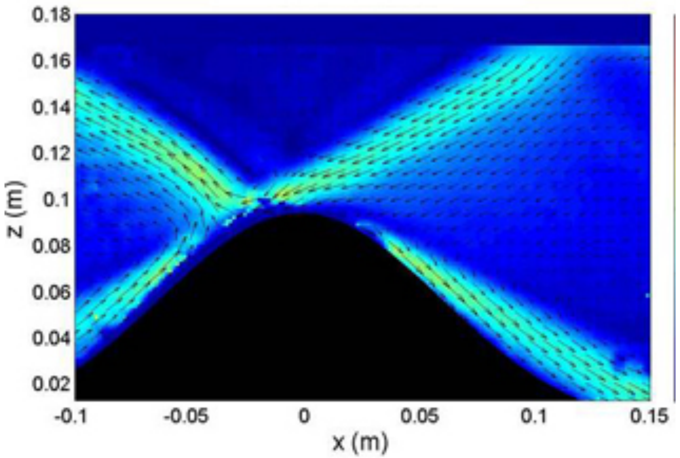


Maugé & Gerkema, NPG 2008

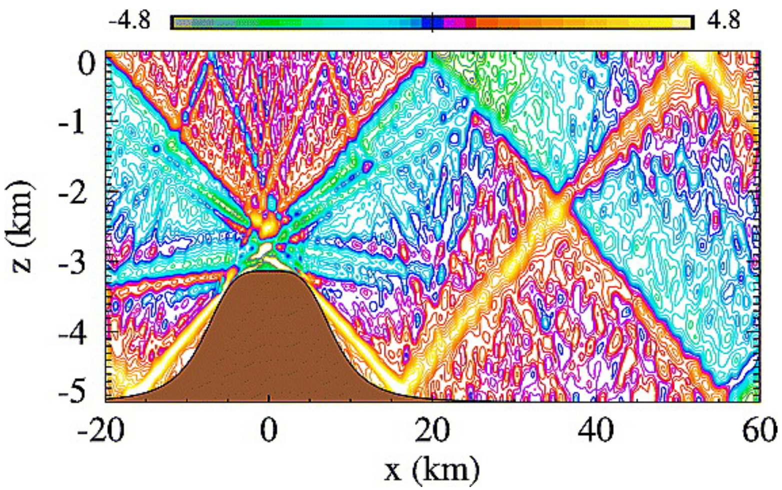
Fields observations



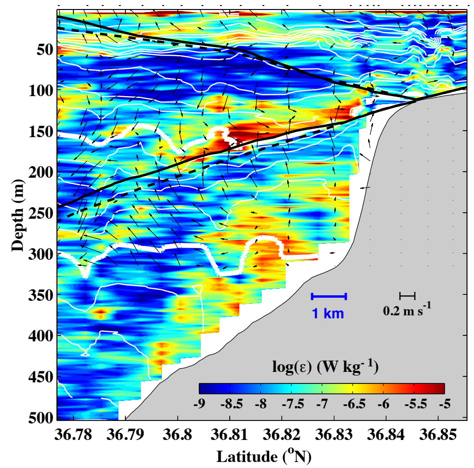
Holbrok & Fer, GRL 2005



Peacock, Echeverri & Balmforth, JPO 2008



Lamb, GRL 2004



Lien & Gregg, JGR 2001

Outline

1. Introduction
2. Internal wave attractors in 2D
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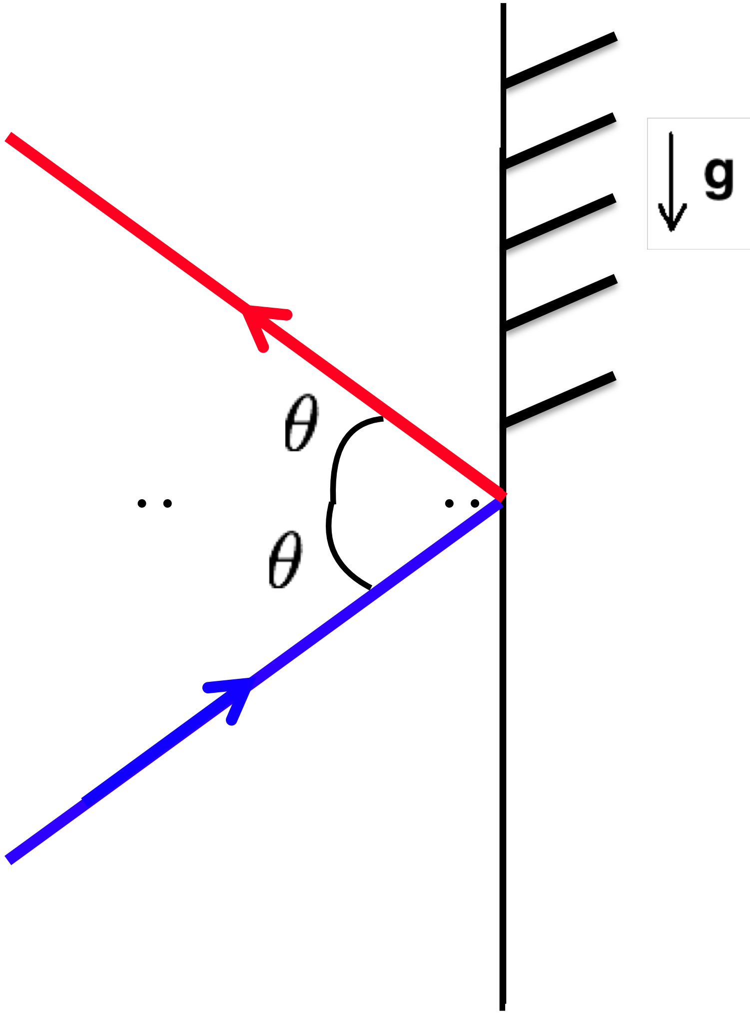
4. Inertial wave attractors in 3D
5. Conclusion and Perspectives

Internal Waves Reflection

Dispersion relation: $\Omega = \frac{\omega}{N} = \pm \sin\theta$

Reflection on a *vertical* wall

1) for a *ray*



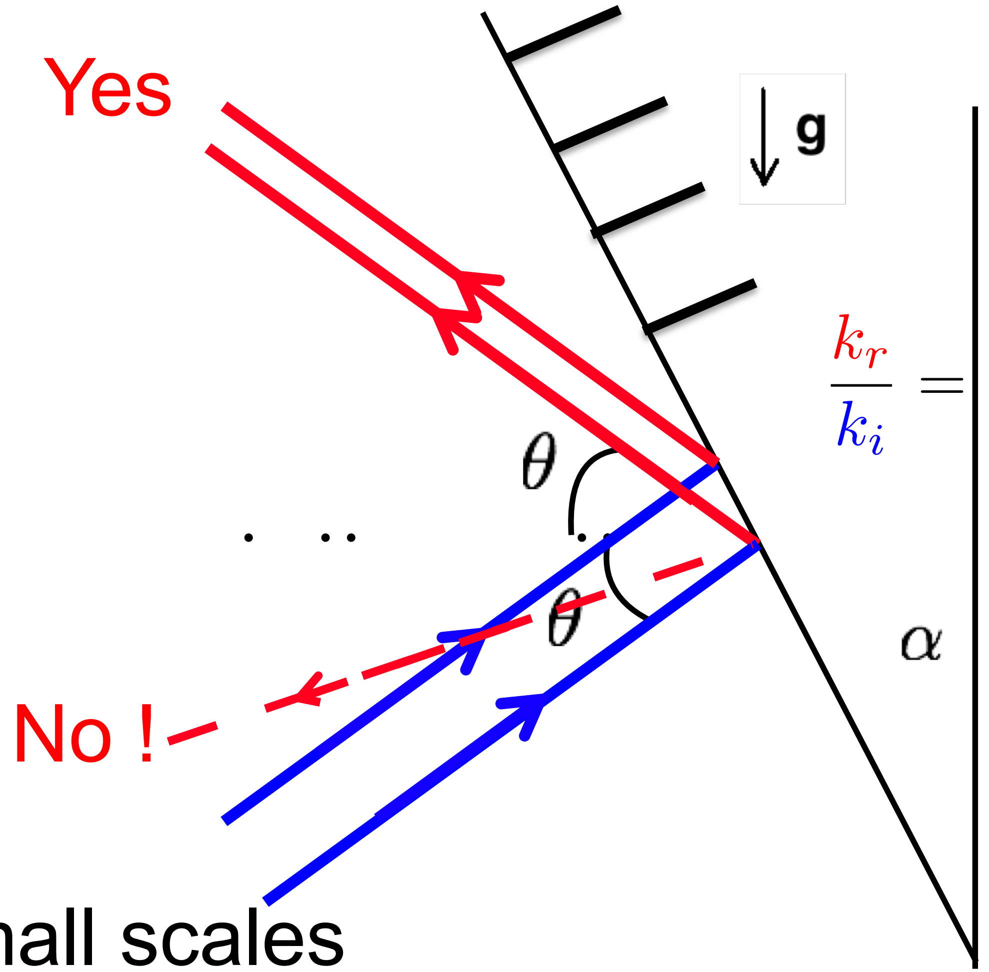
Analogous to the « Snell-Descartes » reflection

Internal Waves Reflection

Dispersion relation: $\Omega = \dfrac{\omega}{N} = \pm \sin\theta$

Reflection on an *inclined* wall

2) For a *beam*



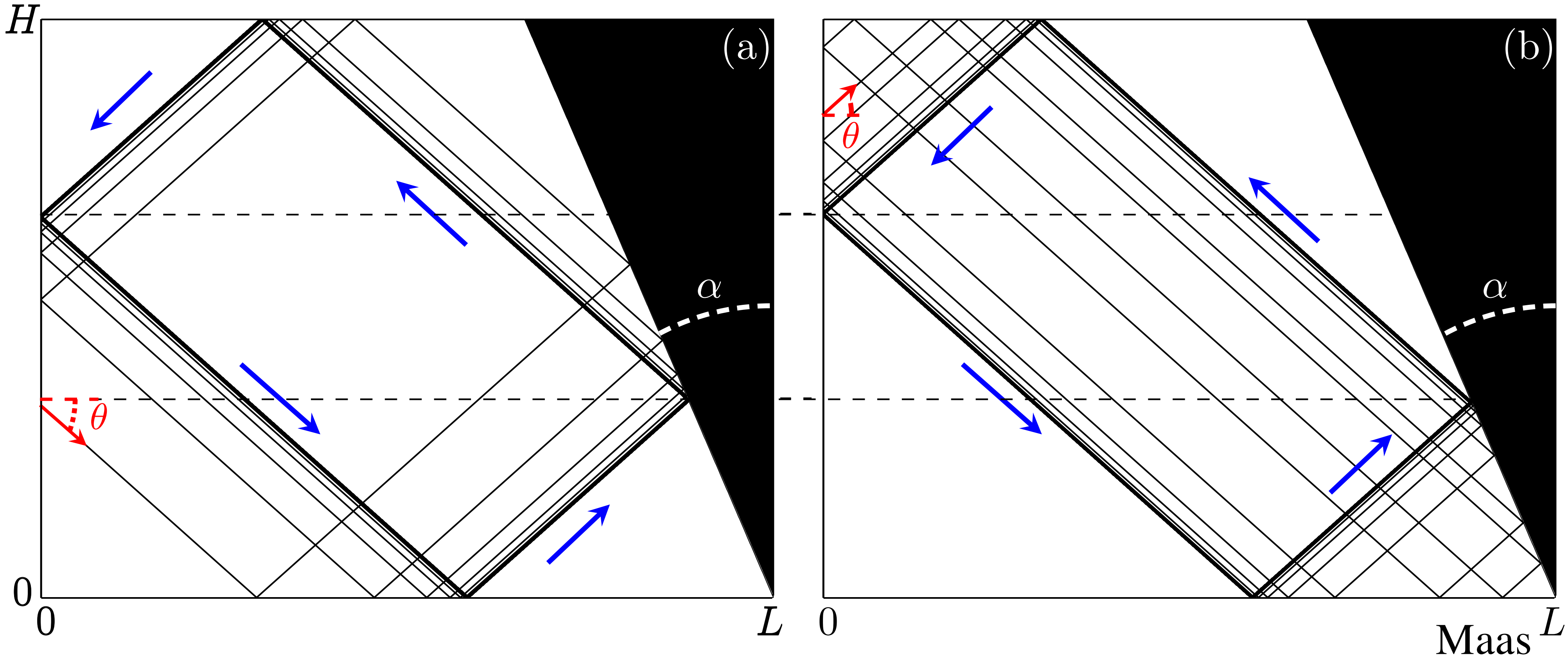
$$\frac{k_r}{k_i} = \left| \frac{\cos(\theta - \alpha)}{\cos(\theta + \alpha)} \right|$$

Energy focalisation:
Linear transfer to small scales

Ingredients for a Cascade?

Internal Wave Attractor

Theoretical prediction



Internal wave attractor corresponds to the existence of a *limit cycle*, depending on the geometrical parameters.

Origin of a cascade in wavelength (or in *k*)

$$\frac{k_r}{k_i} = \left|\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right|$$

Internal Wave Attractor

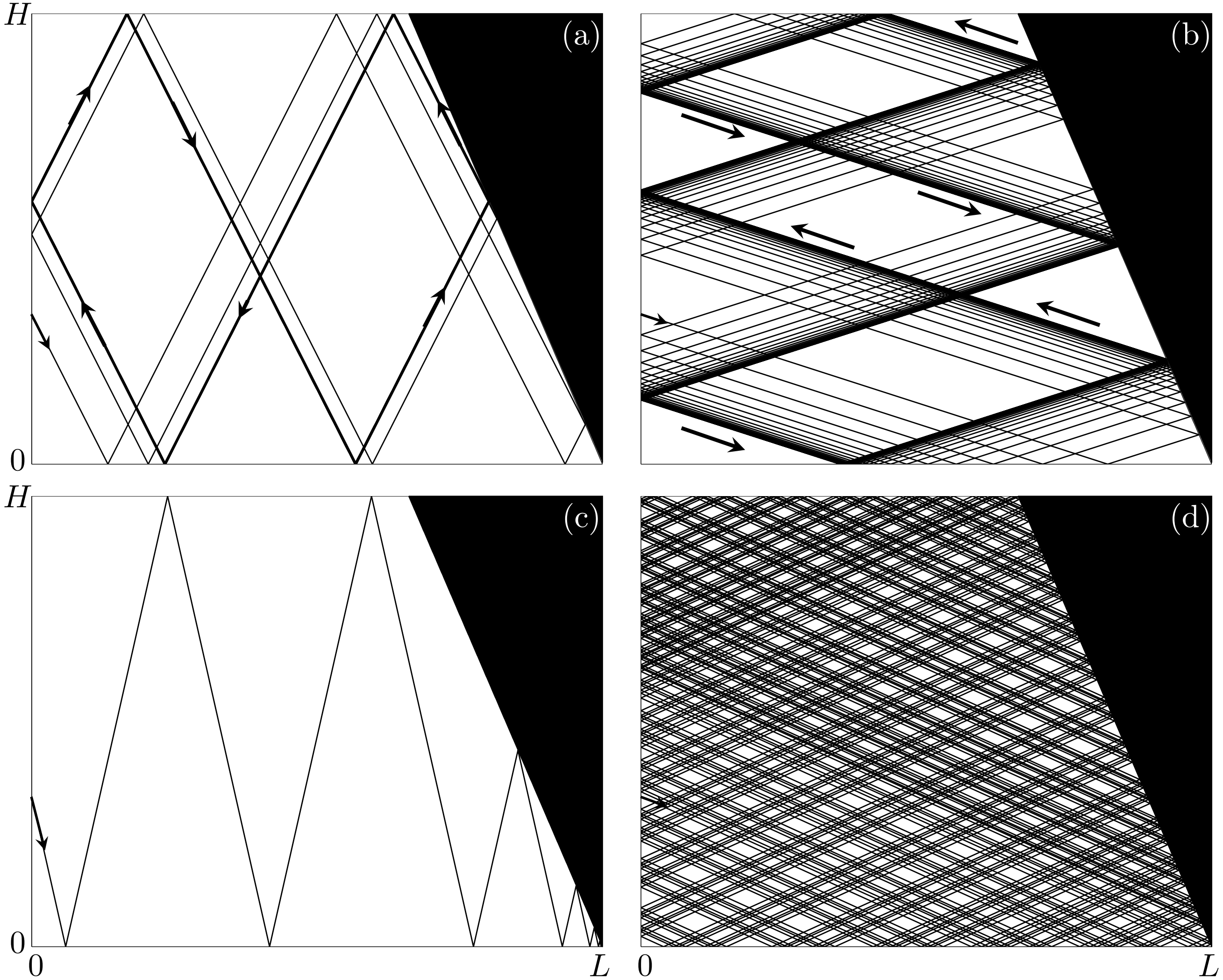
More complex attractors

$\theta = 58^\circ$

(2,1) attractor

$\theta = 15^\circ$

(1,3) attractor



$\theta = 75^\circ$

Point attractor

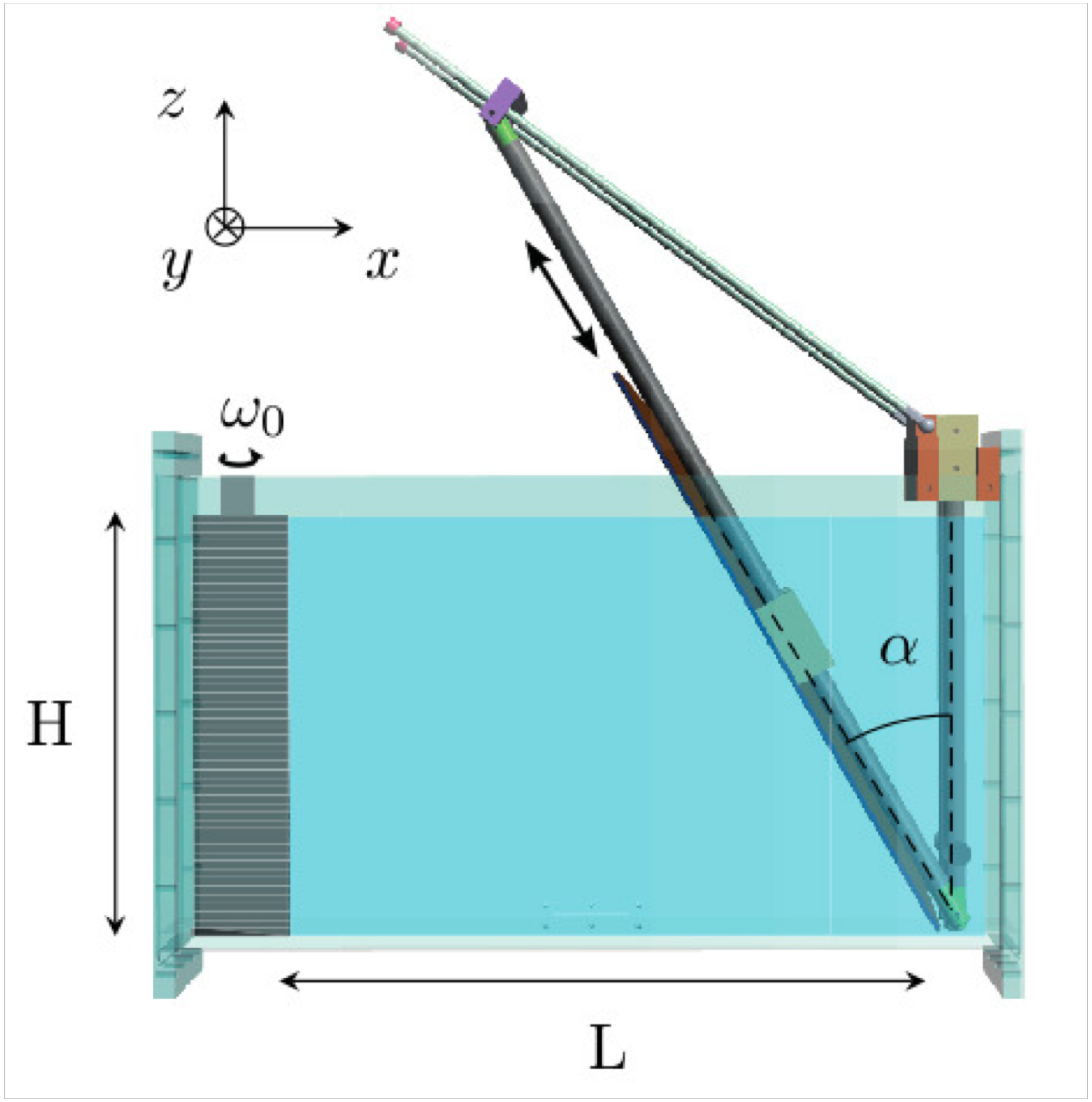
$\theta = 21^\circ$

No attractor

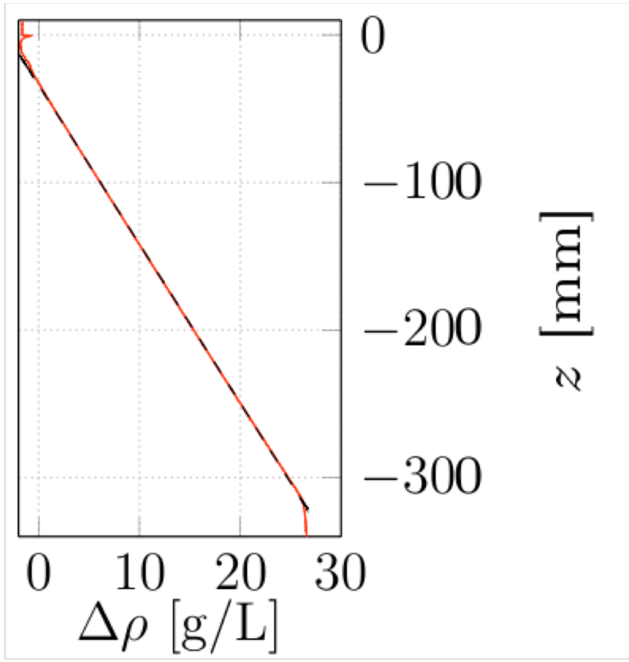
An internal wave billiard

Internal Wave Attractor

Experimental setup:



2D flow

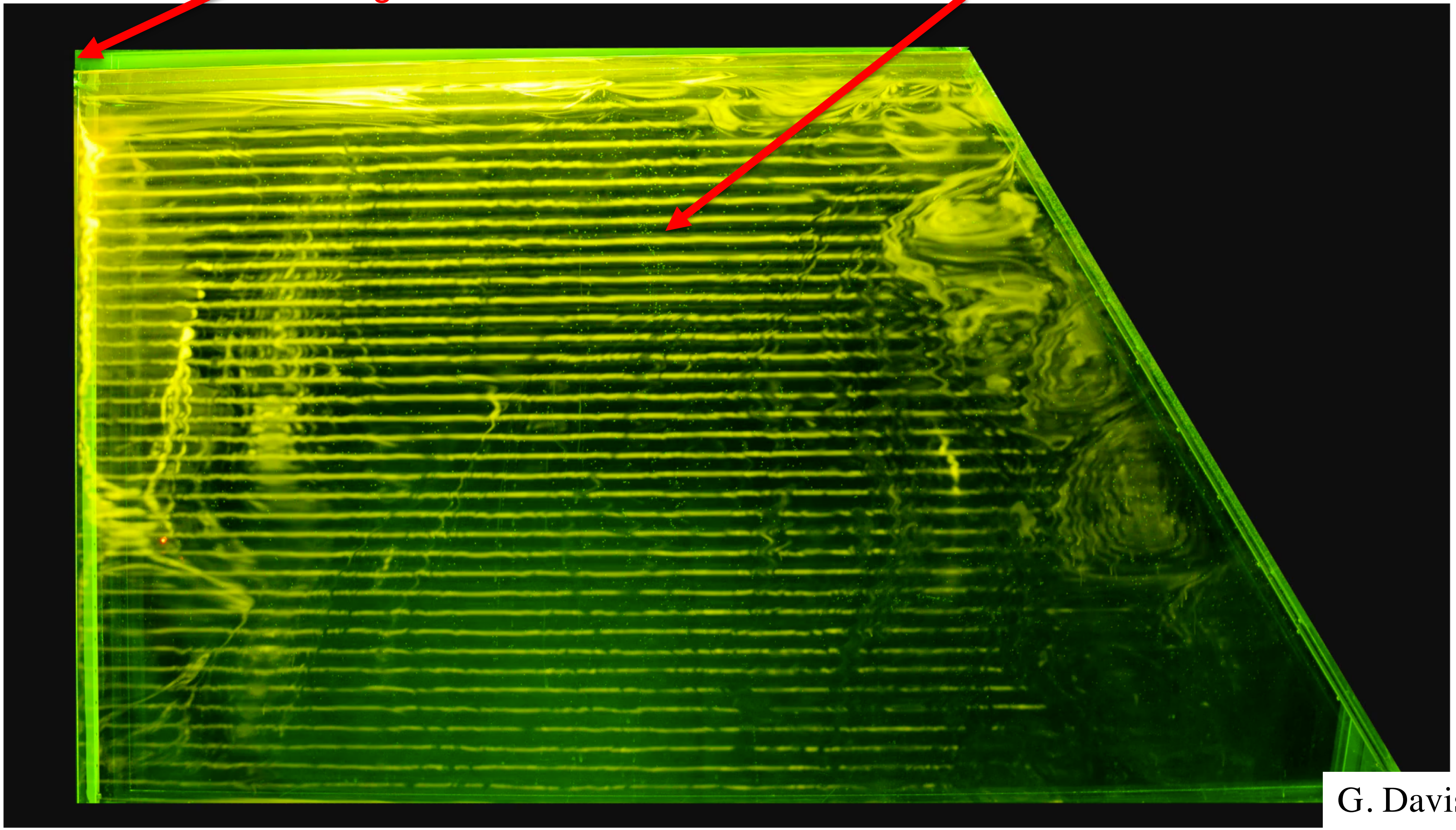


Generator profile: $\eta(z,t) = a\cos(\pi z/H)\cos(\omega t)$

Internal Wave Attractor

Wave generator

Isopycnals shown with fluoresceine

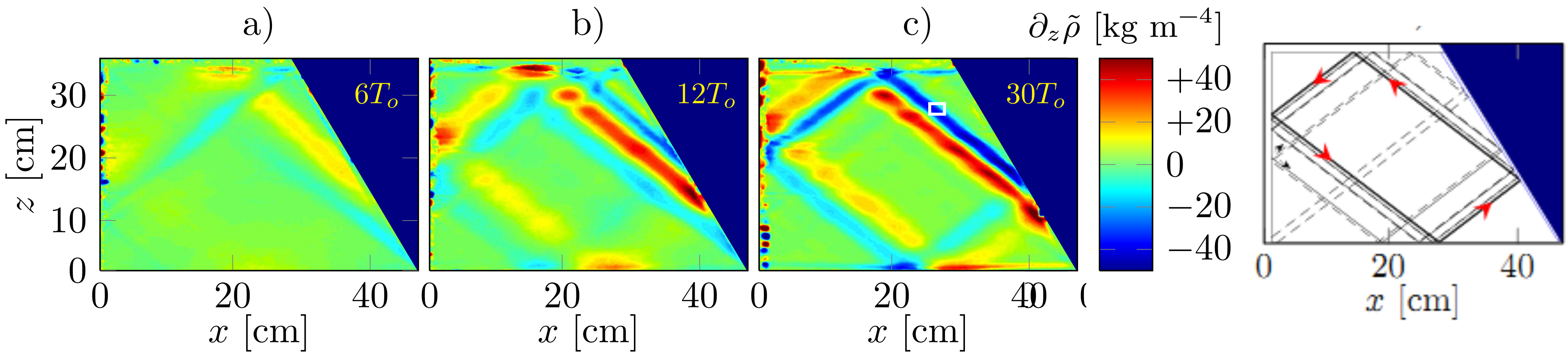


G. Davis

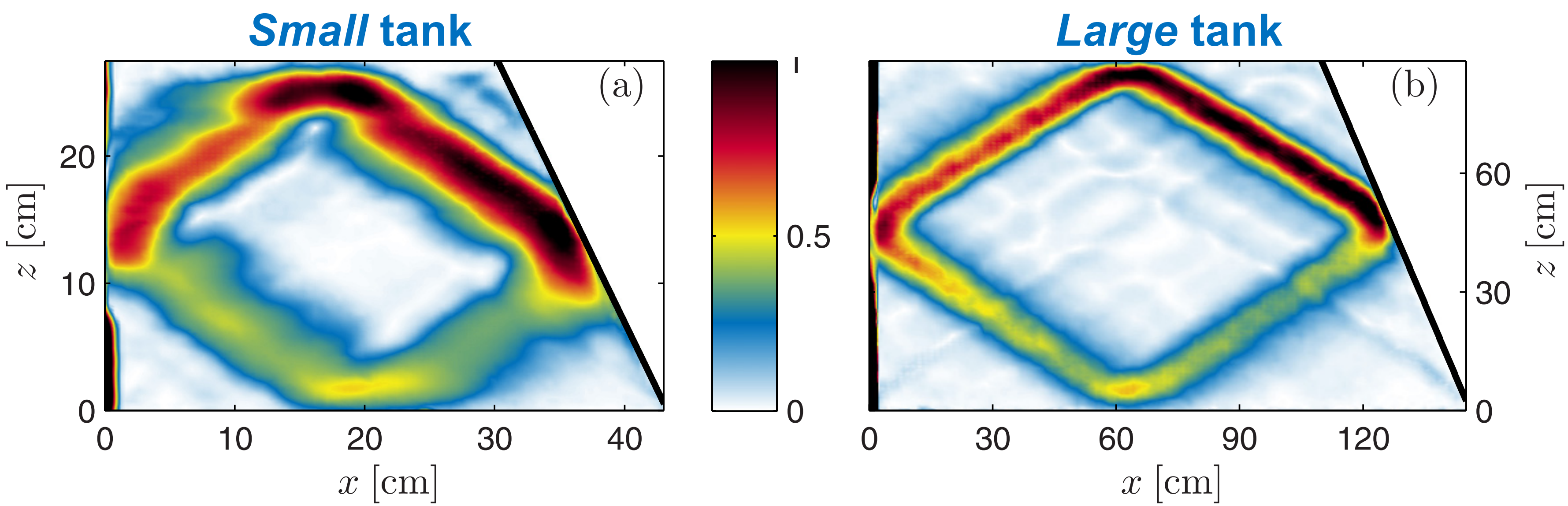
Experimental result

Experimental Results

Scolan, Ermanuyk, Dauxois, *Phys. Rev. Lett. 110*, 234501 (2013).



Balance between viscous damping and focusing.

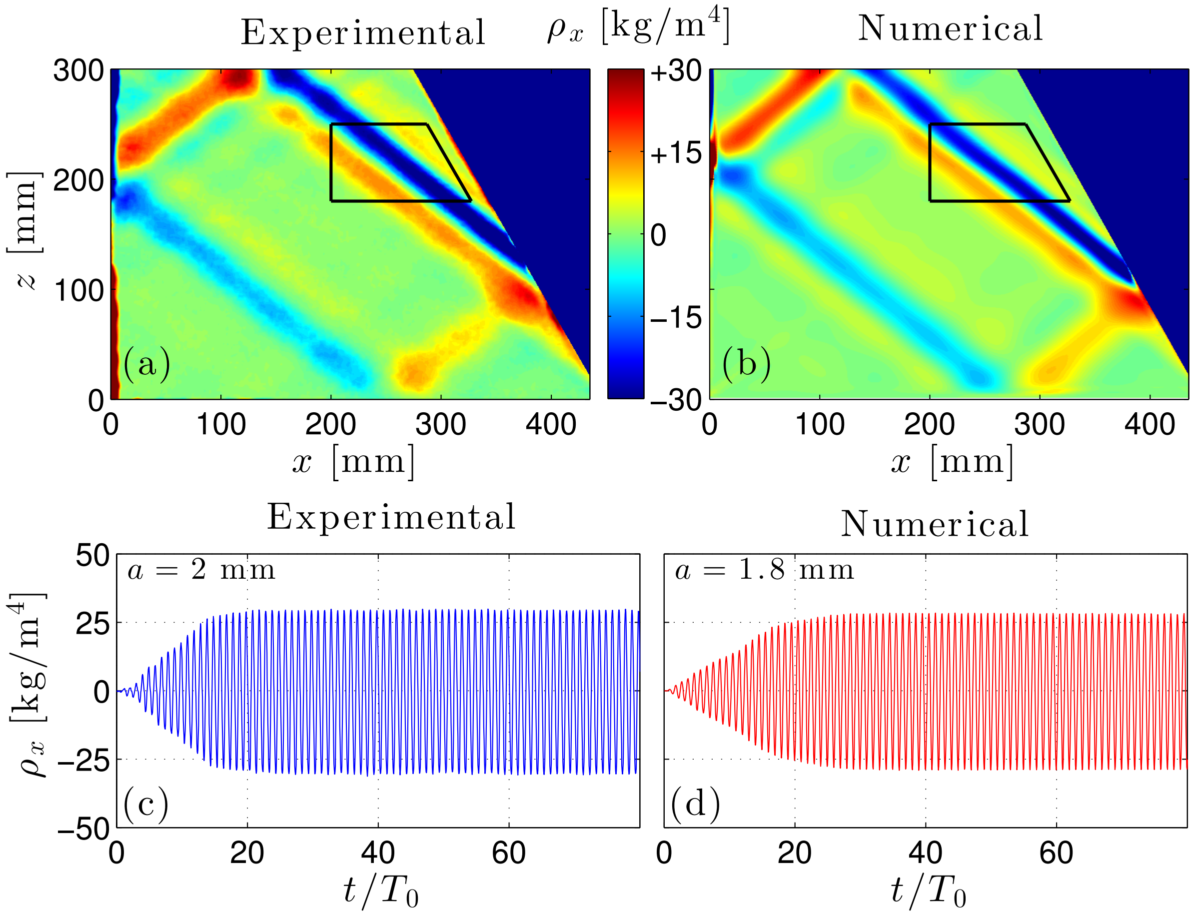


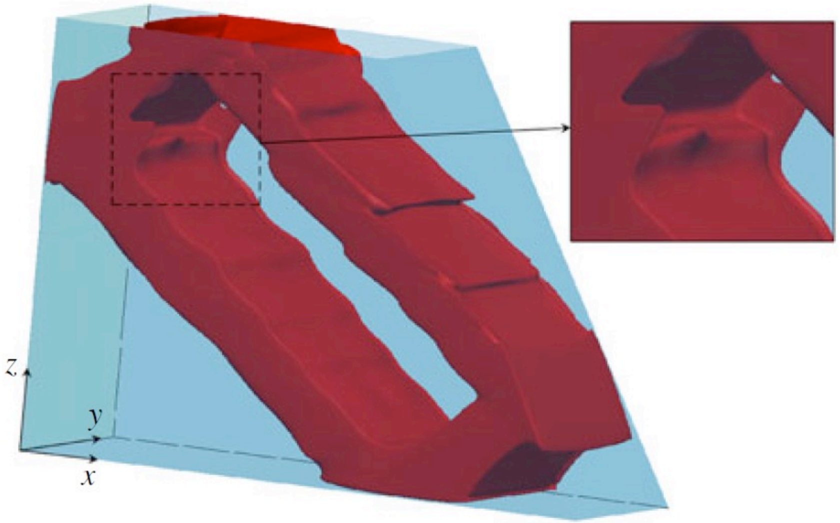
Scaling law ? Towards realistic applications.

Brouzet, Sibgatullin, Ermanyuk, Joubaud, Dauxois, *Phys. Rev. Fluids* **2**, 114803 (2017).

Numerical Calculations

Model: Navier-Stokes in Boussinesq approximation + continuity + salt transport
Method: spectral elements 2D and 3D, code Nek5000 (Fischer & Ronquist 1994)
BC: no-slip at rigid walls, stress-free at free surface





I.Sibgatullin

Brouzet, Sibgatullin, Scolan, Ermanyuk, Dauxois *JFM* 793, 109-131(2016)

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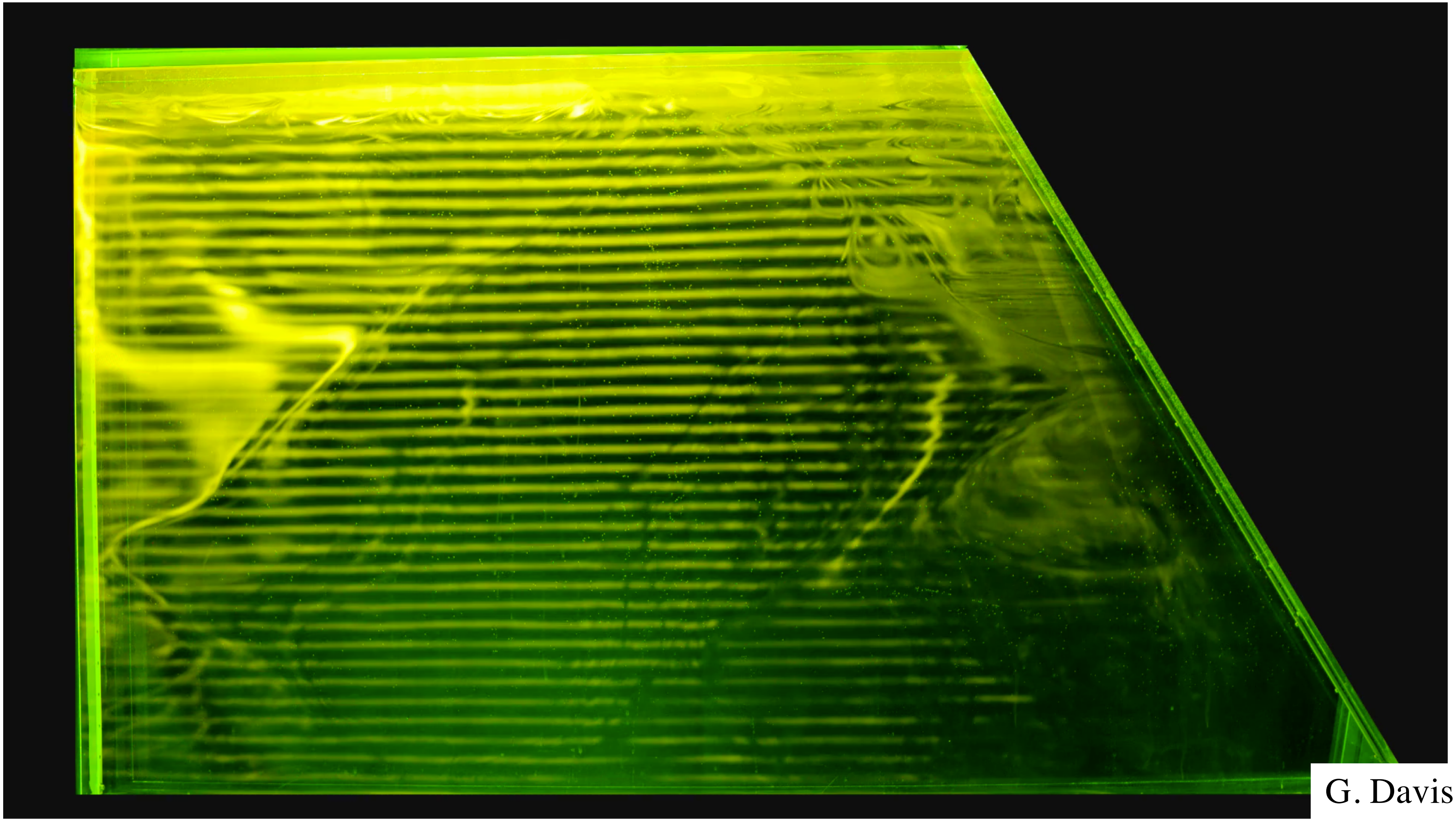
Internal Wave Attractor

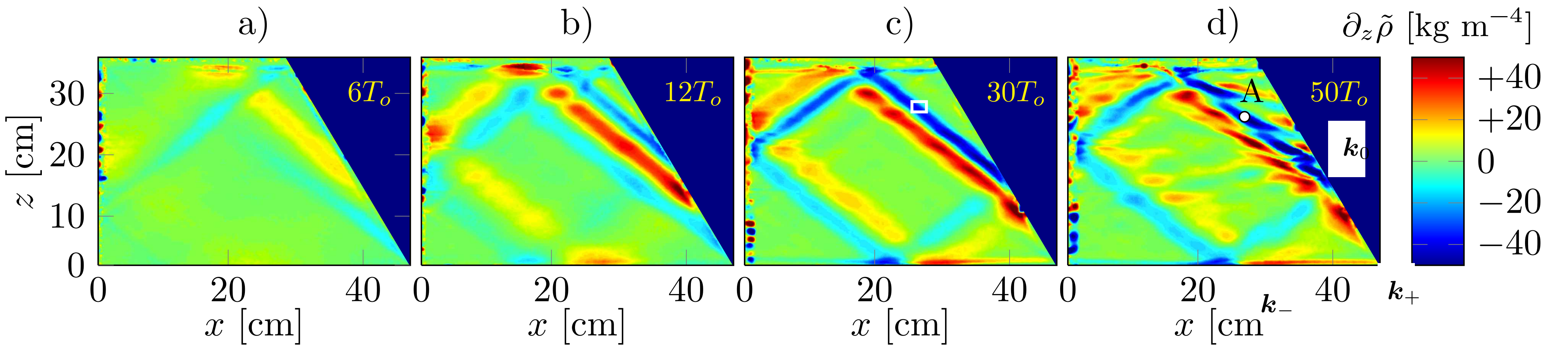


G. Davis

Experimental result

Experimental Results

Scolan, Ermanuyk, Dauxois, *Phys. Rev. Lett. 110*, 234501 (2013).



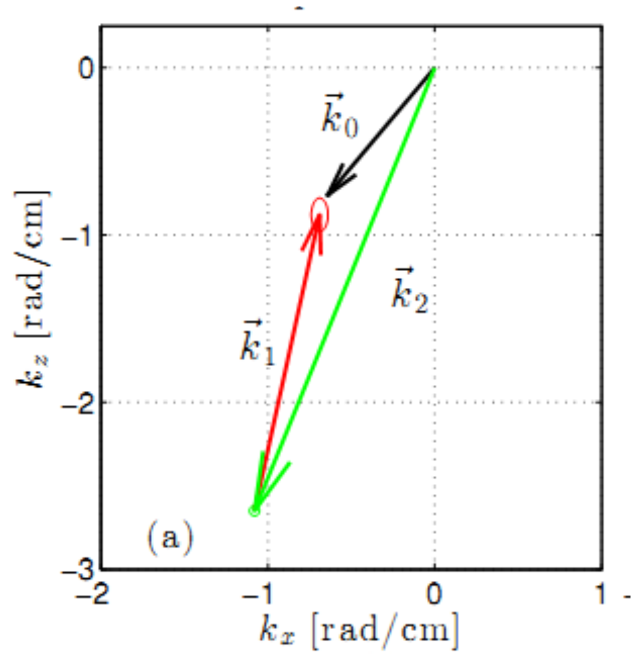
Triadic Resonant Instability

$$\boldsymbol{k}_0 = \boldsymbol{k}_+ + \boldsymbol{k}_-$$

Spatial resonance condition

$$\omega_0 = \omega_+ + \omega_-$$

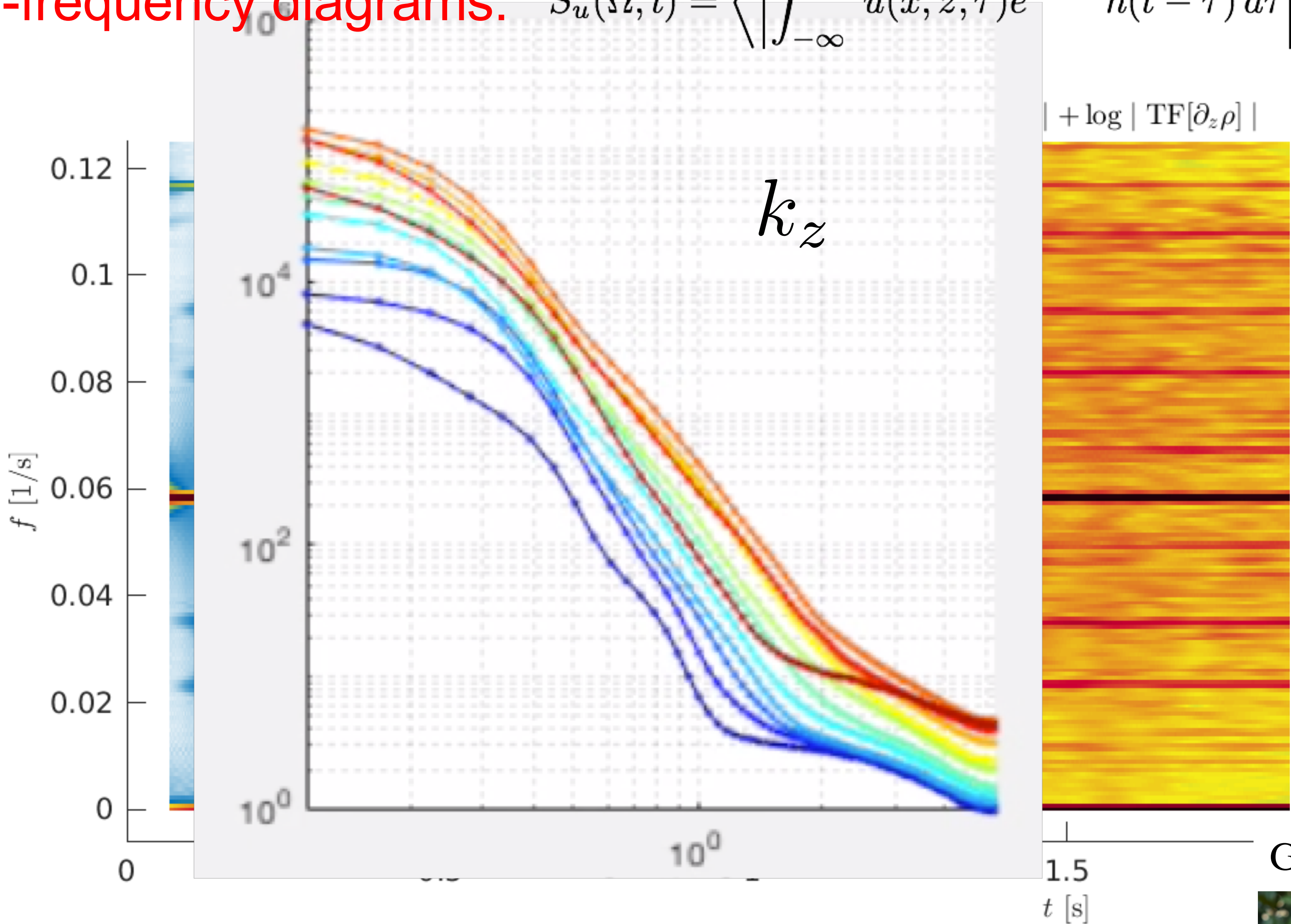
Temporal resonance condition



Brouzet, Sibgatullin, Ermanyuk, Joubaud, Dauxois, *Phys. Rev. Fluids* **2**, 114803 (2017).

Energy Cascade

Time-frequency diagrams: $S_u(\Omega, t) = \left\langle \left| \int_{-\infty}^{+\infty} u(x,z,\tau) e^{i\Omega N \tau} h(t-\tau)\, d\tau \right|^2 \right\rangle_{xz}$

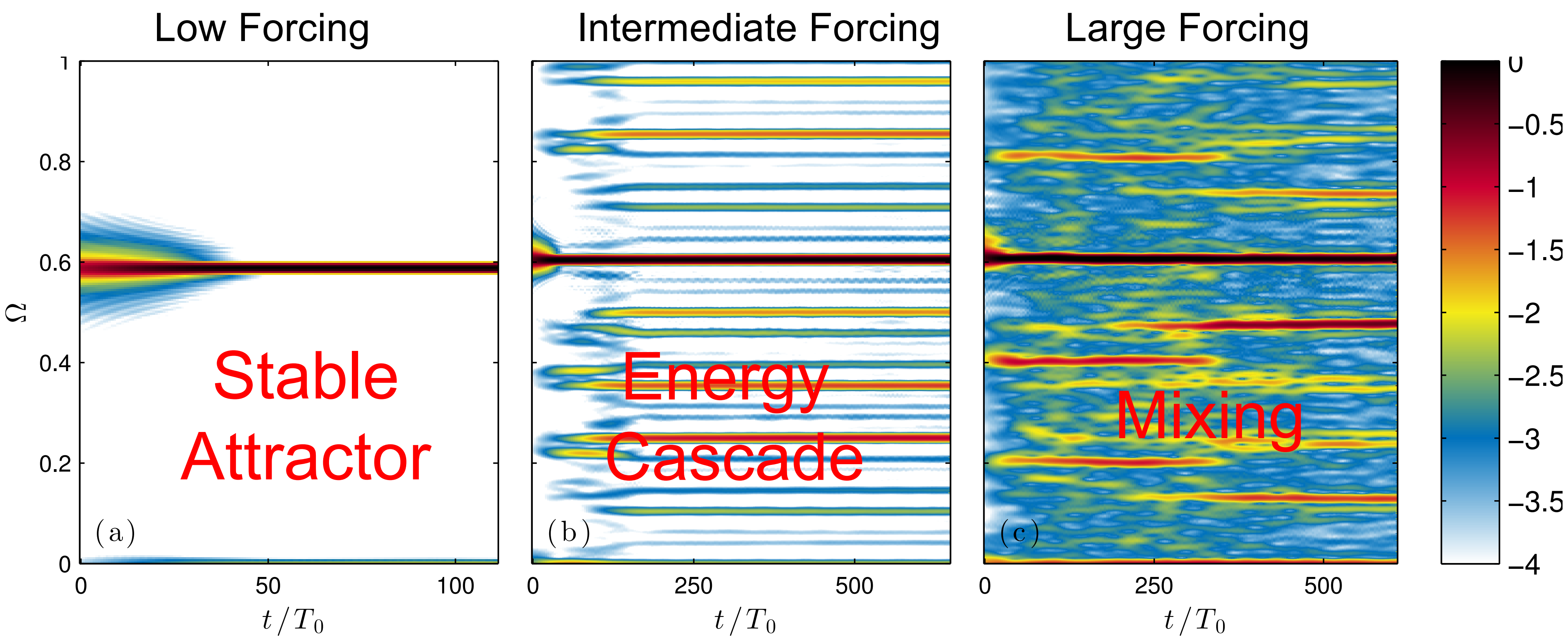


G. Davis

a =3 to 13mm continuously in 4h (=900T_0)

Energy Cascade

Time-frequency diagrams: $S_u(\Omega, t) = \left\langle \left| \int_{-\infty}^{+\infty} u(x,z,\tau) e^{i\Omega N \tau} h(t-\tau)\, d\tau \right|^2 \right\rangle_{xz}$



attractor amplitude

Origin of a cascade in omega

Brouzet, Ermanyuk, Joubaud, Sibgatullin, Dauxois, *EPL* 113, 44001 (2016)

Outline

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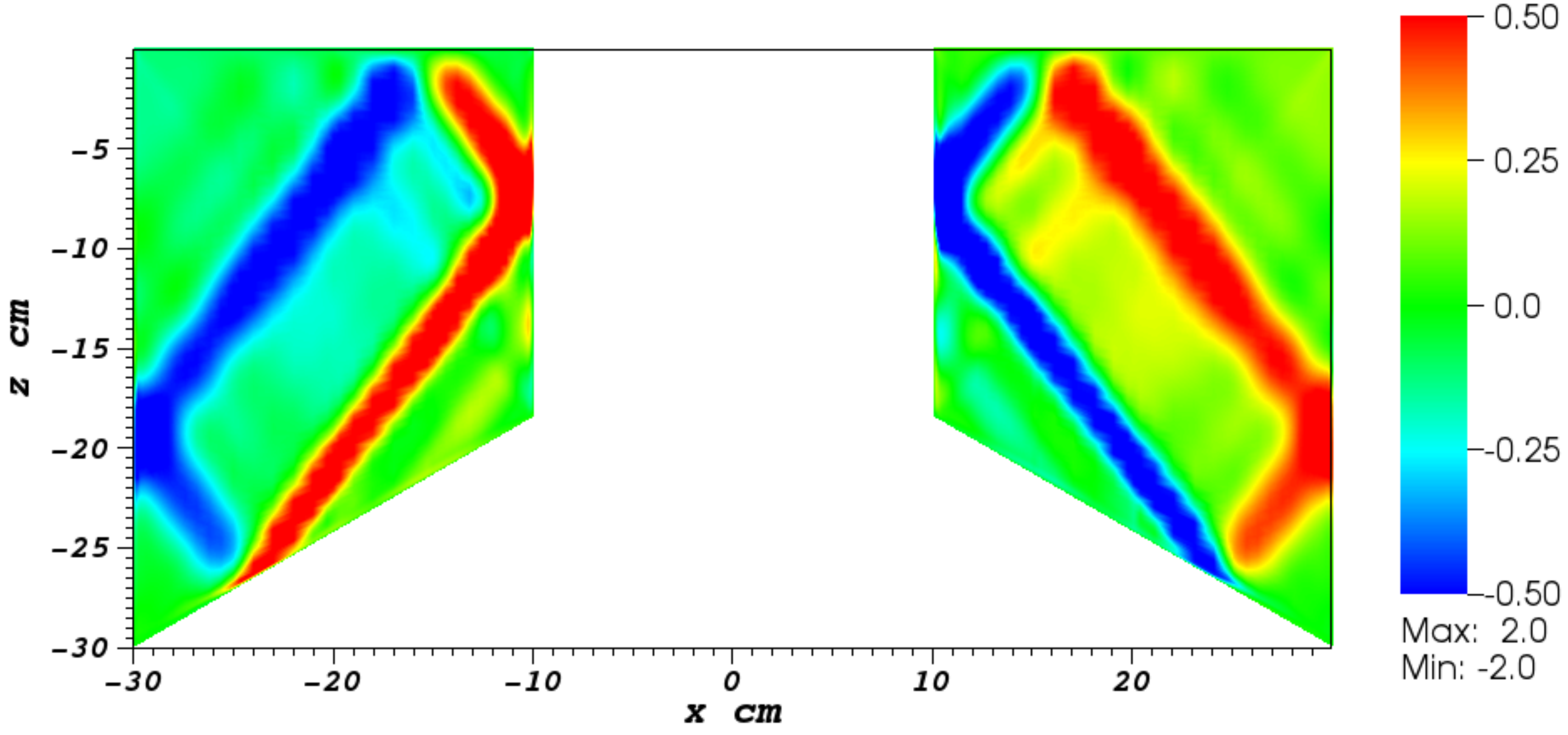
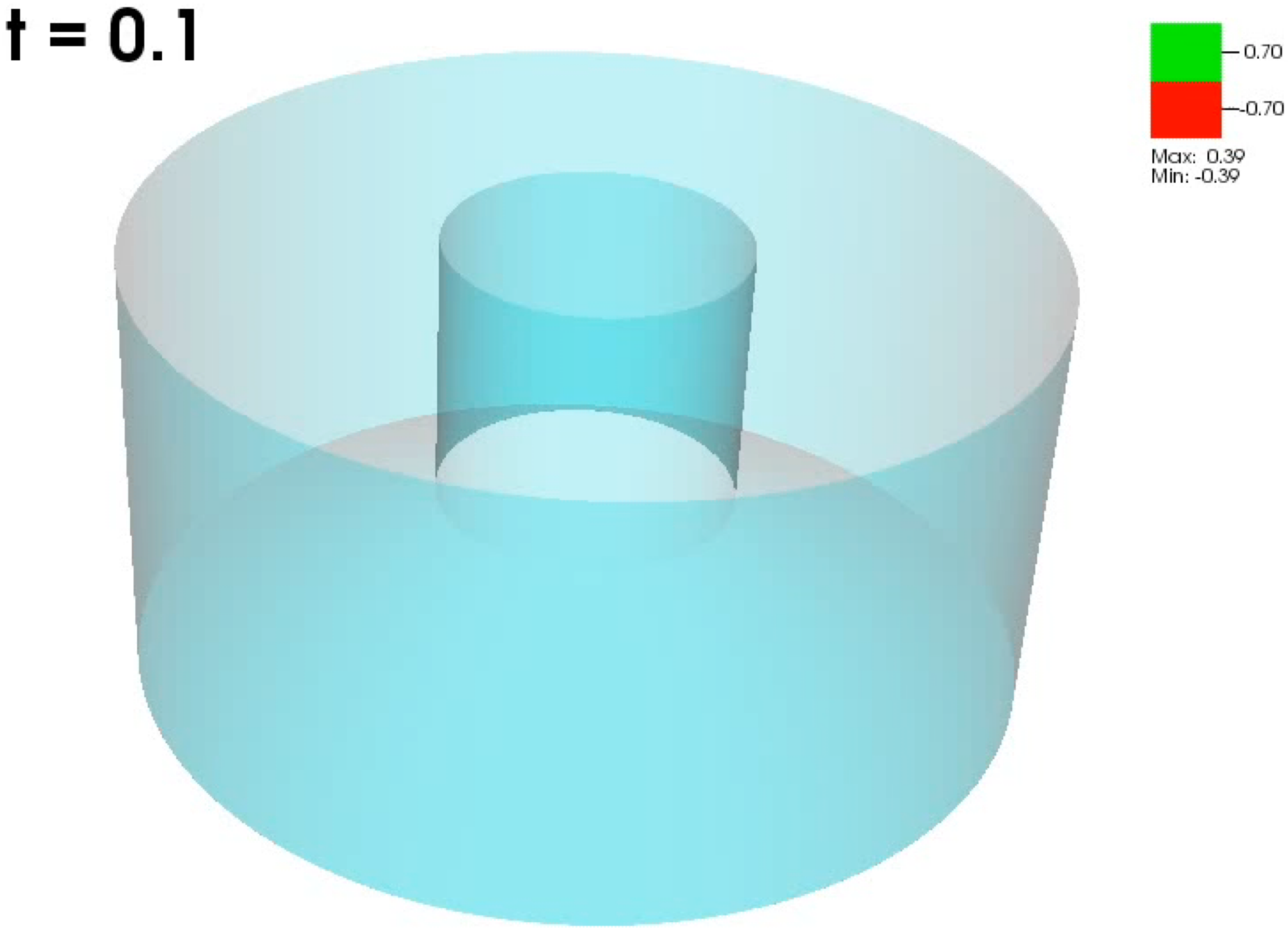
Inertial waves Attractors

Experimental & numerical setups
with rotating homogenous fluid



axisymmetric setup

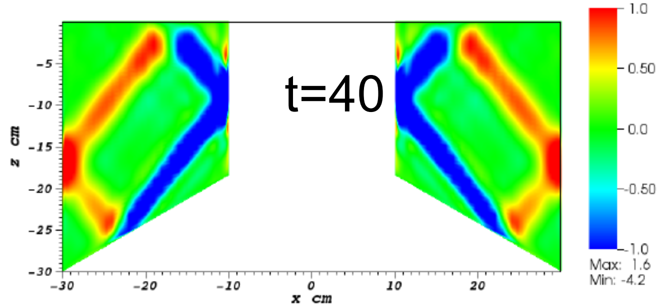
Inertial waves Attractors



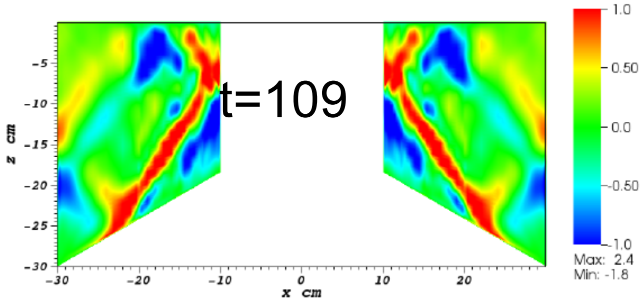
Inertial waves Attractors

Side Views

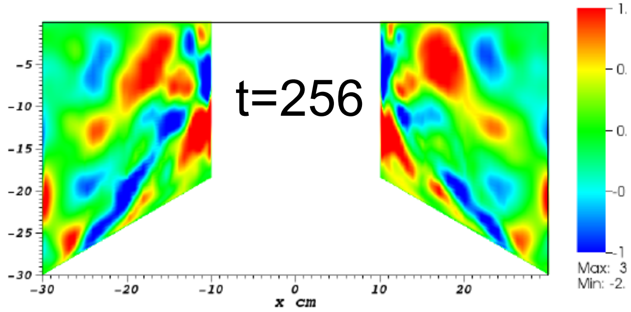
Stable attractor



Development of TRI

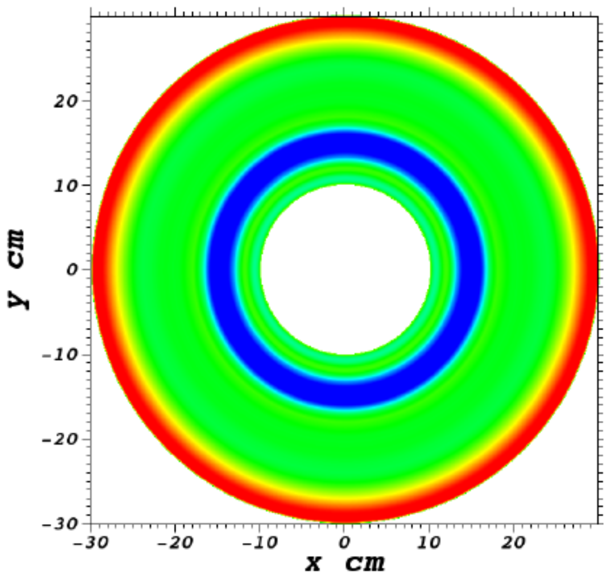


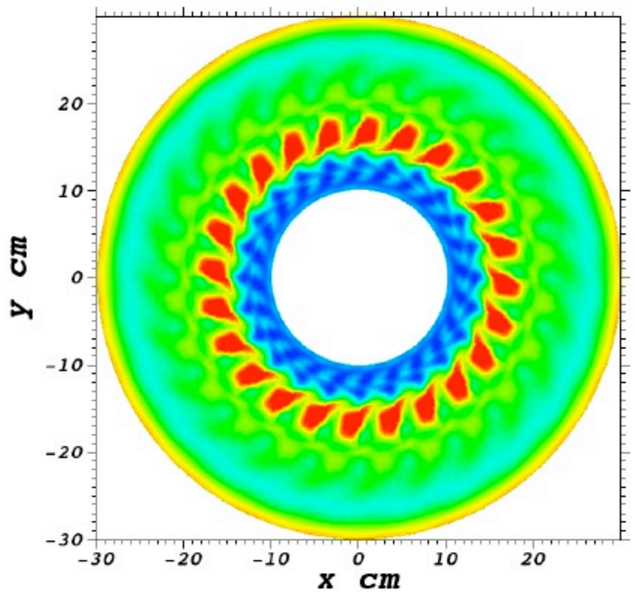
Wave turbulence?

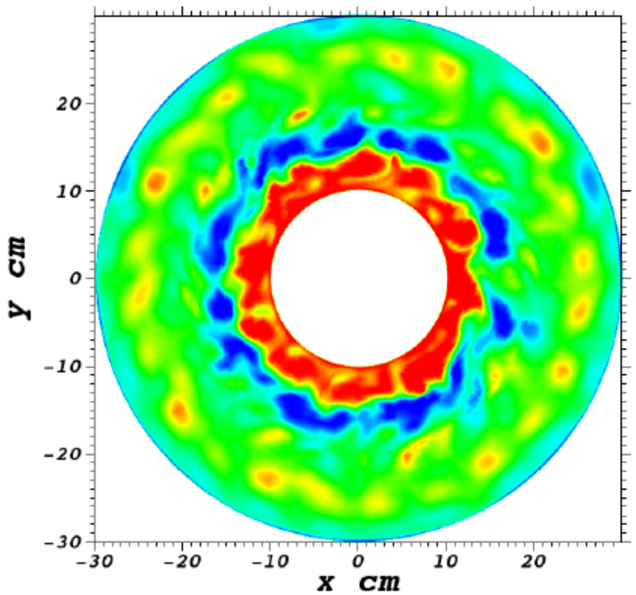


Vertical Velocity

Top Views

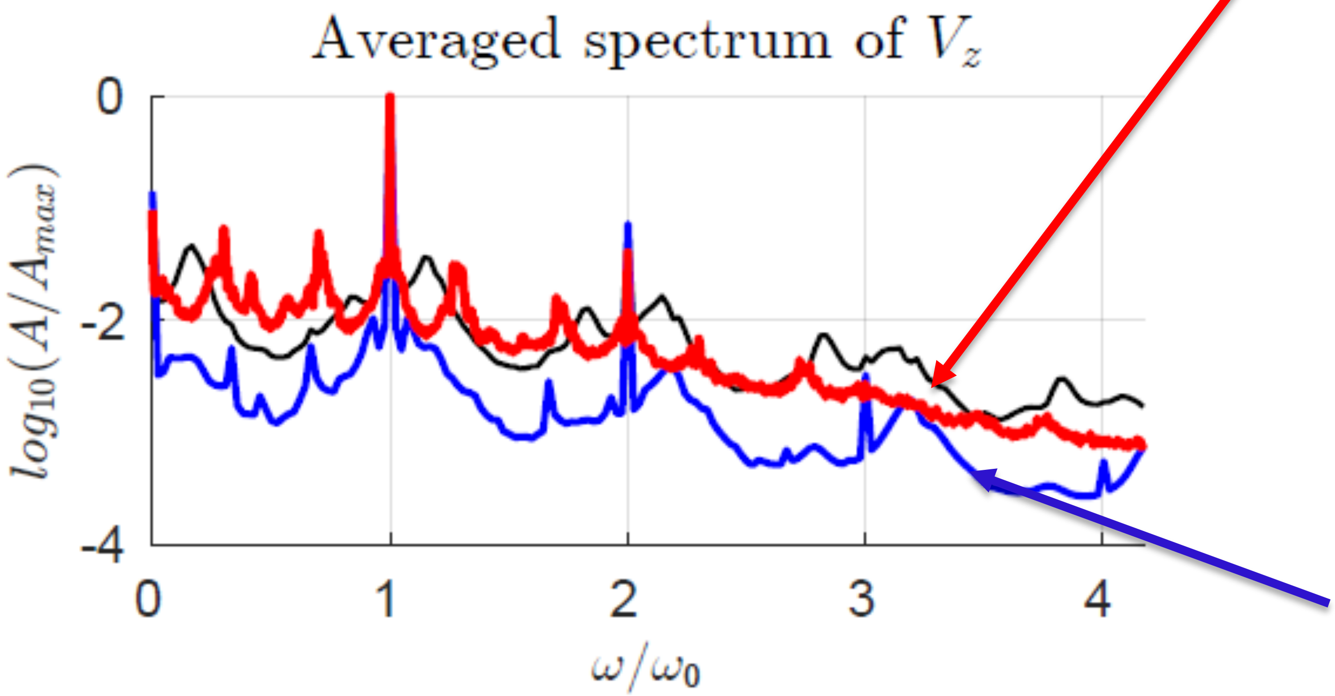


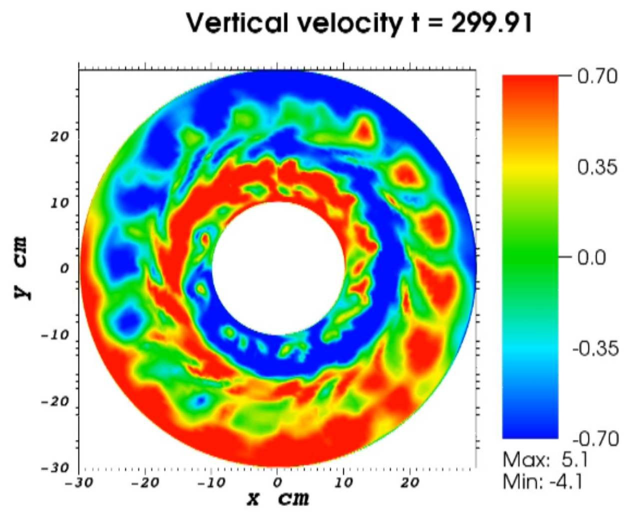


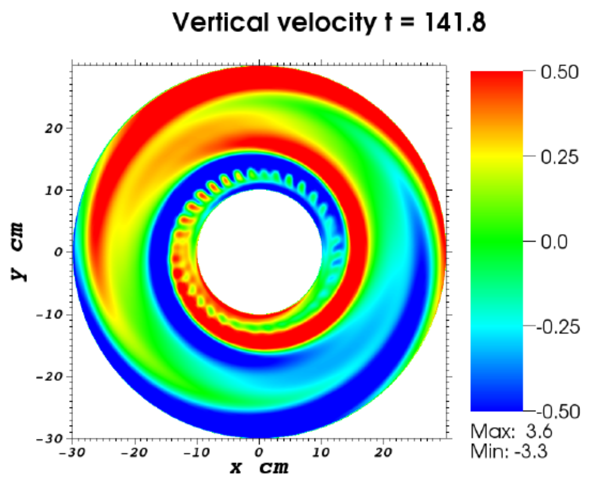


Inertial waves Attractors

Frequency spectra





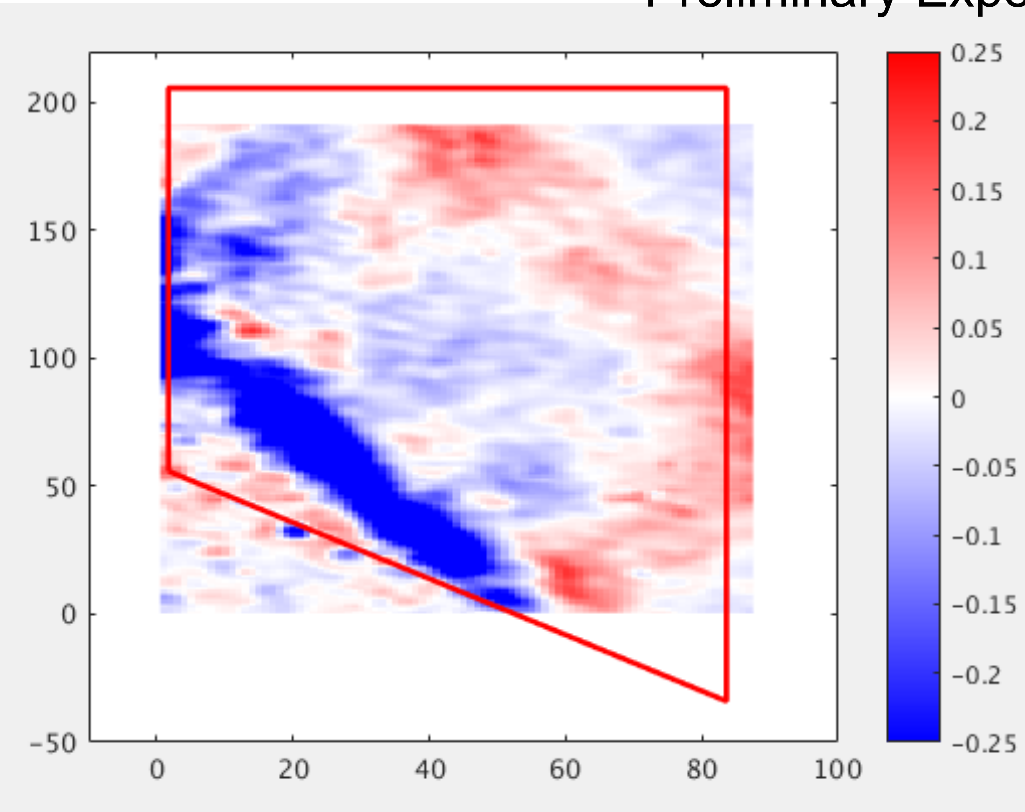


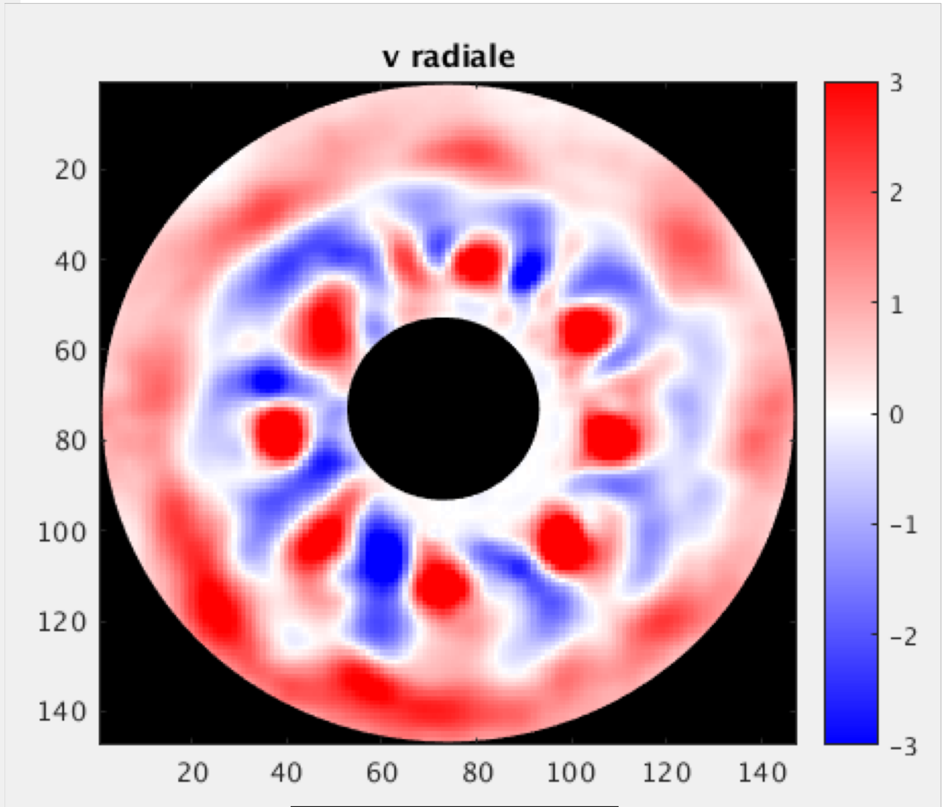
What is the slope at large frequencies?

Relation to Garrett-Munk spectrum and L'vov &Tabak?

Inertial waves Attractors

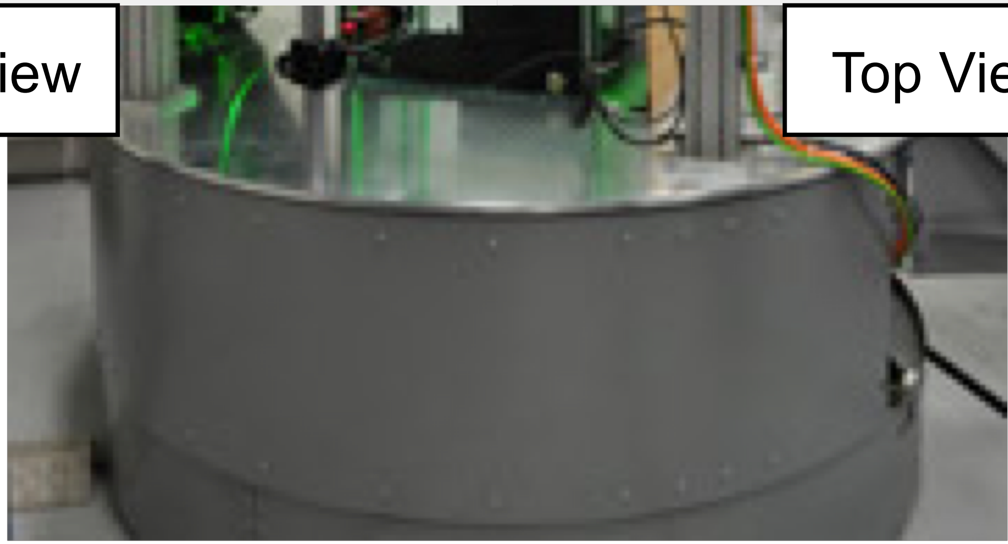
Preliminary Experimental Results





Side View

Top View



S. Boury

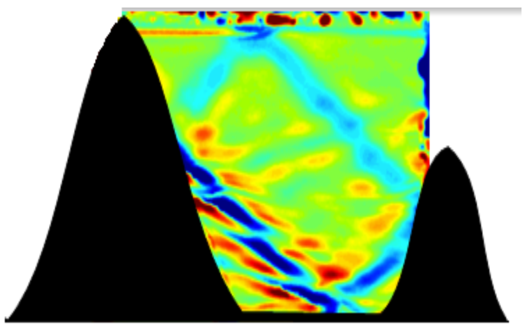
Perspectives

- **Physics**: Excellent experimental set-up to study
  - -Wave-turbulence
  - -Abyssal mixing in the lab

- **Mathematics:**
  - -Spectral properties (modes, quasi-modes)
  - -Viscosity is a mathematical issue
  - -Nonlinearity and validity of the approximations

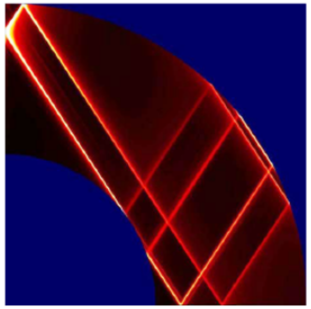
- **Real world analog:**

Attractors between ridges



Source of wave turbulence and mixing

Rotating spherical shells in stars



Analogy rotation vs. stratification

Acknowledgments

S. Boury

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P. Odier

E. Ermanyuk

I. Sibgatullin

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Publications

Brouzet, Ermanyuk, Joubaud, Sibgatullin, Dauxois, *EPL* **113**, 44001 (2016)

Brouzet, Sibgatullin, Scolan, Ermanyuk, Dauxois, *JFM* (2016)

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Brouzet, Sibgatullin, Ermanyuk, Joubaud, Dauxois, *Phys. Rev. Fluids* (2017)

Pillet, Ermanyuk, Maas, Sibgatullin, Dauxois, *JFM* (2018)

Pillet, Maas, Dauxois, *European Journal of Mechanics / B Fluids* (2018)

T. Dauxois, S. Joubaud, P. Odier, A. Venaille, Annual Review of Fluid Mechanics (2018)

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