Against the wind

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The linear and the nonlinear dynamics of open unstable flow in a finite domain of size L is studied on a modified supercritical Ginzburg–Landau equation. When the advection term is nonzero, the bifurcation to a finite-amplitude state occurs when the instability is absolute, even for large L. The standard weakly nonlinear theory is limited to a control parameter domain of size varying as L^{-5} due to the nonnormality of the linear evolution operator. The fully nonlinear solution is given and two generic cases are discussed: a supercritical case in which the instability is absolute and a subcritical case in which the instability is solely convective. The subcritical case gives a mathematical example of a bypass transition due to transient growth. The supercritical case allows a fully quantitative comparison, including the effect of the domain size, with results obtained by Büchel *et al.* for the size of the bifurcated solutions in the Taylor–Couette problem with throughflow. [S1070-6631(99)00510-3]

I. INTRODUCTION

Linear and nonlinear pattern selection in the one-dimensional Ginzburg–Landau equation with homogeneous boundary conditions on an interval of length L has been investigated for over one decade.[1,2] This equation is meant to model the dynamics of, for example, convection rolls in the Rayleigh–Bénard experiment or of chemical reaction waves in the Belousov–Zhabotinsky experiment. Confinement in a box of size L was found to affect the bifurcation threshold weakly by a term of order L^{-2} and the wave number selection by order L^{-1}. This has been understood in terms of the correlation length l in an infinite domain which, generically for a supercritical bifurcation, diverges as $\mu^{-1/2}$ (where μ is the bifurcation parameter). Boundary conditions influence the flow only over the correlation length l and therefore, when $\mu \gg L^{-2}$, their effect is limited to diffusive layers small compared to L.

Confinement effects are not so trivial when an advection term is added to the Ginzburg–Landau equation and cannot be removed since the boundary conditions single out a unique reference frame. This equation qualitatively describes open flows such as the Rayleigh–Bénard[3] or Couette–Taylor experiments[4,5] with crossflow, or more classically jets, wakes, mixing layers, and boundary layers,[6] for which the mean advection is nonzero in the laboratory frame since fluid particles continually enter and leave the domain. A similar equation holds in closed flows when the instability is traveling in the laboratory frame, as in binary convection.[7]

When order one advection is present, confinement in a box of size L,[8–10] or spatial inhomogeneities due to slow variations of the equation parameters,[8,11–13] delay the linear threshold by an order one quantity since the instability has now to become absolute.[6] *Weakly nonlinear* analyses are of little help to describe the bifurcation since their validity is limited to order (L^{-5}) departure from criticality in the finite box case[8,10] and to exponentially small departure from criticality in the variable coefficient case.[14]

The aim of this paper is precisely to lift these restrictions, by illustrating on an elementary model generic bifurcation diagrams which incorporate *strongly nonlinear* effects. More specifically, whereas the linear threshold in a finite box is related to the absolute nature of the instability, it is demonstrated in the present study that this is not so for the nonlinear threshold. By invoking an argument similar to that in Refs. 10, 15–17, the existence of a fully nonlinear solution in a finite domain is expected to depend on the direction of propagation of the front separating the bifurcated state from the basic state in an infinite domain.[18,19] When the front velocity is linearly selected, a nonlinear solution exists only when the instability is absolute, and the bifurcation is shown to be supercritical. When the front selection is nonlinear, a nonlinear solution exists whereas the instability is still convective and the bifurcation is shown to be subcritical. In this case, because of the nonnormality of the linear evolution operator,[14,20–22] initial perturbations of exponentially small amplitude induce large transients which trigger the nonlinear transition.

The present study builds upon the results of Refs. 15–17 where the concept of nonlinear absolute instability is introduced and where solutions in a semi-infinite domain are analytically derived. To construct the fully nonlinear solution in a finite domain, the solution in a semi-infinite domain determined by the method of matched asymptotic expansions[16] is used, allowing us to obtain *analytically* the scaling laws found numerically in Refs. 3, 5, 8, 10.

In a related study[10] the bifurcation structure of the complex Ginzburg–Landau equation in a finite domain has been numerically computed and interpreted in terms of a front solution. In that case, bifurcation is observed to be supercritical and to occur solely when the instability is absolute. From the present point of view, this corresponds to the fact that in the supercritical complex Ginzburg–Landau model the front solution is always *linearly selected*.[19] In this respect the supercritical Ginzburg–Landau equation is not generic.

In the present analysis, we examine instead the dynamics of the different states which prevail in a finite box in the case of an elementary model exhibiting a *nonlinearly selected* front solution. Many extensions of the real or complex Ginzburg–Landau equation (see, for example, Ref. 16) might have been suitable candidates. We have chosen to study the van der Pol–Duffing type equation (1), where the mean advection effect includes a quadratic term in the perturbation amplitude that accounts for nonlinear variations in the wave speed. It should be emphasized that this equation is considered here as a toy model. It is not rationally derived from the Navier–Stokes equations via a multiple scale approach since the asymptotic expansion close to the absolute instability threshold becomes invalid when advection is order unity.[10] Indeed recent numerical simulations of bluff body wakes[23] have demonstrated that such surprisingly large nonlinear modifications to the mean flow are essential in order to understand the structure of the bifurcated wake.

II. NONLINEAR MODEL

The Ginzburg–Landau equation describes the wave amplitude in a bifurcating spatially extended system and has been considered to model the transition of closed[24] as well as open[25] fluid dynamical systems. For simplicity we discuss the real amplitude case, which corresponds to an instability breaking a discrete symmetry, and add the term $\alpha A^2\,\partial_x A$ which represents the lowest-order nonlinear contribution to the mean advection velocity consistent with $A\rightarrow -A$ symmetry. The model reads:

$$\partial_t A=\mu A-(U-\alpha A^2)\partial_x A+\partial_{xx}A-A^3, \tag{1}$$

with $A(x,t)$ the order parameter, U the mean advection velocity, μ the bifurcation parameter. Rescaling x, t, A would bring one of the parameters U or α to unity but we keep both parameters to facilitate the discussion. For $\alpha=0$ the standard Ginzburg–Landau equation with real amplitude and real coefficients is recovered. Solutions of Eq. (1) are sought in a finite domain $(0,L)$ with boundary conditions:

$$A(0,t)=A(L,t)=0. \tag{2}$$

III. LINEAR SOLUTION

Before applying the boundary conditions Eq. (2), consider an infinite domain for which infinitesimal amplitude solutions of Eq. (1) represent instability waves of the form $\Re\{\exp(i(kx-\omega t))\}$ with the wave number k and the frequency ω linked by the dispersion relation $\omega=Uk+i(\mu-k^2)$. The system is linearly stable if any infinitesimal initial condition is damped. This is the case if $\Im(\omega)<0$ for any k real, i.e., if $\mu<0$. In the ''laboratory'' frame [but without applying the boundary conditions Eq. (2)], the group velocity $v_g=d\omega/dk$ discriminates between convectively unstable flow, for which initial transients are advected downstream, and absolutely unstable flow, for which initial transients grow to infinity with time at any fixed location x. These concepts, originally introduced in plasma physics,[26] have been successfully applied to the understanding of open flow dynamics.[6,27]

The wave of zero group velocity is characterized by the following absolute frequency and wave number:

$$\omega_0=i(\mu-U^2/4), \tag{3}$$

$$k_0=-iU/2. \tag{4}$$

The flow is convectively unstable if $\Im(\omega_0)<0$, i.e., if $\mu<\mu_t\equiv U^2/4$, and absolutely unstable if $\mu>\mu_t$. To be consistent with the literature,[6] these notions pertaining to waves in an infinite domain are called *local* since they do not take into account spatial inhomogeneities such as boundary conditions.

When the boundary conditions Eq. (2) are enforced, the streamwise direction x becomes an eigendirection and the linear solutions take the form $A(x,t)=\Re(\exp(-i\omega_n t)\phi_n(x))$, with ω_n the *global* frequency[6] and $\phi_n(x)$ the linear *global* mode (the term *global* refers to physical space and not to phase space as in dynamical systems theory) given by

$$\omega_n=i(\mu-U^2/4-\pi^2n^2/L^2)=\omega_0-i\pi^2n^2/L^2, \tag{5}$$

$$\phi_n(x)=\exp[U(x-L)/2]\sin(\pi n x/L). \tag{6}$$

Equation (5) shows that the global threshold $\mu_G\equiv U^2/4+\pi^2/L^2=\mu_t+\pi^2/L^2$, at which the leading eigenvalue ω_1 is destabilized, differs from the local instability threshold $\mu=0$ by two terms: an order one term μ_t depending only on U due to the advection, and an order L^{-2} term due to the finite size of the box. Therefore, we obtain the seemingly paradoxical result that, no matter what the length of the interval, the *global* bifurcated solution prevails for $\mu>\mu_t$, i.e., when the instability is *locally* absolute, whereas when the interval is taken to be infinite right from the start, the bifurcation takes place at $\mu=0$, i.e., when the instability is *locally* convective.

This singular behavior of the spectrum as $L\rightarrow\infty$ has been discussed, without referring to the concepts of local absolute or convective instability, by Reddy and Trefethen[21] in their study of the advection-diffusion operator. It is related to the transient amplification of initial perturbations associated with the nonnormality of the linear global operator. In other words, the linearized form of Eq. (1) with boundary conditions Eq. (2) is such that its eigenmodes Eq. (6) are nonorthogonal. A practical way to understand the physics associated with nonnormality is to consider the response to time-harmonic forcing at the real frequency ω. When the flow is globally unstable, the amplification is infinite when forcing is applied at the global frequency. As noticed in Ref. 22 it is finite but it is extremely large in a whole band of ω when the flow is locally convectively unstable, and it is smaller than one (no amplification) when the flow is locally stable. When $L\rightarrow\infty$ the amplification goes to infinity for ω inside the unstable band $(-\sqrt{\mu},\sqrt{\mu})$. When the Galilean invariance is broken, then the linear global evolution operator is nonnormal, and its spectrum bifurcates when a finite domain becomes locally absolutely unstable. However, prior to this global bifurcation, ϵ-pseudospectra for small ϵ cross the real ω-axis, when a convectively unstable region appears.

IV. WEAKLY NONLINEAR SOLUTION IN FINITE DOMAIN

We show that this nonnormality associated with large time-harmonic amplification has two effects on the nonlinear analysis: first it strongly limits the validity of the standard weakly nonlinear theory;[14] second, when the global bifurcation is subcritical, it allows the nonlinear solution to be excited by extremely low amplitude perturbations.

Since the linear global modes have well-separated growth rates Eq. (5), a naive idea consists in performing a weakly nonlinear analysis, as in Refs. 8, 10, 14, by introducing a small parameter η such that $\mu-\mu_G=\eta\Delta\mu$. The solution is then expanded in powers of η: $A=\Sigma_i\eta^i A_i$, and a slow time scale $T=\eta^2 t$ is introduced. At first order we obtain $A_1(x,t,T)=\Re\{a(T)\phi(x)\exp(-i\omega t)\}$ with the global amplitude $a(T)$ still unknown. Equations (5) and (6) define $\omega\equiv\omega_1$ and $\phi(x)\equiv\phi_1(x)$. At third order the compatibility condition imposes that $a(T)$ obey the Landau equation

$$da/dT=\Delta\mu a-c\|a\|^2 a, \tag{7}$$

with

$$c=\langle\psi|\phi^3-\alpha\phi^2\,\partial_x\phi\rangle/\langle\psi|\phi\rangle, \tag{8}$$

where $\langle|\rangle$ stands for the usual inner product over the interval $0\leqslant x\leqslant L$ and $\psi(x)\equiv\exp(-Ux/2)\sin(\pi x/L)$ belongs to the kernel of the adjoint operator at $\mu=\mu_G$. We obtain:

$$c=\frac{12\pi^4[1-\exp(-UL)](4-\alpha U)}{UL(4\pi^2+U^2L^2)(16\pi^2+U^2L^2)}. \tag{9}$$

Two limits are important: when L goes to infinity and U is finite $c\sim 12\pi^4(4-\alpha U)(UL)^{-5}$; when U vanishes (as in Ref. 2) $c\sim 3/4$. Note at this stage that the weakly nonlinear nature of the instability changes from supercritical to subcritical when α exceeds $4/U$.

In the above computation, the choice of ϕ specified in Eq. (6) is such that its maximum amplitude is unity (at leading order in $1/L$). Therefore, if we assume the consistency of the expansion, that $\eta A_1\ll 1$ at any location x, the weakly nonlinear theory is limited, when U is finite, to $\mu-\mu_G\ll c\sim L^{-5}$. This restriction, already given in Refs. 8, 10, is in fact due to the nonnormality of the global operator. It is far more severe than the usual restriction $\mu-\mu_G\ll L^{-2}$ given by the separation in the global eigenvalues Eq. (5). When $\mu-\mu_G$ is small but larger than $12\pi^4(4-\alpha U)(UL)^{-5}$, the Landau-type expansion becomes invalid and a different nonlinear description should be used.

V. FULLY NONLINEAR SOLUTION IN FINITE DOMAIN

The basic idea underlying the fully nonlinear description is that, when L is large enough, the solution is similar to that described in Ref. 15 for a semi-infinite domain. The solution consists of a front separating the basic state from the bifurcated region downstream, which would have moved upstream had the domain been infinite, and is prevented from doing so by the upstream boundary condition. This front solution has recently been calculated for model problems in infinite domains.[18,19] For an unstable basic state, an infinity of front solutions corresponding to different velocities exist but a single one is dynamically selected.[18,19] Two cases are possible: either the front is selected by the linear marginal stability principle or it moves faster and is selected by a nonlinear criterion.[19] To each of these cases is associated a different solution in a semi-infinite domain. According to the results derived in Ref. 16, a solution in a semi-infinite domain called a nonlinear global mode may be viewed as a stationary front (or a front able to withstand advection) with a zero amplitude at some location.

It has been shown in Ref. 16 that the application of these results to Eq. (1) leads to the two possible generic cases: When $\alpha<6/U$, the front is linearly selected in an infinite domain. The threshold necessary to obtain a nonlinear global mode in a semi-infinite domain coincides with the absolute instability threshold μ_t. In this case, the distance Δx required to reach saturation is given at leading order by:

$$\Delta x\simeq\pi(\mu-\mu_t)^{-1/2}. \tag{10}$$

Physically, this scaling may be understood by considering that close to $x=0$ the amplitude is small and the solution may be viewed as the superposition of two waves of complex wave number differing only by an order $(\mu-\mu_t)^{1/2}$ complex term.

When $\alpha>6/U$, the front is nonlinearly selected in an infinite domain. A nonlinear mode in a semi-infinite domain may still be viewed as a front blocked by the upstream boundary, but now the nonlinear global mode threshold $\mu_\infty\equiv 3\alpha^{-1}(U-3\alpha^{-1})$ (see Ref. 16 for details) is smaller than μ_t. In this case, the distance Δx required to reach saturation is proportional at leading order to:

$$\Delta x\simeq\ln\left(\frac{1}{\mu-\mu_\infty}\right). \tag{11}$$

Physically, this size corresponds to the fact that the boundary condition at $x=0$ is fulfilled by the linear superposition of two waves, propagating, respectively, upstream and downstream, with spatial growth rates differing by order unity. The value $\alpha=6/U$ which distinguishes between the two cases is given by comparing μ_∞ with μ_t.

Equation (10) has been derived analytically in Refs. 16, 17 and independently obtained numerically in Refs. 8, 10. The scaling law Eq. (11) has recently been observed in a Hele-Shaw cell experiment.[28]

Let us now discuss the implication of these scaling laws on the structure of fully nonlinear solutions in a box of finite size. Intuitively, a solution confined in a box of size L will resemble the solution in the semi-infinite domain modified by a diffusive boundary layer of width of order unity at the downstream boundary $x=L$ if saturation is reached before $x=L$ (Fig. 1). Therefore the condition ''L large'' is not sufficient to insure that the saturation amplitude is indeed attained, and we enforce instead $L-\Delta x\gg 1$. When $\alpha<6/U$, we substitute Eq. (10) in the latter condition and we expand the quantity $\mu-\mu_t$ in powers of $1/L$: $\mu-\mu_t=\pi^2/L^2+\gamma/L^3$, where the first term on the right hand side means that the control parameter μ is close to the threshold μ_G, and γ is still unknown. The condition $L-\Delta x\gg 1$ imposes $\gamma\gg 2\pi^2$ and therefore:

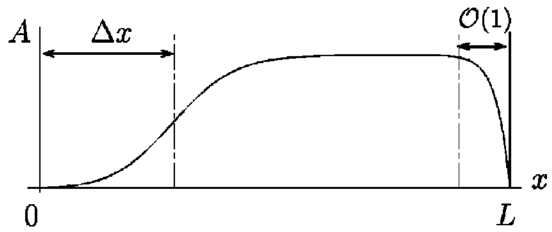

FIG. 1. Nonlinear global mode in a box of size L. Δx is the distance required to reach the saturation amplitude $\mu^{1/2}$.

$$\mu-\mu_t-\pi^2L^{-2}=\mu-\mu_G \gg 2\pi^2L^{-3}. \tag{12}$$

In that case the amplitude of the solution is constant at the value $\mu^{1/2}$ outside two boundary layers, one of size Δx given by Eq. (10) at the upstream boundary $x=0$ and the other of size unity at the downstream boundary $x=L$.

The bifurcation diagram [Fig. 2(a)] is obtained by plotting the maximum amplitude of the numerically determined steady solution of Eq. (1) as a function of μ. The solution has been computed for $L=10\times\pi$. The interval in μ where the flow is locally stable is marked by S, convectively unstable by C, absolutely unstable by A. GS marks the linearly globally stable domain, $\mu<\mu_G\equiv\mu_t+\pi^2/L^2$; W the domain of validity of the weakly nonlinear theory, $|\mu-\mu_G| \ll 48\pi^4(UL)^{-5}$ and K the domain $\mu-\mu_G \gg L^{-3}$ where the solution is obtained as a linear front blocked by the upstream boundary condition. The terminology K is used for Kolmogorov, since the linear front velocity selection was first discovered in Ref. 18.

The bifurcation diagram [Fig. 2(a)] is in sharp contrast with the classical ''easy bifurcation'' case[2] of a finite unstable system with no advection $U=0$: in the latter case, the bifurcation takes place close to $\mu=0$, the linear operator is normal and the Landau constant is order unity.

With a finite advection $(U\neq 0)$, the bifurcation takes place when the flow is absolutely unstable $(\mu_G>\mu_t)$. Furthermore, the Landau constant being order L^{-5}, the domain of validity (W in gray) of the Landau model is extremely limited and the strongly nonlinear saturated solution takes over very rapidly (for $\mu-\mu_G$ larger than order L^{-3}). It is somewhat of a surprise that the domain of existence of the strongly nonlinear solution coincides with the domain of linear absolute instability.

This is no longer true when $\alpha>6/U$ since the nonlinear solution in a semi-infinite domain[16] exists for $\mu>\mu_\infty$ which includes the range $]\mu_\infty,\mu_t[$ where the basic state is still convectively unstable. In that case, because of the logarithmic scaling for Δx [Eq. (11)], the global mode in the box of size L exists when μ exceeds μ_∞ by an exponentially small quantity. In Fig. 2(b), N marks the domain (in gray) where the solution is obtained as a nonlinearly selected front blocked by the upstream boundary condition (and not a linearly selected front as in the region marked K). The bifurcation diagram is radically different from the previous case since, in the finite box, the basic state $A=0$ is linearly stable until μ_G and therefore the flow exhibits hysteresis [heavy lines and arrows in Fig. 2(b)] between μ_∞ and μ_G. In this subcritical range, an unstable solution (dash-dotted curve) exists and

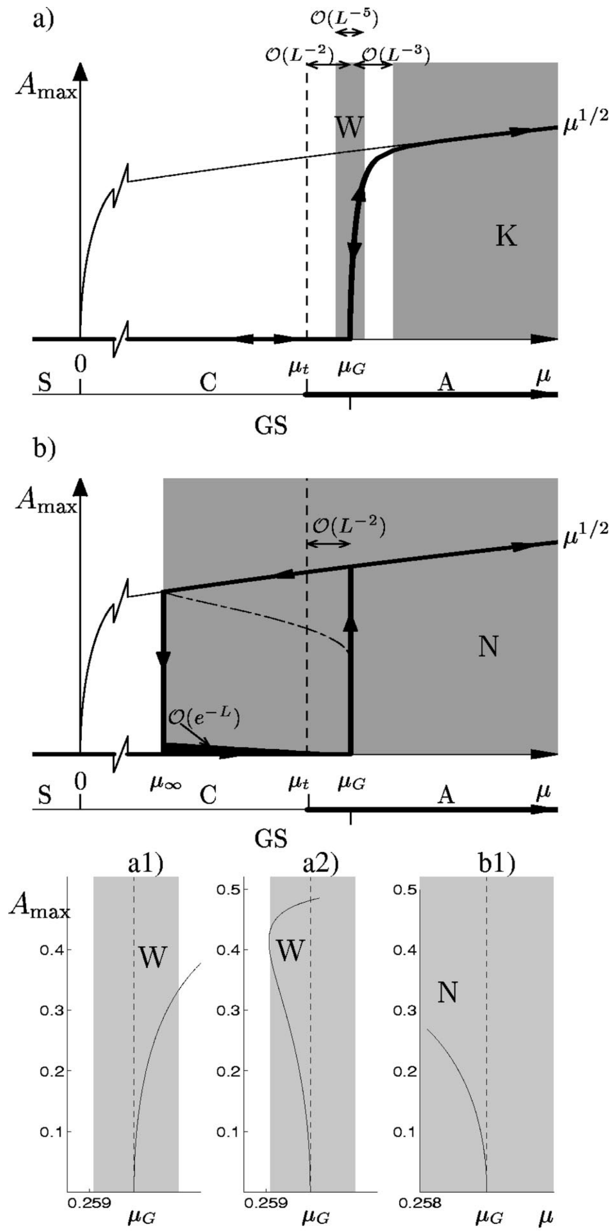

FIG. 2. Numerically computed bifurcation diagrams (heavy line) for U of order unity and $L=10\times\pi$ large but finite. (a) $\alpha<6/U$ $(\alpha=1, U=4)$. (b) $\alpha>6/U$ $(\alpha=1, U=12)$. (a1) Enlargement around μ_G in the case $\alpha<4/U$ $(\alpha=3, U=1)$. (a2) Enlargement around μ_G in the case $4/U<\alpha<6/U$ $(\alpha=5, U=1)$. (b1) Enlargement around μ_G in the case $\alpha>6/U$ $(\alpha=7, U=1)$.

connects the nonlinear solution which bifurcates (saddle node bifurcation) close to $\mu=\mu_\infty$ to the basic state. The basic state is destabilized via a subcritical pitchfork bifurcation at μ_G, with an order L^{-5} nonlinear coefficient in the Landau equation.

To obtain the bifurcation diagrams of Fig. 2, the stationary solutions of Eq. (1) have been computed numerically as trajectories in $(A,dA/dx)$ phase space. They have been obtained by perturbation of the nonlinear global modes already computed in the case of a semi-infinite domain in Ref. 16. The numerical procedure used to compute the stable and unstable branches of the bifurcations diagrams (Fig. 2) is a type of shooting method: in a semi-infinite domain, a nonlinear

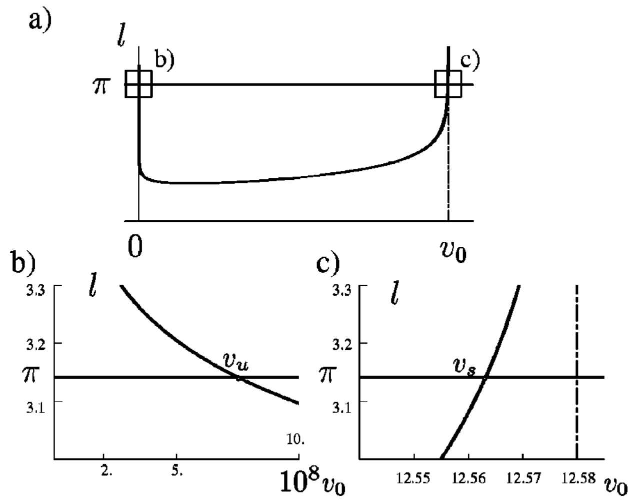

FIG. 3. Principle of determination of the different nonlinear global modes. (a) Size l of solutions as a function of the initial slope v_0. $L=\pi$ is one of the box sizes we consider. (b) and (c) are enlargements around the intersection points each of which represents a solution in a domain of size $L=\pi$.

global mode has a nonzero initial slope $dA/dx(0)$. This particular value of $dA/dx(0)$ with $A(0)=0$ is the single initial condition which converges to $A_2 \equiv (\sqrt{\mu},0)$ at infinity (initial condition on the stable manifold of A_2). When trajectories are integrated forward in space using the boundary condition at the origin $A(0)=0$ and a smaller initial slope $v_0 = dA/dx(0)$ than that of the semi-infinite nonlinear global mode, the size of the solution is finite, i.e., there is a point $x=l$ such that $A(l)=0$. The point l is computed as a function of the initial slope v_0 and presents a concave shape when $\mu_\infty < \mu < \mu_G$ (Fig. 3).

Coming back to the solution in a finite domain, the second boundary condition $A(L)=0$ must be applied in order to single out solutions of size L. These solutions are found by intersecting the previously obtained concave curve (Fig. 3) with the horizontal line $l=L$. Therefore, two solutions of size L exist if $\mu_\infty < \mu < \mu_G$: one possesses a very small initial slope v_u [Figs. 3(b) and 4(b)] and corresponds to the unstable branch of Fig. 2(b). The second one corresponding to the stable branch of Fig. 2(b) possesses an order one initial slope v_s [Figs. 3(c) and 4(a)]. In this example for which $U=12$, $\mu=36$ and $\alpha=1$, the size of the box $L=\pi$ may be considered as large, since the stable bifurcated solution is extremely close to that obtained in a semi-infinite domain [Fig. 4(a)]. The existence of the unstable branch with order one maximum amplitude [Fig. 4(b)] close to μ_G shows that the system is strongly subcritical when $\alpha > 6/U$. This is not in contradiction with the weakly nonlinear analysis in the vicinity of μ_G, which predicts subcriticality for a different but smaller threshold $4/U$ [Eq. (9)]. When $4/U < \alpha < 6/U$, the system is weakly subcritical with a small hysteresis loop which exists in an extremely narrow band limited to the W domain (its size measured in terms of the parameter μ is of order L^{-5}) for $\mu < \mu_G$. In this case, the bifurcation diagram is similar to Fig. 2(a), but with a negative slope at μ_G, as shown in the enlargement [Fig. 2(a2)] of Fig. 2(a).

The subcritical nature of the fully nonlinear instability when $\alpha > 6/U$ leads us to consider how noise may induce the transition. The new kind of global mode in the range $\mu_\infty < \mu < \mu_G$ exists not because the linear wave of zero group velocity is destabilized (linear absolute instability) but because the nonlinear front is able to withstand the advection. Numerical simulations of the evolution equation (1) with boundary conditions Eq. (2) show that such global modes are triggered only if the amplitude of the initial perturbation is large enough for the transient to reach an amplitude of order unity in the finite domain. From the global point of view, the amplification of the initial transient is a linear effect associated with the nonnormality of the global operator. The amplification factor is known[20] to be comparable to the time-harmonic amplification mentioned in the first part of the paper and to increase exponentially with L. It may therefore be expected that the ''activation amplitude'' [symbolized by the small black region in Fig. 2(b), which has been widened to make it visible], i.e., the minimum amplitude of the initial perturbation sufficient to trigger the nonlinear global mode, will decrease exponentially with L. Numerical results are in very good agreement with this interpretation: We have chosen initial perturbations possessing a uniform amplitude in space. For a box size $L=10\pi$, the amplitude of this initial condition must exceed 3.4×10^{-41} to trigger a nonlinear front which moves upstream and saturates in a global mode. If the box length is $L=5\pi$, this threshold becomes 9.5×10^{-20} and if $L=2\pi$, it becomes 7.1×10^{-7}. If such a nonlinearly self-sustained global mode were present in a real experiment, this activation amplitude would be too small to be detected since initial and entrance noise, which are say of order 10^{-4} in a precise experimental setup, would generate order unity transients even for a moderate box size L. In any case, the entrance noise determined by the precision of the experimental setup exceeds by several orders of magnitude the threshold to trigger a large nonlinear response. The bifurcation would therefore seem to effectively take place close to μ_∞ as in the semi-infinite case.

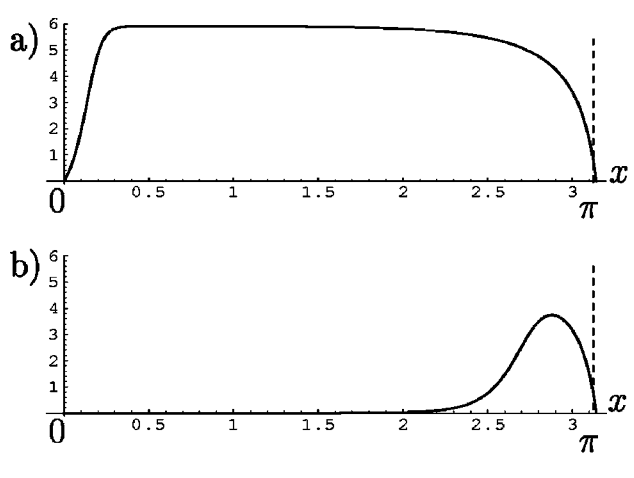

FIG. 4. Nonlinear global modes obtained at the intersection points of Fig. 3. (a) Stable mode with slope v_s. (b) Unstable mode with slope v_u.

VI. DISCUSSION AND COMPARISON WITH OPEN FLOW EXPERIMENTS

This type of bifurcation induced by transient amplification of initial noise is associated with the nonnormality of the linear evolution operator; it has been invoked to explain dynamics in Couette and Poiseuille flow.[20] The present paper shows that a fully nonlinear analysis must be undertaken in order to determine the dynamics of open flows and that the notion of nonnormality only quantifies how the system may be triggered by noise, but tells us nothing about its ultimate response. An elementary partial differential equation (1) has been studied for which two extremely different bifurcation scenarios have been identified, depending on the nonlinearity present in the system (as measured by the α parameter), but without any change in the nonnormality of the linear operator.

Several open flows which follow the bifurcation scenario depicted in Fig. 2(a) have been identified experimentally or numerically: as an example of linearly determined bifurcation scenario, we have analyzed the Taylor–Couette problem with throughflow, within the framework of the complex Ginzburg–Landau equation and we have shown that the dynamics in an open geometry of size L is similar to that in a semi-infinite domain.[17] In particular, the flow behaves like an oscillator when the Taylor number (i.e., the bifurcation parameter which measures the rotation of the inner cylinder) exceeds the absolute instability threshold. For this problem, Büchel *et al.*[5] have compared numerical simulations of the Navier–Stokes equation with those of the Ginzburg–Landau equation in a system of finite length. In Ref. 17, we have demonstrated that when the threshold of absolute instability μ_t is approached, the scaling law Eq. (10) is in good agreement with numerical calculations of the length (i.e., the distance necessary to reach order one amplitude) of the modes obtained above threshold by Büchel *et al.* This indicates that the system possesses the intrinsic dynamics of a semi-infinite system; the influence of the outlet boundary condition is restricted to a very narrow domain near the outlet but does not affect the global dynamics of the system. This analysis is valid because the system considered is sufficiently long and the present analysis determines precisely whether the results obtained by Büchel *et al.* fall within the range of validity of scaling law Eq. (10). A global mode cannot be obtained if the departure from the threshold of absolute instability μ_t is smaller than π^2/L^2. But otherwise, the system behaves as in a semi-infinite domain and a global mode which has saturated over a distance comparable to the size of the box is obtained only if the bifurcation parameter exceeds the global instability threshold μ_G by an additional quantity $2\pi^2/L^3$ [see Eq. (12)]. The domain $\mu-\mu_t \gg \pi^2(1/L^2+2/L^3)$ where the scaling law Eq. (10) pertaining to the semi-infinite interval remains valid for the finite interval is indicated in Fig. 5 by vertical dashed-dotted lines for different values of the size L of the system used by Büchel *et al.* Numerical results for the size of the modes obtained by Büchel *et al.* are also reported. For a box length $L=50$ and $L=25$, all of their data are found to be inside the domain of validity of the scaling law Eq. (10) drawn as a continuous line. Even for a short system of length $L=10$, the data of Büchel *et al.* lie practically all to the right of the corresponding dashed-dotted line. However, the scaling law Eq. (10) should be valid over a shorter range since the departures from criticality in this window are greater than $e^{-2}\sim 0.14$; this corresponds to the observation of Büchel *et al.* that for $L=10$, no bulk region where the flow saturates is observed.

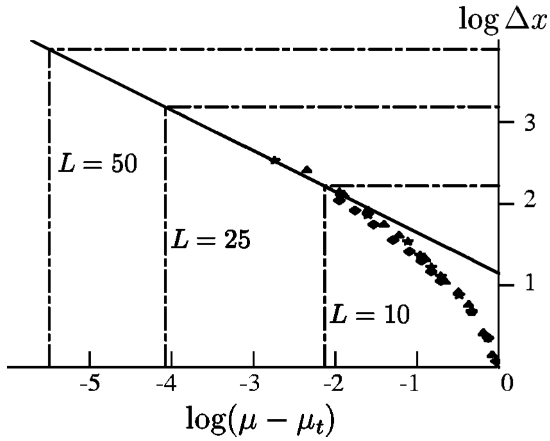

FIG. 5. Size of the global modes versus the deviation from the absolute instability threshold for the Taylor–Couette problem with throughflow. The different symbols show measurements by Büchel *et al.* obtained from numerical simulations of the Navier–Stokes equations. The continuous line represents the scaling law Eq. (10) which is valid as μ tends to μ_t. A global mode effectively saturates over a long distance and Eq. (10) should be valid if the region within the dashed–dotted lines (which depends on the size L of the system) includes sufficiently small departures from the absolute threshold μ_t.

Rayleigh–Bénard convection with throughflow,[3] bluff body wakes,[29] and resonant hot jets[30] also belong to the class for which the bifurcation scenario of Fig. 2(a) holds. Open flows which bifurcate following the fully nonlinear scenario [Fig. 2(b)] are not so common, the only known exception being the shear flow experiment in a Hele-Shaw cell[28] where the scaling law Eq. (11) has recently been reported and for which the nonlinear transition seems to precede the absolute instability threshold. Several experimental situations concerning the problem of front propagation, for example, chemical systems,[31] are known to exhibit nonlinear front selection. Adding a throughflow in this chemical experiment should yield an experimental open flow which bifurcates according to the nonlinear scenario [Fig. 2(b)].

In conclusion, we emphasize that the main ingredient sufficient for the subcritical bifurcation to occur in a finite box is the nonlinear selection of the front velocity. In this respect, model equation (1) displays the necessary minimal features but other models would exhibit similar qualitative dynamics.

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