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An Alternative Method for the Asymptotic Expansion of a Double Integral

A. SELLIER

LADHYX, Ecole Polytechnique, 91128 Palaiseau Cedex, France

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An alternative method is proposed in order to obtain, as $\lambda \to +\infty$, the asymptotic expansion of the double integral

$$K(\lambda) := \int_0^1 \int_0^1 [x^\alpha y^\beta \log^l x \log^j y] f(x, y) g(\lambda x^a y^b)\, dx\, dy,$$

where f is smooth enough and g belongs to $\mathscr{A}(\mathbb{R})$, a space defined in the text. This expansion is derived by employing a powerful tool—the integration in the finite-part sense of Hadamard. © 1995 Academic Press, Inc.

1. INTRODUCTION

Suppose that C designates the set of complex numbers and introduce the set of complex functions $\mathscr{A}(\mathbb{R}) := \{g, g$ is bounded in a neighborhood on the right of zero, $g \in L^1_{\text{loc}}\,(]0, +\infty\,[, C)$, and for all $q \in \mathbb{N}$, $\lim_{x \to +\infty} x^q\, |g(x)| = 0\}$. Observe that if $\mathscr{S}(\mathbb{R})$ denotes the Schwartz space, $\mathscr{S}(\mathbb{R}) \subset \mathscr{A}(\mathbb{R})$. Consider now positive integers j and l, reals a and b with $a > 0$ and $b > 0$, and also two complex numbers α and β with $\text{Re}(\alpha) \geq 0$ and $\text{Re}(\beta) \geq 0$. For a real λ and two complex functions f and g belonging respectively to $\mathscr{C}^\infty(\mathbb{R}^2)$ and to $\mathscr{A}(\mathbb{R})$, the quantity $K(\lambda)$ is defined as

$$K(\lambda) := \int_0^1 \int_0^1 [x^\alpha y^\beta \log^l x \log^j y] f(x, y) g(\lambda x^a y^b)\, dx\, dy. \tag{1.1}$$

It may be useful to establish the asymptotic expansion of the double integral $K(\lambda)$ with respect to the real λ as $\lambda \to +\infty$.

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If $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta) = 0$ and $g \in \mathcal{S}(\mathbb{R})$, Barlet [1] and Bruning [3] showed that such an expansion only involves terms such as $K^{a\alpha}_{nm}(f)\lambda^{-(\alpha+n+1)/a}\log^m \lambda$ and $K^{b\beta}_{nm}(f)\lambda^{-(\beta+n+1)/b}\log^m \lambda$ where $m \in \{1, ..., j + l + 1\}$, n is a positive integer, and $K^{a\alpha}_{nk}$, $K^{b\beta}_{nk}$ are generalized functions with support in the set $\{(x, y) \in [0, 1] \times [0, 1]: xy = 0\}$. For $j = l = 0$, McClure and Wong [7] derived those generalized functions. Their work is based on the use of a convenient and widely employed tool, the Mellin transform (see for instance Bleistein and Handelsman [2] and Wong [10]). Unfortunately they did not propose the general expansion when j and l are any positive integers, although they mention a device to derive it.

Nowadays many approaches are available to deal with the asymptotic expansion of an integral. One may think of the above-mentioned Mellin transform but also of the distributional point of view as recently developed by Estrada and Kanwal in the one-dimensional case [4], but also for multi-dimensional generalized functions [6, 7]. The aim of this paper is to give the general asymptotic expansion of $K(\lambda)$, for g belonging to $\mathcal{A}(\mathbb{R})$ and $\lambda \to +\infty$. This is performed by using an alternative method detailed in Sellier [9].

This paper is organized as follows. In Section 2, the mathematical framework and two basic theorems are exposed. For r real and f smooth enough, the asymptotic expansion of $K(\lambda)$, up to order $o(\lambda^{-r})$, is stated in Section 3. The derivation of this result is established in Section 4. Finally, some examples are proposed in Section 5.

2. Mathematical Framework

The important concept in this work is the integration in the finite part sense of Hadamard, noted $fp \int h(x)\, dx$. For further details the reader is referred to Schwartz [8] and Sellier [9]. It is recalled that C designates the set of complex numbers.

Definition 1. For $\eta > 0$, a complex function g is of the second kind on the open set $]0, \eta[$ if and only if there exist a family of positive integers $(M(n))$, two complex families (β_n), (g_{nm}), and a complex function G such that

$$\forall \varepsilon \in\,]0,\eta[, g(\varepsilon) = \sum_{n=0}^{N} \sum_{m=K(n)}^{m=M(n)} g_{nm}\varepsilon^{\beta_n} \log^m(\varepsilon) + G(\varepsilon), \tag{2.1}$$

$$\mathrm{Re}(\beta_N) < \mathrm{Re}(\beta_{N-1}) < \cdots < \mathrm{Re}(\beta_1) < \mathrm{Re}(\beta_0) := 0,$$
$$\lim_{\varepsilon \to 0} G(\varepsilon) \in C \text{ and } g_{00} := 0 \qquad \text{if } \beta_0 = 0. \tag{2.2}$$

Following Schwartz [8], the complex $\lim_{\varepsilon\to 0} G(\varepsilon)$ is called the finite part in the Hadamard sense of the quantity $g(\varepsilon)$ and denoted by $\mathrm{fp}[g(\varepsilon)]$. If $g_{nm} = 0$ for all (n, m), $\mathrm{fp}[g(\varepsilon)]$ reduces to $\lim_{\varepsilon\to 0} g(\varepsilon)$.

DEFINITION 2. A complex function f is of the first kind on the right at real x_0 if and only if there exist a real $\eta^+ > 0$, a family of positive integers $(J^+(i))$, two complex families (α_i^+), (f_{ij}^+), and a complex function F^+ such that

$$f(x) = \sum_{i=0}^{I} \sum_{j=0}^{J^+(i)} f_{ij}^+ |x - x_0|^{\alpha_i^+} \log^j |x - x_0| + F^+(x - x_0), \text{a.e. in }]x_0, x_0 + \eta^+],$$

$$\mathrm{Re}(\alpha_I^+) < \mathrm{Re}(\alpha_{I-1}^+) < \cdots < \mathrm{Re}(\alpha_1^+) < \mathrm{Re}(\alpha_0^+) := -1, \tag{2.3}$$

$$F^+ \in L^1_{\mathrm{loc}}([x_0, x_0 + \eta^+], C).$$

This definition is extended to the case $x_0 = +\infty$ by stating that f is of the first kind at infinity if and only if there exist a positive real A, a family of positive integers $(J(i))$, two complex families (γ_i), (f_{ij}^∞), and a complex function F^∞ such that

$$f(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} f_{ij}^\infty x^{-\gamma_i} \log^j x + F^\infty(x), \qquad \text{a.e. in } [A, +\infty[,$$

$$\mathrm{Re}(\gamma_I) < \mathrm{Re}(\gamma_{I-1}) < \cdots < \mathrm{Re}(\gamma_1) < \mathrm{Re}(\gamma_0) := 1 \tag{2.4}$$

$$\text{and} \qquad F^\infty \in L^1_{\mathrm{loc}}([A, +\infty[, C).$$

Moreover, for two complex families (α_n), (a_{nm}) and a family of positive integers $(M(n))$, if the sequence $(\mathrm{Re}(\alpha_n))$ is strictly increasing, the abridged notation below is used, for any real r, to designate the specific finite sum of terms

$$\sum_{m,\mathrm{Re}(\alpha)\le r} a_{nm}\varepsilon^{\alpha_n} \log^m \varepsilon := \sum_{n=0}^{N} \sum_{m=0}^{M(n)} a_{nm}\varepsilon^{\alpha_n} \log^m \varepsilon,$$

$$\mathrm{Re}(\alpha_0) < \mathrm{Re}(\alpha_1) < \cdots < \mathrm{Re}(\alpha_{N-1}) < \mathrm{Re}(\alpha_N) \le r \tag{2.5}$$

$$\text{and} \qquad N := \sup\{n, \mathrm{Re}(\alpha_n) \le r\}.$$

If g is a complex function we shall write $\lim_{\varepsilon\to 0} g(\varepsilon) = \sum_{n,\mathrm{Re}(\alpha_n)\le r}^{m,M(n)} a_{nm}\varepsilon^{\alpha_n} \log^m \varepsilon$ if and only if there exist a real $s > r$ and a complex function G_r bounded in a neighborhood of zero in which $g(\varepsilon) = \sum_{m,\mathrm{Re}(\alpha)\le r} a_{nm}\varepsilon^{\alpha_n} \log^m \varepsilon + \varepsilon^s G_r(\varepsilon)$. When there exists m, with $a_{0m} \neq 0$, it is possible to introduce $S_0(g) := \mathrm{Re}(\alpha_0)$. Naturally, we will note $\lim_{u\to+\infty} g(u) = \sum_{n,\mathrm{Re}(\gamma_n)\le r}^{m,M(n)} a_{nm}u^{-\gamma_n} \log^m u$ if and only if there exist a real $s > r$ and a complex function G_r' bounded in a neighborhood of infinity in which $g(u) = \sum_{m,\mathrm{Re}(\gamma)\le r} a_{nm} u^{-\gamma_n} \log^m u + u^{-s} G_r'(u)$. Here if there exists m with $a_{0m} \neq 0$, we set $S_\infty(g) := \mathrm{Re}(\gamma_0)$.

If b is a real with $b > 0$, two spaces are now defined.

DEFINITION 3. For real values r_1 and r_2,

1. $\mathcal{P}(]0, b[, C) := \{f, f$ is a complex pseudofunction of the first kind on the right at zero; if $b < +\infty$ then $f \in L^1_{\mathrm{loc}}(]0, b], C)$, else $f \in L^1_{\mathrm{loc}}(]0, +\infty[, C)$ and f is also of the first kind at infinity$\}$,

2. $\mathcal{E}^{r_2}_{r_1}(]0, b[, C) := \{f, f$ is a complex pseudofunction and there exist complex families (α_i), (A_{ij}) with $(\mathrm{Re}(\alpha_i))$ strictly increasing, a family of positive integers $J(i)$ with $\lim_{x\to 0} f(x) = \sum^{j,J(i)}_{i,\mathrm{Re}(\alpha_i)\le r_1} A_{ij} x^{\alpha_i} \log^j x$; if $b < +\infty$ then $f \in L^1_{\mathrm{loc}}(]0, b], C)$ else $f \in L^1_{\mathrm{loc}}(]0, +\infty[, C)$ and there exist complex families (γ_n), (B_{nm}) with $(\mathrm{Re}(\gamma_n))$ strictly increasing, a family $(M(n))$ with $\lim_{x\to+\infty} f(x) = \sum^{m,M(n)}_{n,\mathrm{Re}(\gamma_n)<r_2} B_{nm} x^{-\gamma_n} \log^m x\}$.

Obviously, for $r_1 \ge -1$ and $r_2 \ge 1$, $\mathcal{E}^{r_2}_{r_1}(]0, b[, C) \subset \mathcal{P}(]0, b[, C)$. Moreover, the operation (for its definition consult Schwartz [8] and Sellier [9]) fp: $f \mapsto \mathrm{fp} \int_0^b f(x)\,dx$ is a linear transformation acting on $\mathcal{P}(]0, b[, C)$. Another set of pseudofunctions, corresponding to an extension of the definition of $\mathcal{E}^{r_2}_{r_1}(]0, b[, C)$ to the dimension two, is introduced.

DEFINITION 4. For reals r_1 and r_2 a pseudofunction $K(x, u)$ belongs to $\mathcal{F}^{r_2}_{r_1}(]0, b[, C)$ if and only if, for λ large enough, $K(x, \lambda x) \in \mathcal{E}(]0, b[, C)$ and it satisfies the following properties:

1. There exist a positive integer N, a complex family (γ_n) with $\mathrm{Re}(\gamma_0) < \cdots < \mathrm{Re}(\gamma_n) := r_2$, families of positive integers $(M(n))$ and of complex pseudofunctions $(K_{nm}(x))$, a real $s_2 > r_2$, a complex function $G_{r_2}(x, u)$, a real $B \ge 0$, and a real $\eta > 0$ such that for any $(x, u) \in]0, b[\times [\eta, +\infty[$,

1.1. $$K(x, u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}(x) u^{-\gamma_n} \log^m u + u^{-s_2} G_{r_2}(x, u), \tag{2.6}$$

1.2. $$\left| \int_\eta^b x^{-s_2} G_{r_2}(x, \lambda x)\, dx \right| \le B < +\infty, \tag{2.7}$$

1.3. there exist a positive integer I, a complex family (α_i) with $\mathrm{Re}(\alpha_0) < \cdots < \mathrm{Re}(\alpha_I) := r_1$, a family of positive integer $(J(i))$, with, for $n \in \{0, ..., N\}$, $m \in \{0, ..., M(n)\}$, $K_{nm} \in \mathcal{E}^{1\ \mathrm{Re}(\gamma_n)}_{\mathrm{Re}(\gamma_n)-1}(]0, b[, C)$, and there also exist a complex family (K^{ij}_{nm}), a real $s_1 < r_1$, a complex function L_{nm} bounded in a neighborhood of zero in which

$$K_{nm}(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K_{nm}^{ij} x^{\alpha_i} \log^j x + x^{s_1} L_{nm}(x). \tag{2.8}$$

2. For the same η, I, s_1 families (α_i) and $(J(i))$, there exist a real $A \geq \eta > 0$, a family of complex pseudofunctions (h^{ij}), a complex function $H_{r_1}(x, u)$, a real $B' \geq 0$, and a real $W > 0$ such that for $0 < x \leq W$ and $u > 0$,

2.1. $$K(x, u) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} h^{ij}(u) x^{\alpha_i} \log^j x + x^{s_1} H_{r_1}(x, u), \tag{2.9}$$

2.2. $$\left| \int_0^A u^{s_1} H_{r_1}(u/\lambda, u)\, du \right| \leq B' < +\infty, \tag{2.10}$$

2.3. for $i \in \{0, \ldots, I\}$ and $j \in \{0, \ldots, J(i)\}$, $h^{ij} \in \mathscr{C}^{1+\mathrm{Re}(\alpha_i)}_{-1-\mathrm{Re}(\alpha_i)}(]0, +\infty[, C)$. More precisely, $\lim_{u\to 0} h^{ij}(u) = \sum_{p,\mathrm{Re}(\beta_p)\leq -1-\mathrm{Re}(\alpha_i)}^{q,Q(p)} H_{pq}^{ij} u^{\beta_p} \log^q u$ and also $\lim_{u\to+\infty} h^{ij}(u) = \sum_{n,\mathrm{Re}(\gamma_n)\leq 1+\mathrm{Re}(\alpha_i)}^{m,M(n)} K_{nm}^{ij} u^{-\gamma_n} \log^m u$. Moreover, there exists a complex function O_{ij} bounded in a neighborhood of infinity in which

$$h^{ij}(u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}^{ij} u^{-\gamma_n} \log^m u + u^{-s_2} O_{ij}(u). \tag{2.11}$$

3. Finally, there exists a complex function $W_{r_1,r_2}(x, u)$ bounded in $]0, \eta] \times [A, +\infty[$, defined as

$$\begin{aligned} x^{s_1} u^{-s_2} W_{r_1,r_2}(x, u) := K(x, u) &- \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}(x) u^{-\gamma_n} \log^m u \\ &- \sum_{i=0}^{I} \sum_{j=0}^{J(i)} \left[h^{ij}(u) - \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}^{ij} u^{-\gamma_n} \log^m u \right] x^{\alpha_i} \log^j x. \end{aligned} \tag{2.12}$$

Taking into account all these notations or definitions and setting (if n and p are positive integers with $p \leq n$) $C_n^p := n!/[p!(n - p)!]$, we state two basic theorems whose derivations are provided in Sellier [9].

THEOREM 1. *For a real r, if there exist $r_1 \geq r - 1$ and $r_2 \geq r$ such that $K(x, u) \in \mathscr{F}_{r_1}^{r_2}([0, b[, C)$, then the integral $I(\lambda)$ admits the following expansion with respect to the large parameter λ:*

$$
\begin{aligned}
I(\lambda) &:= \mathrm{fp}\int_0^b K(x, \lambda x)\,dx \\
&= \sum_{m,\mathrm{Re}(\gamma)\le r}\sum_{l=0}^{m} C_m^l\left[\mathrm{fp}\int_0^b K_{nm}(x)x^{-\gamma_n}\log^{m-l}(x)\,dx\right]\lambda^{-\gamma_n}\log^l\lambda \\
&+ \sum_{j,\mathrm{Re}(\alpha)\le r-1}\sum_{l=0}^{j} C_j^l(-1)^l\left[\mathrm{fp}\int_0^\infty h^{ij}(v)v^{\alpha_i}\log^{j-l}(v)\,dv\right. \qquad (2.13)\\
&- \sum_{\{p;\beta_p=-\alpha_i-1\}}\sum_{q=0}^{Q(p)}\frac{H_{pq}^{ij}}{1+j+q-l}\log^{1+j+q-l}\lambda \\
&+ \left.\sum_{\{n;\gamma_n=1+\alpha_i\}}\sum_{m=0}^{M(n)}\frac{K_{nm}^{ij}}{1+j+m-l}\log^{1+j+m-l}\lambda\right]\lambda^{-(\alpha_i+1)}\log^l\lambda + o(\lambda^{-r}),
\end{aligned}
$$

where each sum $\Sigma_{m,\mathrm{Re}(\gamma)\le v}$ *is defined by* (2.5).

It is important to note that coefficients occurring in the expansion given by this theorem are integrals in the finite-part sense of Hadamard, even if the initial quantity $I(\lambda)$ reduces to a usual integration (in Lebesgue's sense). This is one of the advantages of this approach: the concept of integration in the finite part sense of Hadamard appears naturally.

In fact, one often encounters the particular case where $K(x, \lambda x) = h(x)H(\lambda x)$, i.e., the variables x and $u := \lambda x$ are separated. The theorem below deals with these circumstances.

THEOREM 2. *Consider two complex pseudofunctions* $h \in \mathscr{C}(]0, b[, C)$ *and also* $H \in \mathscr{C}([0, +\infty[, C)$. *Assume that there exist three reals* t, v, w *with* $t \ge -S_0(H)$, $w \ge \mathrm{Max}(-t, -1 - S_0(h))$ *and also if* $b = +\infty$, $v \ge \mathrm{Max}(1 - S_\infty(H), 1 - t)$, *and* $t \ge 1 - S_\infty(h)$. *Moreover, assume that also* $h \in \mathscr{C}_{t-1}^{v}(]0, b[, C)$, $H \in \mathscr{C}_w^t(]0, +\infty[, C)$, *and* $h(x)H(\lambda x) \in \mathscr{P}(]0, b[, C)$. *Then, for any real* $r \le t$, *the integral* $J(\lambda)$ *admits the expansion*:

$$
\begin{aligned}
J(\lambda) &:= \mathrm{fp}\int_0^b h(x)H(\lambda x)\,dx \\
&= \sum_{m,\mathrm{Re}(\gamma)\le r}\sum_{l=0}^{m} C_m^l H_{nm}^\infty\left[\mathrm{fp}\int_0^b h(x)x^{-\gamma_n}\log^{m-l}(x)\,dx\right]\lambda^{-\gamma_n}\log^l\lambda \\
&+ \sum_{j,\mathrm{Re}(\alpha)\le r-1}\sum_{l=0}^{j} C_j^l(-1)^l h_{ij}^0\left[\mathrm{fp}\int_0^\infty H(v)v^{\alpha_i}\log^{j-l}(v)\,dv\right. \qquad (2.14)\\
&- \sum_{\{p;\beta_p=-\alpha_i-1\}}\sum_{q=0}^{Q(p)}\frac{H_{pq}^0}{1+j+q-l}\log^{1+j+q-l}\lambda \\
&+ \left.\sum_{\{n;\gamma_n=1+\alpha_i\}}\sum_{m=0}^{M(n)}\frac{H_{nm}^\infty}{1+j+m-l}\log^{1+j+m-l}\lambda\right]\lambda^{-(\alpha_i+1)}\log^l\lambda + o(\lambda^{-r}),
\end{aligned}
$$

where it is understood that $\lim_{x\to 0} h(x) = \sum_{j,\mathrm{Re}(\alpha)\le l-1} h^0_{ij} x^{\alpha_i} \log^j x$ *and for function* H, $\lim_{u\to\infty} H(u) = \sum_{m,\mathrm{Re}(\gamma)\le r} H^{\infty}_{nm} u^{-\gamma_n} \log^m u$, $\lim_{u\to 0} H(u) = \sum_{q,\mathrm{Re}(\beta)\le w} H^0_{pq} u^{\beta_p} \log^q u$.

3. The Expansion of $K(\lambda)$

For $(p, q) \in \mathbb{N}^2$, $\mathcal{D}^{(p,q)}(]0, 1[^2)$ is the set of complex functions f such that $\forall (i, j) \in \mathbb{N}^2$ with $0 \le i \le p$ and $0 \le j \le q$, then $\partial^i\partial^j[f]/(\partial x^i \partial y^j)$ exists and is continuous on an open set containing $]0, 1[^2$.

The expansion of $K(\lambda)$ is stated in the following theorem where the quantities a, b, α, β, j, and l remain those defined in the introduction.

Theorem 3. *Consider* $r \ge \mathrm{Max}(\mathrm{Re}(\alpha)/a, \mathrm{Re}(\beta)/b) + 1$ *and integers* $p_x := [\![ar - \mathrm{Re}(\alpha)]\!]$, $p_y := [\![br - \mathrm{Re}(\beta)]\!]$ *where* $[\![d]\!]$ *denotes the integer part of real* d, $N := \sup\{q \in \mathbb{N}, 1 + q + \mathrm{Re}(\beta) \le br\} = [\![br - \mathrm{Re}(\beta)]\!] - 1$, $I := \sup\{q \in \mathbb{N}, 1 + q + \mathrm{Re}(\alpha) \le ar\} = [\![ar - \mathrm{Re}(\alpha)]\!] - 1$ *and* $N_i := \inf\{q \in \mathbb{N}, a[\mathrm{Re}(\beta) + q + 1] > b[\mathrm{Re}(\alpha) + i + 1]\}$. *If* $f \in \mathcal{D}^{(p_x,p_y)}(]0, 1[^2)$, *the integral* $K(\lambda)$ *defined by* (1.1) *admits, as* λ *tends to infinity, the expansion*

$$K(\lambda) = \sum_{n=0}^{N} \sum_{m=0}^{j} K^{b\beta}_{nm}(f)\lambda^{-(\beta+n+1)/b} \log^m \lambda + \sum_{i=0}^{I} \sum_{m=0}^{l} K^{a\alpha}_{im}(f)\lambda^{-(\alpha+i+1)/a} \log^m \lambda$$
$$+ \sum_{i=0}^{I} \sum_{n=0}^{N_i-1} \sum_{m=0}^{1+j+l} S_{inm}(f)\lambda^{-(\alpha+i+1)/a} \log^m \lambda + o(\lambda^{-r}), \tag{3.1}$$

where $K^{b\beta}_{nm}$, $K^{a\alpha}_{im}$, *and* S_{inm} *are generalized functions defined as*

$$K^{b\beta}_{nm}(f) := \frac{C^m_j}{n!} \sum_{k=0}^{j-m} C^k_{j-m} \left(-\frac{1}{b}\right)^m \left(-\frac{a}{b}\right)^k \left[\int_0^\infty u^{\beta+n} \log^{j-m-k}(u) g(u^b)\, du\right]$$
$$\times \left[\mathrm{fp} \int_0^1 x^{\alpha-(a/b)(\beta+n+1)} \log^{l+k}(x) f^n_y(x, 0)\, dx\right], \tag{3.2}$$

$$K^{a\alpha}_{im}(f) := \frac{C^m_l}{i!} \sum_{k=0}^{l-m} C^k_{l-m} \left(-\frac{1}{a}\right)^m \left(-\frac{b}{a}\right)^k \left[\int_0^\infty u^{\alpha+i} \log^{l-m-k}(u) g(u^a)\, du\right]$$
$$\times \left[\mathrm{fp} \int_0^1 y^{\beta-(b/a)(\alpha+i+1)} \log^{j+k}(y) f^i_x(0, y)\, dy\right], \tag{3.3}$$

and $S_{inm}(f) := 0$ *if* $a(\beta + n + 1) \ne b(\alpha + i + 1)$; *else*

$$S_{inm}(f) := \frac{(-1)^{j+l} j! l!}{(1+j+l)!} \left(\frac{1}{a}\right)^{l+1} \left(\frac{1}{b}\right)^{j+1} \frac{f_{xy}^{in}(0,0)}{n! i!} C_{1+j+l}^{m} \times \left[\int_0^\infty u^{(\beta+n+1)/b-1} \log^{1+j+l-m}(u^{-1}) g(u)\, du \right]. \tag{3.4}$$

Since $g \in \mathcal{A}(\mathbb{R})$, the integrals involving the function g in formulas (3.2), (3.3), and (3.4) are legitimate. Moreover, the assumption $f \in \mathcal{D}^{(p_x, p_y)}(]0,1[^2)$ guarantees the existence of each integration appearing in the finite part sense of Hadamard.

Observe that formula (3.1) provides an asymptotic expansion up to order $o(\lambda^{-r})$ when function f is smooth enough. In fact, if $r < \mathrm{Max}(\mathrm{Re}(\alpha)/a, \mathrm{Re}(\beta)/b) + 1$ and $f \in \mathcal{D}^{(1,1)}(]0,1[^2)$ then $K(\lambda) = o(\lambda^{-r})$. If $f \in \mathcal{C}^{\infty}(\mathbb{R}^2)$, this theorem agrees with the results of Barlet [1] and McClure and Wong [7] (when $j = l = 0$) and extends their validity to the case $g \in \mathcal{A}(\mathbb{R})$.

4. Derivation of the Expansion

This derivation is achieved in several steps and uses Theorems 1 and 2.

4.1.

Step 1: Choice of $K(x, u)$. Suppose that λ is a large and positive real and set $\lambda' := \lambda^{1/a}$. If a new complex function G defined by $G(x) := g(x^a)$ is introduced, then $K(\lambda)$ rewrites as

$$\begin{aligned} K(\lambda) &= \int_0^1 x^\alpha \log^l x \left[\int_0^1 y^\beta \log^j y f(x,y) G(\lambda' x y^{b/a})\, dy \right] dx \\ &= \int_0^1 K(x, \lambda' x)\, dx \end{aligned} \tag{4.1}$$

where

$$K(x,u) := x^\alpha \log^l x \int_0^1 y^\beta \log^j y f(x,y) G(u y^{b/a})\, dy. \tag{4.2}$$

Obviously $K(x, \lambda' x) \in \mathcal{P}(]0,1[, C)$. Now each property of Definition 4 is investigated.

4.1.1. *Property* 1. We set $u' = u^{a/b}$ and use the function H with $H(x) := G(x^{b/a}) = g(x^b)$. Hence

$$K(x,u) = K(x,u') = x^{\alpha}\log^{l} x \int_0^1 y^{\beta}\log^{j} y f(x,y) H(u'y)\,dy. \tag{4.3}$$

Since $u' \to +\infty$ when $u \to +\infty$, for any $x \in]0,1[$ the asymptotic expansion of $K(x,u)$ as $u \to +\infty$ is obtained by applying Theorem 2 to the expansion of $K(x,u')$ as $u' \to +\infty$. Here, the definition of H and the assumption $g \in \mathcal{A}(\mathbb{R})$ show that $H^{\infty}_{nm} = 0$ (since $\lim_{x\to+\infty} x^n\,|H(x)| = 0$, $\forall n \in \mathbb{N}$), $h(y)$ which presents an adequate expansion near $y = 0$ with $\alpha_i := \beta + i$ and $h^0_{ij} := f^i_y(x,0)/i!$, and finally that $\mathrm{fp}\int_0^{\infty} H(v)v^{\beta+i}\log^{j-l'}(v)\,dv = \int_0^{\infty} v^{\beta+i}\log^{j-l'}(v)g(v^b)\,dv$ and thereby exists. Actually, for positive integers i and n we have adopted the usual notation: $f^{in}_{xy} := \partial^i\partial^n f/\partial x^i\partial y^n$. Keeping in mind the definition $N := \sup\{q \in \mathbb{N},\ \mathrm{Re}(\beta) + q + 1 \le br\}$, the expansion of $K(x,u)$ for $x > 0$, as $u \to +\infty$ and up to order $o(u'^{-br})$, is

$$\frac{K(x,u)}{x^{\alpha}\log^{l}(x)} \sim \sum_{i=0}^{N} \frac{f^i_y(x,0)}{i!} \sum_{l'=0}^{j} C^{l'}_j(-1)^{l'} \left[\int_0^{\infty} v^{\beta+i}\log^{j-l'}(v)H(v)\,dv\right] u'^{-(\beta+i+1)}\log^{l'} u'.$$

Consequently, the reader may check that the relations (2.6) and (2.7) are satisfied by $K(x,u)$ with $\gamma_n := a(\beta + n + 1)/b$, $M(n) := j$, $r_2 := a[\mathrm{Re}(\beta) + N + 1]/b$, and

$$\begin{aligned} K_{nm}(x) &:= \frac{C^m_j(-1)^m}{n!}\left(\frac{a}{b}\right)^m \left[\int_0^{\infty} v^{\beta+n}\log^{j-m}(v)g(v^b)\,dv\right] x^{\alpha}\log^{l} x f^n_y(x,0) \\ &= E_{nm}x^{\alpha}\log^{l} x f^n_y(x,0). \end{aligned} \tag{4.4}$$

Because $f \in \mathcal{D}^{(l+1,N+1)}([0,1]^2)$ this last equality (4.4) shows that $K_{nm}(x)$ admits an appropriate expansion near $x = 0$; i.e., condition (2.8) is fulfilled with $\alpha_i := \alpha + i$, $j := l$, $K^{il}_{nm} := E_{nm}f^{in}_{xy}(0,0)/i!$, and also $r_1 := \mathrm{Re}(\alpha) + l$.

4.1.2. *Properties* 2 *and* 3. The expansion of $K(x,u)$ as $x \to 0$ is required. Since $f \in \mathcal{D}^{(l+1,N+1)}([0,1]^2)$ it is legitimate to derivate the integral appearing in relation (4.2) with respect to the variable x and thereby to obtain the required expansion. The reader may verify that relations (2.9) and (2.10) are fulfilled with $\alpha_i := \alpha + i$, $j = l$ and

$$h^{il}(u) := \frac{1}{i!}\int_0^1 y^{\beta}\log^{j} y f^i_x(0,y)G(uy^{b/a})\,dy, \tag{4.5}$$

$$x^{r_1}H_{r_1}(x,u) := \frac{x^{\alpha+l+1}\log^{l}(x)}{(l+1)!}\left[\int_0^1 y^{\beta}\log^{j} y f^{l+1}_x(\theta(x)x,y)G(uy^{b/a})\,dy\right], \tag{4.6}$$

where $0 < \theta(x) < 1$, for all $x \in [0,1]$.

Since g is bounded in a neighborhood on the right of zero, $h^{il}(u)$ satisfies the same property. If $u \to +\infty$, the expansion of $h^{il}(u)$ is deduced from an application of Theorem 2 (as already achieved for the expansion of $K(x, u)$ for x given and $u \to +\infty$). Hence, if $u \to +\infty$ and up to order $o(u^{-ar})$,

$$h^{il}(u) \sim \sum_{n=0}^{N} \sum_{m=0}^{j} \frac{C_j^m(-1)^m}{i!n!} \left(\frac{a}{b}\right)^m \left[\int_0^\infty v^{\beta+n} \log^{j-m}(v) g(v^b)\, dv\right] f_{xy}^{in}(0,0) u^{-\gamma_n} \log^m u. \tag{4.7}$$

Consequently, h^{il} obeys relation (2.11). Moreover, use of Definition (4.6) ensures property 3 of Definition 2.3.

4.2.

Step 2: Application of Theorem 1 to $K(x, u)$. As $K(x, u)$ satisfies all the required properties, Theorem 1 applies. However, since $h^{il}(u)$ is bounded in a neighborhood of zero and $\mathrm{Re}(\alpha_i) = \mathrm{Re}(\alpha) + i \geq 0$, the term of the right-hand side of (2.13) containing H_{pq}^{ij} is zero. Thus the expansion of $K(\lambda) = K(\lambda')$ reduces to $K(\lambda') = T_1(\lambda') + T_2(\lambda') + T_3(\lambda') + o(\lambda'^{-R})$ with $R := \mathrm{Min}(r_1 + 1, r_2)$, and:

(i) $T_1(\lambda')$ is the first sum arising on the right-hand side of (2.13), i.e.,

$$T_1(\lambda') = \sum_{n=0}^{N} \sum_{m=0}^{j} \sum_{k=0}^{m} C_m^k \left[\mathrm{fp} \int_0^1 K_{nm}(x) x^{-\gamma_n} \log^{m-k}(x)\, dx\right] \lambda'^{-\gamma_n} \log^k \lambda'; \tag{4.8}$$

(ii) $T_2(\lambda')$ is the second sum on the right-hand side of (2.13), i.e.,

$$T_2(\lambda') = \sum_{i=0}^{I} \sum_{k=0}^{l} C_l^k(-1)^k \left[\mathrm{fp} \int_0^\infty h^{il}(u) u^{\alpha_i} \log^{l-k}(u)\, du\right] \lambda'^{-(\alpha_i+1)} \log^k \lambda'; \tag{4.9}$$

(iii) $T_3(\lambda')$ is the last sum occurring on the right-hand side of (2.13) which is related to the behavior of h^{il} near infinity. Here,

$$T_3(\lambda') = \sum_{i=0}^{I} \sum_{k=0}^{l} C_l^k(-1)^k \sum_{\{n;\gamma_n=\alpha_i+1\}} \sum_{m=0}^{j} \frac{K_{nm}^{il}}{1+l+m-k} \lambda'^{-(\alpha_i+1)} \log^{1+l+m} \lambda'. \tag{4.10}$$

Note that the definition of R, r_1, r_2, and $\lambda' := \lambda^{1/a}$ leads to $o(\lambda'^{-R}) = o(\lambda^{-r})$. Each contribution $T_1(\lambda')$, $T_2(\lambda')$, and $T_3(\lambda')$ is treated below.

in which Q_n is the displacement of the nth mass from equilibrium and P_n is its momentum. (For the convenience of the reader, in this section we shall denote the space variable by n, $n \in \mathbf{Z}$, whereas the time variable will be t, $t \geq 0$.) To simplify the notation, we omit the dependence on the time variable.

Considering (6.1) for a lattice infinitely long in both directions we set

$$\begin{aligned} a(n) &= \tfrac{1}{2}e^{(Q_{n+1}-Q_n)/2} \\ b(n) &= -P_n/2, \qquad n \in \mathbf{Z}. \end{aligned} \tag{6.2}$$

We emphasize that $a(n)$ and $b(n)$ depend smoothly on a parameter t. In addition, it is evident that $a(n) > 0$ for all n.

We turn our consideration to motion which is confined in some finite region of the lattice, assuming no motion in the distance. Therefore, for $|n| \gg 1$, we have

$$Q_{n+1} - Q_n = 0, \qquad P_n = 0$$

and hence, we can think of $a(n) \to \frac{1}{2}$ and $b(n) \to 0$ rapidly as $|n| \to \infty$.

To derive the connection between the Toda lattice and the Schrödinger discrete second-order eigenvalue problem, we introduce the self-adjoint operator $\mathbf{L}$ and the skew-adjoint operator $\mathbf{B}$, acting on $H = l^2(\mathbf{Z},)$ by the formulas

$$\begin{aligned} \mathbf{L}y(n) &= a(n-1)y(n-1) + b(n)y(n) + a(n)y(n+1) \\ \mathbf{B}y(n) &= a(n)y(n+1) - a(n-1)y(n-1). \end{aligned}$$

In view of the setting given by (6.2), the *Lax representation* of the equations of motion (6.1) is given by

$$\frac{d\mathbf{L}}{dt} = [\mathbf{B}, \mathbf{L}] = \mathbf{BL} - \mathbf{LB}.$$

It implies that all the eigenvalues λ of $\mathbf{L}$ are time-independent (they are constants of the motion). In addition, since the operator $\mathbf{L}$ is self-adjoint, they are also real.

From the asymptotic behavior of the coefficients $a(n)$ and $b(n)$ of the operator $\mathbf{L}$, the eigenvalue equation

$$\mathbf{L}\varphi(n) = \lambda\varphi(n) \tag{6.3}$$

is asymptotically close to

$$\tfrac{1}{2}[\varphi(n-1) + \varphi(n+1)] = \lambda\varphi(n). \tag{6.4}$$

The translation $\mathbf{L} \mapsto \mathbf{LE}$ transforms (6.3) to

$$\varphi(n+2) + \tilde{b}(n)\varphi(n+1) + \tilde{a}(n-1)\varphi(n) = \lambda\frac{1}{a(n)}\varphi(n+1), \tag{6.5}$$

where $\tilde{b}(n) = b(n)/a(n)$ and $\tilde{a}(n-1) = a(n-1)/a(n)$, satisfying $\tilde{b}(n) \to 0$ and $\tilde{a}(n) \to 1$ rapidly as $|n| \to \infty$. Hence, Eq. (6.5) can be written as

$$\varphi(n+2) - 2\lambda\varphi(n+1) + \varphi(n) = v_1(n)\varphi(n) + v_2(n)\varphi(n+1) \tag{6.6}$$

for some functions $v_1(n)$ and $v_2(n)$ that decay sufficiently fast as $|n| \to \infty$. In our notation from Section 3, the matrix formulation of Eq. (6.6) gives

$$\mathbf{x}(n+1) = [\mathcal{A}_\lambda + \mathcal{B}(n)]\mathbf{x}(n), \tag{6.7}$$

where

$$\mathbf{x}(n) = \begin{bmatrix} \varphi(n) \\ \varphi(n+1) \end{bmatrix}$$

and

$$\mathcal{A}_\lambda = \begin{bmatrix} 0 & 1 \\ -1 & 2\lambda \end{bmatrix} \quad \text{and} \quad \mathcal{B}(n) = \begin{bmatrix} 0 & 0 \\ v_1(n) & v_2(n) \end{bmatrix}.$$

Set $\lambda = \frac{1}{2}(z + z^{-1})$. In light of our previous results, the Jost solutions $\varphi_j(n, z)$ of (6.6) are characterized by

$$\begin{aligned} \varphi_1(n, z) &\sim z^n && \text{as } n \to \infty \\ \varphi_2(n, z) &\sim z^{-n} && \text{as } n \to -\infty. \end{aligned}$$

It is evident that, for $|z| = 1$, the pairs of functions $\{\varphi_1(n, z), \varphi_1(n, z^{-1})\}$ and $\{\varphi_2(n, z), \varphi_2(n, z^{-1})\}$ form a distinguished basis of solutions for the difference operator $\mathbf{L} - \lambda\mathbf{I}$ as $n \to \infty$ and $n \to -\infty$, as long as $z \neq \pm 1$. We have one may be tempted both to invert the integrations and to use the change of variable $u^a := v^a y^b$ without caution. Unfortunately, L_{ik} denotes an integral in the finite part sense of Hadamard and this concept requires great care when attempting any transformation, such as change of variable, inversion of the integrations. First a useful result is recalled.

PROPOSITION 1. *For reals c and d with* $0 < c < c, j \in \mathbb{N}$, *and* $\alpha \in C$

$$\int_c^d x^\alpha \log^j x\, dx = P^j_\alpha(d) - P^j_\alpha(c), \tag{4.21}$$

with

$$P^j_{-1}(t) = \frac{\log^{j+1}(t)}{j+1} \text{ and } P^j_\alpha(t) = t^{\alpha+1} \sum_{k=0}^{j} \frac{(-1)^{j-k} j!}{k!(\alpha+1)^{1+j-k}} \log^k(t), \qquad \text{for } \alpha \neq -1. \tag{4.22}$$

For i and k given and positive integers we set $N_i := \inf\{q \in \mathbb{N}, a[\mathrm{Re}(\beta) + q + 1] > b[\mathrm{Re}(\alpha) + i + 1]\}$, $F_{N_i}(y) := f^i_x(0, y) - \sum_{n=0}^{N_i-1} f^{in}_{xy}(0,0)y^n/n!$, and introduce the function $W_{N_i}(v) := \int_0^1 y^\beta \log^j yF_{N_i}(y)\, g(v^a y^b)\, dy$. Observe that for $\mathrm{Re}(\alpha) + i + 1 \leq ar$ this definition of N_i ensures $a(\mathrm{Re}(\beta) + N_i) \leq b(\mathrm{Re}(\alpha) + i + 1) \leq abr$, i.e., $N_i \leq br - \mathrm{Re}(\beta) \leq N + 1$. Because $f \in \mathscr{D}^{(I+1,N+1)}(]0,1[^2)$, this remark shows that such a definition of F_{N_i} is legitimate. Application of Theorem 2 provides the behavior of this function $W_{N_i}(v)$, as $v \to +\infty$. More precisely,

$$W_{N_i}(v) \sim \sum_{p=0}^{P} \sum_{m'=0}^{j} A_{pm'} \frac{\partial^p F_{N_i}}{\partial y^p}(0) v^{-\gamma_p} \log^{m'} v; \gamma_p := a(\beta + p + 1)/b, \tag{4.23}$$

and $(A_{pm'})$ are complex coefficients.

These notations allow L_{ik} to be cast into the following form

$$L_{ik} = \int_0^\infty \left[\int_0^1 y^\beta \log^j yF_{N_i}(y) g(v^a y^b)\, dy \right] v^{\alpha+i} \log^{l-k}(v)\, dv + \sum_{n=0}^{N_i-1} \frac{f^{in}_{xy}(0,0)}{n!} fp \int_0^\infty \left[\int_0^1 y^{\beta+n} \log^j(y) g(v^a y^b)\, dy \right] v^{\alpha+i} \log^{l-k}(v)\, dv. \tag{4.24}$$

Observe that the first term on the right-hand side of (4.24), noted A, is a usual integral. Indeed, according to (4.22) and (4.23), the possible divergent contribution as $v \to +\infty$ corresponds to the circumstances $\mathrm{Re}(\alpha) + i + 1 - \mathrm{Re}(\gamma_p) \geq 0$, i.e., $a[\mathrm{Re}(\beta) + p + 1] \leq b[\mathrm{Re}(\alpha) + i + 1]$, which may be only obtained for $p \leq N_i - 1$ (see definition of N_i). But if $p \leq N_i - 1$, $\partial^p F_{N_i}/\partial y^p(0) = 0$. It is also legitimate to apply successively Fubini's theorem and a change of variable $u^a := v^a y^b$ to this integral A. Hence

$$A = \int_0^1 y^{\beta-(b/a)(\alpha+i+1)} \log^j y F_{N_i}(y) \left[\int_0^\infty u^{\alpha+i} \log^{l-k}(uy^{-b/a}) g(u^a)\, du \right] dy. \tag{4.25}$$

Actually, this is a usual integration since $\mathrm{Re}(\beta) - \mathrm{Re}[b(\alpha + i + 1)]/a + N_i + 1 > 0$ (see the above definitions of integer N_i and function F_{N_i}). Taking into account the linearity of the transformation fp: $h \mapsto \mathrm{fp} \int_0^1 h(x)\, dx$ and (4.25), L_{ik} is rewritten as

$$\begin{aligned} L_{ik} = {} & \mathrm{fp} \int_0^1 y^{\beta-(b/a)(\alpha+i+1)} \log^j y f_x^i(0, y) \left[\int_0^\infty u^{\alpha+i} \log^{l-k}(uy^{-b/a}) g(u^a)\, du \right] dy \\ & - \sum_{n=0}^{N_i-1} \frac{f_{xy}^{in}(0,0)}{n!} \left\{ \mathrm{fp} \int_0^1 y^{\beta+n-(b/a)(\alpha+i+1)} \log^j y \right. \\ & \left[\int_0^\infty v^{\alpha+i} \log^{l-k}(vy^{-b/a}) g(v^a)\, dv \right] dy \\ & \left. - \mathrm{fp} \int_0^\infty \left[\int_0^1 y^{\beta+n} \log^j(y) g(v^a y^b)\, dy \right] v^{\alpha+i} \log^{l-k}(v)\, dv \right\}. \end{aligned} \tag{4.26}$$

The first term on the right-hand side of (4.26) is noted M_{ik}. Use of the Newton binomial formula ensures

$$\begin{aligned} T_4(\lambda') := {} & \sum_{i=0}^{I} \sum_{k=0}^{l} \frac{C_l^k (-1)^k}{i!} M_{ik} \lambda'^{-(\alpha+i+1)} \log^k \lambda' \\ = {} & \sum_{i=0}^{I} \frac{1}{i!} \left\{ \mathrm{fp} \int_0^1 y^{\beta-(b/a)(\alpha+i+1)} \log^j(y) f_x^i(0, y) \right. \\ & \left. \left[\int_0^\infty u^{\alpha+i} E_i^l(x, u, \lambda') g(u^a)\, du \right] dy \right\} \end{aligned} \tag{4.27}$$

with

$$\lambda'^{(\alpha+i+1)}E_i^l(x,u,\lambda') := \sum_{k=0}^{l} C_l^k(-1)^k \log^k(\lambda^{1/a}) \log^{l-k}(uy^{-b/a})$$

$$= \log^l(uy^{-b/a}\lambda^{-1/a}) = \sum_{m=0}^{l}\sum_{k=0}^{l-m} C_l^m C_{l-m}^k \tag{4.28}$$

$$\left(-\frac{1}{a}\right)^m \left(-\frac{b}{a}\right)^k \log^k(y) \log^{l-m-k}(u) \log^m \lambda.$$

Thus,

$$T_4(\lambda') = \sum_{i=0}^{l} \frac{1}{i!} \sum_{m=0}^{l} \left[\sum_{k=o}^{l-m} E_{a,b,\alpha,\beta}^{l,m,k} \text{fp} \int_0^1 y^{\beta-(b/a)(\alpha+i+1)} \log^{j+k}(y) f_x^i(0,y)\, dy \right] \tag{4.29}$$
$$\lambda^{-(\alpha+i+1)/a} \log^m \lambda$$

with

$$E_{a,b,\alpha,\beta}^{l,m,k} := C_l^m C_{l-m}^k \left(-\frac{1}{a}\right)^m \left(-\frac{b}{a}\right)^k \left[\int_0^\infty u^{\alpha+i} \log^{l-m-k}(u) g(u^a)\, du\right]. \tag{4.30}$$

Note that $T_4(\lambda')$ is the second term on the right-hand side of (3.1).

Now, in order to deal with the remaining terms on the right-hand side of (4.26) a useful lemma is stated. The set $\mathcal{B}(\mathbb{R})$ is introduced as $\mathcal{B}(\mathbb{R}) := \{g, \forall \gamma \in C$ with $\text{Re}(\gamma) > -1, \forall q \in \mathbb{N}\ x^\gamma \log^q(x) g(x) \in L^1_{\text{loc}}([0, +\infty[, C)$, $\lim_{x\to+\infty} x^q\, |g(x)| = 0$ and for any real $r > 0$, $X^r \int_0^1 y^\gamma \log^q(y) g(Xy)\, dy$ is bounded as $X \to 0^+\}$. Observe that $\mathcal{S}(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$.

LEMMA 1. *Consider reals $a > 0$, $b > 0$, complex numbers γ and δ with* $\text{Re}(\gamma) \geq 0$, $\text{Re}(\delta) \geq 0$, *positive integers j and q and a function $g \in \mathcal{B}(\mathbb{R})$. If we set*

$$\mathcal{D}_j^q(\gamma,\delta,g) := \text{fp} \int_0^1 y^{\gamma-(b/a)(\delta+1)} \log^j(y) \left[\int_0^\infty v^\delta \log^q(vy^{-b/a}) g(v^a)\, dv\right] dy, \tag{4.31}$$

$$\mathcal{D}'^q_j(\gamma,\delta,g) := \text{fp} \int_0^\infty \left[\int_0^1 y^\gamma \log^j(y) g(v^a y^b)\, dy\right] v^\delta \log^q(v)\, dv, \tag{4.32}$$

then $\Delta_j^q(\gamma,\delta,g) := \mathcal{D}_j^q(\gamma,\delta,g) - \mathcal{D}'^q_j(\gamma,\delta,g) = 0$; *except if* $a(\gamma+1) = b(\delta+1)$. *In this latter case*

$$\Delta_j^q(\gamma, \delta, g) = \left(\frac{1}{a}\right)^{1+q}\left(\frac{1}{b}\right)^{1+j}\left[\sum_{m=0}^{j} \frac{C_j^m(-1)^m}{m+q+1}\right] \left[\int_0^\infty u^{(\gamma+1)/b-1} \log^{j+q+1}(u) g(u)\, du\right]. \tag{4.33}$$

Proof. For the sake of simplicity we note $\mathcal{D}_j^q(g) = \mathcal{D}_j^q(\gamma, \delta, g)$. Observe that

$$\mathcal{D}_j^q(g) = \sum_{m=0}^{q} C_q^m \left(-\frac{b}{a}\right)^m \left[\mathrm{fp}\int_0^1 y^{\gamma-(b/a)(\delta+1)} \log^{m+j}(y)\, dy\right] \left[\int_0^\infty v^\delta \log^{q-m}(v) g(v^a)\, dv\right]. \tag{4.34}$$

First suppose that $a(\gamma + 1) = b(\delta + 1)$. Then, see (4.34) and (4.22), $\mathcal{D}_j^q(g) = 0$. Use of the change of variable $z := v^a y^b$ gives for the integral $d := \int_0^1 y^\gamma \log^j(y) g(v^a y^b)\, dy$,

$$d = \sum_{m=0}^{j} C_j^m \left(-\frac{a}{b}\right)^m \left[\frac{\log^m(v)}{b} v^{-(a/b)(\gamma+1)} \int_0^{v^a} z^{(\gamma+1)/b-1} \log^{j-m}(z^{1/b}) g(z)\, dz\right]. \tag{4.35}$$

Taking into account the assumption $a(\gamma + 1) = b(\delta + 1)$ and relations (4.22) and (4.32), an integration by parts (always valid when dealing with integration in the finite part sense of Hadamard) makes it possible to write

$$\mathcal{D}'^q_j(g) = \mathrm{fp}[\mathcal{F}(v)]_0^\infty - \frac{1}{b}\sum_{m=0}^{j} \frac{C_j^m(-1)^m}{m+q+1}\left(\frac{a}{b}\right)^m \left[\int_0^\infty a v^{(a/b)(\gamma+1)-1} \log^{m+q+1}(v) \log^{j-m}(v^{a/b}) g(v^a)\, dv\right] \tag{4.36}$$

with

$$\mathcal{F}(v) := \frac{1}{b}\sum_{m=0}^{j} C_j^m \left(-\frac{a}{b}\right)^m \frac{\log^{m+q+1}(v)}{m+q+1} v^{\delta+1-(a/b)(\gamma+1)} \int_0^{v^a} z^{(\gamma+1)/b-1} \log^{j-m}(z^{1/b}) g(z)\, dz = \sum_{m=0}^{j}\sum_{n=0}^{j-m} \frac{C_j^m C_{j-m}^n}{m+q+1} \left(-\frac{a}{b}\right)^m v^{\delta+1} \log^{m+q+1}(v) \log^{j-m-n}(v^{a/b}) \left[\int_0^1 y^\gamma \log^n(y) g(v^a y^b)\, dy\right].$$

Since $\int_0^1 y^\gamma \log^n(y) g(v^a y^b)\, dy = b^{-(n+1)} \int_0^1 Y^{(1/b)(\gamma+1)-1} \log^n Y g(v^a Y)\, dY$ with $\mathrm{Re}[(\gamma+1)/b - 1] > -1$ and g belongs to $\mathcal{B}(\mathbb{R})$ (see the assumption bearing on $X^r \int_0^1 y^\gamma \log^q(y) g(Xy)\, dy$ as $X \to 0^+$), it is easy to obtain $\mathcal{F}(0) = 0$. Moreover, application of Theorem 2 ensures that for all real $r > a(\mathrm{Re}(\gamma) + 1)/b$, as $v \to \infty$,

$$\int_0^1 y^\gamma \log^n(y) g(v^a y^b)\, dy = \sum_{l'=0}^{n} C_n^{l'} \left(-\frac{a}{b}\right)^{l'} \left[\int_0^\infty t^\gamma \log^{n-l'}(t) g(t^b)\, dt\right] v^{-(a/b)(\gamma+1)} \log^{l'} v + o(v^{-r}). \tag{4.37}$$

Taking into account the assumption $a(\gamma+1) = b(\delta+1)$, $\mathcal{F}(v)$ rewrites as $v \to \infty$: $\mathcal{F}(v) = \sum_{m=0}^{j} \sum_{n=0}^{j-m} \sum_{l'=0}^{n} F_{m,n,l'} \log^{1+q+j-n+l'}(v) + o(v^{-1})$. Consequently (according to Definition 1 with $\varepsilon := v^{-1}$), $\mathrm{fp}[\mathcal{F}(v)]_0^\infty = 0$. Finally, change of variable $u := v^a$ in the integrals appearing on the right-hand side of (4.36) provides the equality (4.33).

Now suppose that $a(\gamma+1) \neq b(\delta+1)$ and that $j = 0$. With relation (4.22) equality (4.34) rewrites as

$$\mathcal{D}_0^q(g) = -q! \left(\frac{a}{b}\right) \sum_{m=0}^{q} \frac{(-1)^m}{(q-m)!} \frac{1}{[\delta + 1 - (a/b)(\gamma+1)]^{m+1}} \left[\int_0^\infty v^\delta \log^{q-m}(v) g(v^a)\, dv\right]. \tag{4.38}$$

Change of variable $z := v^a y^b$, integration by parts, and relation (4.22) lead to

$$\mathcal{D}'^q_0(g) = \mathrm{fp}[\xi(v)]_0^\infty - \frac{a}{b} \int_0^\infty \left[\sum_{m'=0}^{q} \frac{q!(-1)^{q-m'} v^\delta \log^{m'}(v)}{m'![1 + \delta - (a/b)(\gamma+1)]^{1+q-m'}}\right] g(v^a)\, dv, \tag{4.39}$$

with

$$\xi(v) := \sum_{m'=0}^{q} \frac{q!(-1)^{q-m'} v^{\delta+1} \log^{m'}(v)}{m'![1 + \delta - (a/b)(\gamma+1)]^{1+q-m'}} \left[\int_0^1 y^\gamma g(v^a y^b)\, dy\right]. \tag{4.40}$$

Here (see asymptotic expansion (4.37), with $n = 0$), the function ξ takes the form $\xi(v) = \sum_{m'=0}^{q} E_{m'} v^{\delta+1-a(\gamma+1)/b} \log^{m'} v + o(v^{-1})$, with $b(\delta+1) \neq a(\gamma+1)$ and $\xi(0) = 0$. Thus, $\mathrm{fp}[\xi(v)]_0^\infty = 0$. If we set $m := q - m'$ in equality (4.38), $\mathcal{D}'^q_0(g) = \mathcal{D}_0^q(g)$. The case $j \neq 0$ is obtained by induction. Suppose that

$a(\gamma + 1) \neq b(\delta + 1)$ and that for all positive integer q and all functions $h \in \mathcal{B}(\mathbb{R})$, $\mathcal{D}_j^q(h) = \mathcal{D}'^q_j(h)$. This is true for $j = 0$. As $\log^{j+1}(y) = -a \log^j(y)[\log(vy^{-b/a}) - \log(v)]/b$, Definition (4.31) takes the following form:

$$\mathcal{D}^q_{j+1}(g) = -\frac{a}{b}\mathcal{D}^{q+1}_j(g) + \frac{1}{b}\,\mathrm{fp}\int_0^1 y^{\gamma-(b/a)(\delta+1)} \log^j(y) \left[\int_0^\infty v^\delta \log^q(vy^{-b/a}) g(v^a) \log(v^a)\, dv\right] dy. \tag{4.41}$$

On the other hand, as $\log(y) = \log(y^b v^a)/b - a\log(v)/b$, Definition (4.32) is rewritten as

$$\mathcal{D}'^q_{j+1}(g) = -\frac{a}{b}\mathcal{D}'^{q+1}_j(g) + \frac{1}{b}\,\mathrm{fp}\int_0^\infty \left[\int_0^1 y^\gamma \log^j(y) \log(v^a y^b) g(v^a y^b)\, dy\right] v^\delta \log^q(v)\, dv. \tag{4.42}$$

Observe that if $g \in \mathcal{B}(\mathbb{R})$ and $w(x) := \log(x)g(x)$ then $w \in \mathcal{B}(\mathbb{R})$. Consequently, if we choose $h(x) := \log(x)g(x)$ or $h(x) := g(x)$, the induction assumption shows that $\mathcal{D}^q_{j+1}(g) = \mathcal{D}'^q_{j+1}(g)$.

Definitions (4.31) and (4.32) give

$$\begin{aligned} T_5(\lambda') &:= \sum_{i=0}^{l}\sum_{k=0}^{l} \frac{C_l^k(-1)^k}{i!}(L_{ik} - M_{ik})\lambda'^{-(\alpha+i+1)} \log^k \lambda' \\ &= -\sum_{k=0}^{l} C_l^k(-1)^k \sum_{i=0}^{l}\sum_{n=0}^{N_i-1} \frac{f_{xy}^{in}(0,0)}{i!n!}\Big\{\mathcal{D}_j^{l-k}(\beta+n, \alpha+i, g) \\ &\quad - \mathcal{D}'^{l-k}_j(\beta+n, \alpha+i, g)\Big\}\lambda'^{-(\alpha+i+1)} \log^k \lambda'. \end{aligned} \tag{4.43}$$

For $0 \le n \le N_i - 1$, it may be possible to have $a(\beta + n + 1) = b(\alpha + i + 1)$, if $a[\mathrm{Im}(\beta)] = b[\mathrm{Im}(\alpha)]$ and $a[\mathrm{Re}(\beta) + n + 1] = b[\mathrm{Re}(\alpha) + i + 1]$. Application of Lemma 1 leads to

$$\begin{aligned} T_5(\lambda') = &-\left(\frac{1}{a}\right)^{l+1}\left(\frac{1}{b}\right)^{j+1} \sum_{i=0}^{l}\sum_{\{n:\Delta(i,n)=0\}} \frac{f_{xy}^{in}(0,0)}{i!n!}\sum_{k=0}^{l} C_l^k(-1)^k \\ &\left[\sum_{m=0}^{j}\frac{C_j^m(-1)^m}{m+l-k+1}\right] \times \left[\int_0^\infty u^{(\beta+n+1)/b-1}\log^{1+j+l-k}(u) g(u)\, du\right] \\ &\lambda^{-(\alpha+i+1)/a}\log^k \lambda. \end{aligned} \tag{4.44}$$

In fact, see treatment of quantity I_{ml}, $\sum_{m=0}^{j} C_j^m(-1)^m(m+l+k+1)^{-1} = j!(l-k)!/(1+j+l-k)!$. Thus

$$T_5(\lambda') = \frac{(-1)^{j+l}l!j!}{(1+j+l)!}\left(\frac{1}{a}\right)^{l+1}\left(\frac{1}{b}\right)^{j+1}\sum_{i=0}^{l}\sum_{\{n:\Delta(i,n)=0\}}\frac{f_{xy}^{in}(0,0)}{i!n!}$$
$$\left[\int_0^\infty u^{(\beta+n+1)/b-1}S_5(u,\lambda)g(u)\,du\right] \tag{4.45}$$

with

$$S_5(u,\lambda) := \sum_{k'=j+1}^{1+j+l} C_{1+j+l}^{k'}(-1)^{k'}\log^{k'}(u)\lambda^{-(\alpha+i+1)/a}\log^{1+j+l-k'}\lambda. \tag{4.46}$$

Finally, $T_3(\lambda') + T_5(\lambda')$ is the last contribution in Eq. (3.1).

5. Applications

To conclude a few illustrations of formula (3.1) are proposed. First, it is worth noting that Theorem 3 applies if $g(u) = G(u)e^{-\delta u}$ where $\delta \in C$ with $\mathrm{Re}(\delta) > 0$ and G belongs to $L^1_{\mathrm{loc}}(]0,+\infty[, C)$ and is also bounded in a neighborhood on the right of zero and at infinity. Each of the given examples will satisfy this property.

Example 1. Assume that $g(u) := 0$ if $u \geq 1$, $g \in L^1_{\mathrm{loc}}(]0,+\infty[, C)$, and g is bounded near zero. If $r \geq 1$ and $f \in \mathcal{D}^{(s,s)}(]0,1[^2)$ with $s := [\![r]\!]$, then

$$\begin{aligned} M(\lambda) &= \int_0^1\int_0^1 \ln(x)f(x,y)g(\lambda xy)\,dx\,dy \\ &= o(\lambda^{-r}) + \sum_{n=0}^{[r]-1}\frac{1}{n!}\Bigg\{\left[\int_0^1 u^n g(u)\,du\right]\Bigg[\mathrm{fp}\int_0^1 x^{-(n+1)}\log(x)f_y^n(x,0)\,dx \\ &\quad - \mathrm{fp}\int_0^1 y^{-(n+1)}[\log(y)+1]f_x^n(0,y)\,dy\Bigg] \\ &\quad + \left[\mathrm{fp}\int_0^1 y^{-(n+1)}f_x^n(0,y)\,dy\right]\left[\int_0^1 u^n\log(u)g(u)\,du\right] \\ &\quad - \frac{f_{xy}^{nn}}{2n!}\Bigg[\int_0^1 u^n\log^2(u)g(u)\,du - 2\log\lambda\int_0^1 u^n\log(u)g(u)\,du \\ &\quad + \log^2(\lambda)\int_0^1 u^n g(u)\,du\Bigg]\Bigg\}\lambda^{-(n+1)}. \end{aligned}$$

This result is obtained by applying (3.1) with $a = b = 1$, $\alpha = \beta = 0$, $j = 0$, and $l = 1$.

EXAMPLE 2. If $g(u) := e^{-u}$ and, for complex γ such that $\mathrm{Re}(\gamma) > 0$, the Gamma and Digamma functions obey the usual definitions $\Gamma(\gamma) := \int_0^\infty u^{\gamma-1}e^{-u}\,du$, $\Psi(\gamma) := \Gamma'(\gamma)/\Gamma(\gamma)$ (it is recalled that $\Psi(1) = -C_e$ and for $n \in \mathbb{N}^*$, $\Psi(n+1) = -C_e + \sum_{k=1}^{n} k^{-1}$ where C_e denotes the Euler's constant), Theorem 3 ensures for $r \geq 1$ and $f \in \mathcal{D}^{([\![r]\!],[\![2r]\!])}(]0,1[^2)$ the expansion

$$\int_0^1\int_0^1 f(x,y)e^{-\lambda xy^2}\,dx\,dy = \sum_{n=0}^{[\![2r]\!]-1} \frac{\Gamma((n+1)/2)}{2n!} \left[\mathrm{fp}\int_0^1 x^{-(n+1)/2} f_y^n(x,0)\,dx\right]\lambda^{-(n+1)/2}$$
$$+ \sum_{i=0}^{[\![r]\!]-1}\left[\mathrm{fp}\int_0^1 y^{-2(i+1)} f_x^i(0,y)\,dy\right]\lambda^{-(i+1)} \tag{5.1}$$
$$+ \sum_{k=0}^{I([\![2r]\!]-1)} \frac{f_{x,y}^{k,2k+1}(0,0)}{2(2k+1)!}(-\Psi(k+1)$$
$$+ \log[\lambda])\lambda^{-(k+1)} + o(\lambda^{-r}),$$

where for $M \in \mathbb{N}^*$: $I(M) := (M-1)/2$ if M is odd, else $I(M) := (M-2)/2$. For $k \in \mathbb{N}$, the well known relations $\Gamma(k) = (k-1)!$ and $\int_0^\infty u^k \log(u)e^{-u}\,du = \Gamma'(k+1)/\Gamma(k+1)$ have been employed.

EXAMPLE 3. It may be useful to extend the treatment of the above example to the case of a complex parameter λ. More precisely, it is assumed that $\lambda \in C$ with $\mathrm{Re}(\lambda) \to +\infty$. We introduce the real s such that $\lambda = \mathrm{Re}(\lambda)[1 + is]$ (if i designates the complex such that $i^2 = -1$) and choose $g(u) = g_s(u) := e^{-(1+is)u}$. For $a > 0$, $b > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $(l,j) \in \mathbb{N}^2$, $r \geq \mathrm{Max}(\mathrm{Re}(\alpha)/a, \mathrm{Re}(\beta)/b) + 1$, $H := [\![ar - \mathrm{Re}(\alpha) - 1]\!]$, $N := [\![br - \mathrm{Re}(\beta) - 1]\!]$, $p_x := [\![ar - \mathrm{Re}(\alpha)]\!]$, $p_y := [\![br - \mathrm{Re}(\beta)]\!]$, and $f \in \mathcal{D}^{(p_x,p_y)}(]0,1[^2)$ the next expansion holds:

$$I(\lambda) = \int_0^1\int_0^1 x^\alpha y^\beta \log^l x \log^j y f(x,y)e^{-\mathrm{Re}(\lambda)(1+is)x^a y^b}\,dx\,dy$$
$$= \sum_{n=0}^{N}\sum_{m=0}^{j}\frac{C_j^m j^{-m}}{n!}\sum_{k=0}C_{j-m}^k\frac{(-1)^{m+k}a^k}{b^{j+1}}\mathcal{L}_s^{j-m-k}\left(\frac{\beta+n+1}{b}\right)$$
$$\times\left[\mathrm{fp}\int_0^1 x^{\alpha-(a/b)(\beta+n+1)}\log^{l+k}(x)f_y^n(x,0)\,dx\right]$$

$$[\mathrm{Re}(\lambda]^{-(\beta+n+1)/b}\log^m[\mathrm{Re}(\lambda)]$$

$$+\sum_{h=0}^{H}\sum_{m=0}^{l}\frac{C_l^m}{h!}\sum_{k=0}^{l-m}C_{l-m}^k\frac{(-1)^{m+k}b^k}{a^{l+1}}\mathcal{L}_s^{l-m-k}\left(\frac{\alpha+h+1}{a}\right)$$

$$\times\left[\mathrm{fp}\int_0^1 y^{\beta-(b/a)(\alpha+h+1)}\log^{j+k}(y)f_x^h(0,y)\,dy\right]$$

$$[\mathrm{Re}(\lambda)]^{-(\alpha+h+1)/a}\log^m[\mathrm{Re}(\lambda)]$$

$$+\sum_{n=0}^{N}\sum_{h=0}^{H}\frac{\Delta_{(h,n)}f_{xy}^{hn}(0,0)j!l!}{n!h!(1+h+l)!}\left(\frac{1}{a}\right)^{l+1}\left(\frac{1}{b}\right)^{j+1}\sum_{m=0}^{1+j+l}C_{1+j+l}^m(-1)^{1+m}$$

$$\times\mathcal{L}_s^{1+j+l-m}\left(\frac{\beta+n+1}{b}\right)[\mathrm{Re}(\lambda)]^{-(\beta+n+1)/b}\log^m[\mathrm{Re}(\lambda)]$$

$$+o([\mathrm{Re}(\lambda)]^{-r}),$$

where $\Delta_{(h,n)} := 1$ if $b(\alpha+h+1) = a(\beta+n+1)$, else $\Delta_{(h,n)} := 0$ and for $\gamma \in C$ with $\mathrm{Re}(\gamma) > 0$ and $k \in \mathbb{N}$,

$$\mathcal{L}_s^k(\gamma) := \int_0^\infty u^{\gamma-1}\log^k(u)e^{-(1+is)u}\,du$$

$$= (1+is)^{-\gamma}\sum_{l=0}^{k}C_k^l(-1)^{k-l}\log^{k-l}(1+is)\Gamma^{(l)}(\gamma)$$

where $\Gamma^{(l)}$ designates the derivative of order l of complex function Gamma.

By replacing λ by $\mathrm{Re}(\lambda) \to +\infty$ and choosing $g(u) := e^{-(1+is)u}$, Theorem 3 ensures this result. In fact, $\mathcal{L}_s^k(\gamma) = (1+is)^{-\gamma}\sum_{l=0}^{k}C_k^l(-1)^{k-1}\log^{k-1}(1+is)E_l(\gamma)$ where each complex $E_l(\gamma)$ is defined as $E_l(\gamma) := \int_0^\infty[(1+is)u]^{\gamma-1}\log^l[(1+is)u]e^{-(1+is)u}(1+is)\,du$. Introduction of the complex variable $z := (1+is)u$ and of the set $\mathcal{C} := \{z \in C;\, z = (1+is)t \text{ for } t \in \mathbb{R}_+\}$ allows us to write $E_l(\gamma) = \int_{\mathcal{C}} F_l(z)\,dz$ with $F_l(z) := z^{\gamma-1}\log^l(z)e^{-z}$. Observe that F_l turns out to be analytic in $C\backslash\{0\}$. Moreover, for $0 < \varepsilon < R$ and if $\mathcal{P}$ is a path of C defined as $\mathcal{P} := \mathcal{C}_{\varepsilon,R} \cup \mathcal{C}'_{\varepsilon,R} \cup \mathcal{T}_\varepsilon \cup \mathcal{T}_R$ with $\mathcal{C}_{\varepsilon,R} := \{z \in \mathcal{C};\, \varepsilon \le |z| \le R\}$, $\mathcal{C}'_{\varepsilon,R} := \{t \in \mathbb{R};\, \varepsilon \le t \le R\}$, $\mathcal{T}_\varepsilon := \{\varepsilon e^{i\theta} \text{ for } 0 \le \theta \le \arctan(s)\}$, $\mathcal{T}_R := \{\mathrm{Re}^{i\theta} \text{ for } 0 \le \theta \le \arctan(s)\}$ the assumption $\mathrm{Re}(\gamma) > 0$ ensures that $\lim_{\varepsilon\to0}\int_{\mathcal{T}_\varepsilon} F_l(z)\,dz = 0 = \lim_{R\to+\infty}\int_{\mathcal{T}_R} F_l(z)\,dz$. Consequently, $E_l(\gamma) = \lim_{\varepsilon\to0,R\to+\infty}\int_{\mathcal{C}_{\varepsilon,R}} F_l(z)\,dz = \lim_{\varepsilon\to0,R\to+\infty}\int_\varepsilon^R F_l(t)\,dt = \int_0^\infty t^{\gamma-1}\log(t)e^{-t}\,dt = \Gamma^{(l)}(\gamma)$.

For instance, if $r > 1$ and real $\lambda \to +\infty$, choice of $s = 1$, $a = 4$, $b = 1$, $\alpha = 0 = \beta$, $f(x, y) = \sin(x)$, $l = 0$, and $j = 1$ leads (with $\Gamma^{(2)}(1) = \pi^2/6 + C$) to

$$P(\lambda) = \int_0^1 \int_0^1 \sin(x) \log(y) e^{-\lambda(1+i)x^4 y}\, dx\, dy$$

$$= -\left\{\left[\log(1+i) + C_e + \log[\lambda]\right]\left[\mathrm{fp}\int_0^1 x^{-4}\sin(x)\, dx\right]\right.$$

$$\left. + 4\,\mathrm{fp}\int_0^1 x^{-4}\log(x)\sin(x)\, dx\right\}\frac{\lambda^{-1}}{1+i}$$

$$-\left[\left(\frac{\pi^2}{6} + C_e^2\right)\log^2(1+i) - \log(1+i) + 1 - 2[C_e \log(1+i)\right.$$

$$\left. + 1]\log[\lambda] + \log^2[\lambda]\right]\frac{\lambda^{-1}}{48(1+i)}$$

$$+ \sum_{k=0}^{I([4r]-1)} \frac{(1+i)^{-(k+1)/2}\Gamma((k+1)/2)(-1)^{k+1}}{2(2k+1)!(1-k)}\lambda^{-(k+1)/2} + o(\lambda^{-r}).$$

This result enables us to deal with the asymptotic expansion of the following integrals:

$$I_1(\lambda) = \int_0^1 \int_0^1 \sin(x)\log(y)\cos(\lambda x^4 y) e^{-\lambda x^4 y}\, dx\, dy = \mathrm{Re}[P(\lambda)],$$

$$I_2(\lambda) = \int_0^1 \int_0^1 \sin(x)\log(y)\sin(\lambda x^4 y) e^{-\lambda x^4 y}\, dx\, dy = -\mathrm{Im}[P(\lambda)].$$

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