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NON LINEAR STABILITY OF SLENDER ACCRETION DISKS BY BIFURCATION METHOD

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The nonlinear evolution of a constant angular momentum accretion disk subject to Papaloizou and Pringle (1984; hereafter PP) instability is investigated. The analysis is performed on an inviscid incompressible two-dimensional model using a formalism suitable for Hamiltonian systems. Only the most unstable modes are taken into account. It is found that relevant nonlinear terms have a stabilizing influence on the system. This supports recent numerical experiments showing a transition towards a quasi-stable planet configuration. The extension of the method to fat disk instabilities, more relevant to AGN disks, is discussed.

KEY WORDS: Accretion discs, nonlinear stability, Hamiltonian systems.

1. INTRODUCTION

Accretion Disks are a generic feature in the vicinity of compact objects-black holes, neutron stars and white dwarfs. They are also found in symbiotic systems, and in star-forming regions. Disks around white dwarfs are used to explain the cataclysmic variables (CVs; Horne 1991). These disks are geometrically thin and consequently Keplerian ($\Omega \propto R^{-3/2}$). Since their angular momentum decreases outwards, they are locally linearly stable with respect to axisymmetric disturbances (Rayleigh, 1880; Dubrulle and Knobloch, 1992). Disks around putative black holes ($M_{BH} \approx 10^{6-9} M_{\odot}$) in the central regions of the active galaxies (AGNs), e.g. radio galaxies and quasars, are thought to be vertically thick, i.e. tori. Pressure gradients then cause the rotation law to depart from Keplerian: $\Omega \sim r^{-q}$; $q > 3/2$ (see Ulrich 1989 for a survey of the multi-frequency observations, and Blandford 1991 for a fine introduction and survey of AGN theory). However, as long as their rotation index q is less than 2, Rayleigh criterion is still satisfied and these accretion tori are also linearly stable to axisymmetric perturbations.

It was however discovered by Papaloizou and Pringle (1985) that thick tori whose rotation index satisfies $q > \sqrt{3}$ can be unstable with respect to linear, global non-axisymmetric perturbations. The principal mode of instability occurs on a dynamical time-scale Ω^{-1}. It is therefore likely to disrupt the torus, or at least to generate sufficiently efficient angular momentum to preclude the existence of jets as in quasar models. This explains the interest generated by the discovery of this instability and the amount of effort put into the understanding of unstability. Drury (1985), Blaes (1985), Goldreich *et al.* (1986) and Glatzel (1987a, b) clarified the nature of the principal branch of PP instability. They showed that the unstable modes are the result of the coupling of two edge waves across a forbidden region around corotation. Neither compressibility nor Kelvin-Helmholtz mechanism enter into consideration. Goldreich *et al.* also proved that in an incompressible three dimensional torus, with inner radius r_-, outer radius r_+ and orbital radius r_0, vertical hydrostatic equilibrium is an excellent approximation for low β ($\beta = mr/r_0$, m being the azimuthal wave number). Therefore, two-dimensional hydrodynamic calculations may suffice to describe the non-linear evolution of these long wavelength (low m) modes laying on the principal branch of PP instability.

Even with this approximation, the investigation of the non-linear regime of PP instability is not simple. Most analytical investigations are performed in the limit of "slender" tori, for which $(r_+ - r_-)/r_0$ is small, using this ratio as an expansion parameter. Interesting results are also obtained by numerical investigation of the nonlinear development of the instability in slender compressible (Hawley 1987, Blaes and Hawley 1988) or self-gravitating tori (Christodoulou and Narayan, 1992). It was found that a torus in which a single mode with azimuthal wavenumber m is excited breaks up into m blobs which orbit around the central mass at the same velocity Ω as the original torus. These blobs, called "planets" by Hawley, seem to persist for a relatively long time. This led Goodman *et al.* (1987) to postulate that the planet configuration might be a new underlying equilibrium of the two-dimensional configuration. In this light, PP instability would be only the consequence of the bifurcation from one equilibrium solution to another. However, the planet configuration itself is subject to various instabilities. Longer integration of the solutions proves that the planets eventually merge, and a new, roughly axisymmetric structure emerges,but with changed rotation profile ($q \leqslant 1.8$). This result is consistent with results obtained by Zurek and Benz (1986) using a smooth-particle hydrodynamics code with 1000 particles to follow the evolution of a three-dimensional thick torus orbiting a Newtonian point mass. Within two orbits, a constant angular momentum torus is found to redistribute angular momentum so that the average rotation index approaches 1.75. This seems to prove that, at least in the slender case, PP instability eventually saturates and turns into a stable configuration.

The purpose of this work is to confirm and clarify this point via an analytical investigation of the nonlinear regime. Contrary to previous studies we shall not use the (small) width of the torus as an expansion parameter, but rather the amplitude of the perturbation as in standard bifurcation methods. This method can therefore be applied in more general cases, such as in "fat" tori (see discussion in Section 4). To make the analysis tractable, we shall use results derived by Goldreich *et al.* (1986). For low azimuthal modes, a "not too fat" torus can be made equivalent to compressible two-dimensional cylindrical shells by height averaging. Moreover, compressibility is not an essential ingredient of the instability. Blaes (1985) showed that growth rates for the incompressible and compressible models only differ by a factor of order unity. This suggests that a simple, two-dimensional incompressible model may keep the feature of a three-dimensional torus. We shall therefore use the model presented by Blaes and Glatzel (1986), which focuses on incompressible constant angular momentum cylinders.

An interesting feature of PP instability is the occurrence of a sort of "bifurcation" as the width of the torus is varied. Such a feature is conserved in Blaes and Glatzel (Section 2.2). The bifurcation bears some analogy with a standard Hopf bifurcation: it occurs when complex imaginary eigenvalues cross the imaginary axis. In dissipative systems, this bifurcation is well documented and its normal form can be computed via methods based on the "Central Manifold Theorem" (e.g. Krylov-Bogolyubov-Mitropolsky method, see Thual, 1988). This normal form reads to lower order

$$\dot{A}=(\nu+i\Omega)A-\alpha|A|^2A, \tag{1}$$

where A is the complex amplitude of the perturbation, $i^2=-1$, ν is the bifurcation parameter and α a complex number. The sign of the real part of alpha determines whether the first order non-linearities stabilize [$\mathrm{Re}(\alpha)>0$] or destabilize [$\mathrm{Re}(\alpha)<0$] the system. Only when $\mathrm{Re}(\alpha)>0$ do the linear and weakly non-linear analysis describe correctly the behaviour of the bifurcating solution. When $\mathrm{Re}(\alpha)<0$, finite amplitude perturbations may destabilize the system in the linearly stable regime.

In our analysis, we shall however use a non-dissipative (non-viscous) model, in which energy is conserved. In this case, the word bifurcation cannot be strictly applied to the crossing of imaginary eigenvalues because the notion of "attractor" is irrelevant. If the system admits an Hamiltonian function, such crossing is referred to as "resonance". In the special scenario we are considering, it is also customary (however not rigorous) to speak about an "Hamiltonian Hopf bifurcation". Methods have been developed to study this resonance. Van der Meer (1982) proved that the non-linear fate of the system in the vicinity of the resonance depends only on a parameter, a_2, which is the analog of the parameter α in (1) (Section 2.3). He also gave an algorithm to compute this parameter. To be able to use this algorithm, it will be necessary to re-express the model of Blaes and Glatzel, summarized in Section 2.1, using an Hamiltonian formulation. This is done in Section 2.4, using the analogy of our model with surface gravity waves on sea, for which work has been done by Miles (1977). The algorithm of Van der Meer, described in Section 2.3, is then applied to our system and the parameter relevant to the non-linear evolution is computed in Section 2.4. Various approximations used during the computation are discussed in Section 3. Interpretation of our results and connection with previous works are finally provided in Section 4.

2. THE HAMILTONIAN HOPF BIFURCATION

2.1 The model

As in Blaes and Glatzel (1986), the model consists of an incompressible disk of fluid rotating around a Newtonian potential well produced by a central point mass (see Figure 1). The structure parallel to the rotation axis is ignored and thus perturbations

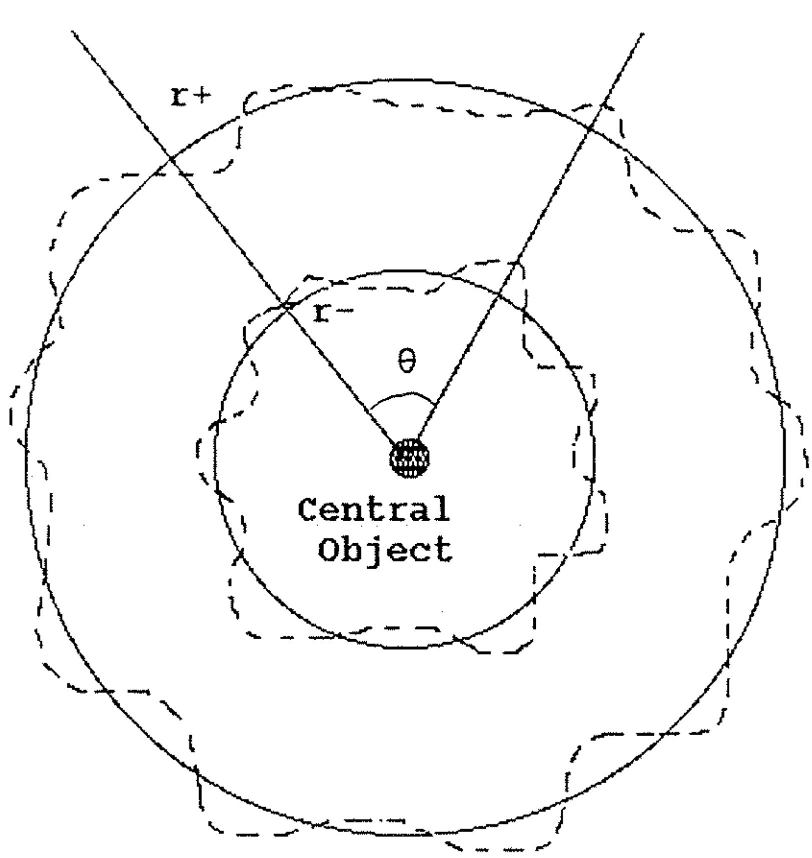

Figure 1 Geometry of the problem. The equilibrium state (solid lines) consists of an annulus. The perturbed state (dashed line) is defined by its inner (r_-) and outer (r_+) boundaries.

are only two-dimensional. Since the ordinary viscosity in the accretion torus is usually very small, we shall restrict our investigation to the inviscid regime. In that case, both the vorticity and the total energy of the fluid are conserved. After a suitable shift in the central potential well, the fluid can therefore be taken irrotational. The velocity field is then the gradient of a scalar potential, Φ. Because of the incompressibility condition, Φ must satisfy the following condition throughout the fluid:

$$\Delta\Phi = 0. \tag{2}$$

The two-dimensional Navier-Stokes equation describing the evolution of the fluid reduces then to the Bernouilli equation:

$$\partial_t\Phi + \tfrac{1}{2}|\nabla\Phi|^2 - V + P = C, \tag{3}$$

where C is a constant, P is the pressure and V is the gravitational potential due to the central mass M:

$$V(r) = GM/r,$$

and ∇ is the gradient operator. To complete the description of the system, two boundary conditions corresponding to the inner ($-$label) and the outer surfaces ($+$label) must be added to (3). Those surfaces are physically defined as the location where the pressure vanishes. In polar coordinates (r, θ), they can be specified by the equations $r=\zeta_-(\theta, t)$ and $r=\zeta_+(\theta, t)$. If we assume that the two surfaces are free, the boundary kinematics is:

$$\nabla\Phi\cdot d\mathbf{s}_\pm = \partial_t\zeta_\pm \zeta_\pm \, d\theta \qquad \text{at the boundaries.} \tag{4}$$

Here, ds is the element of surface area at the boundary, oriented along the unit normal vector $\mathbf{n}_\pm$. The latter are given by:

$$[1+|\nabla\zeta_\pm|^2]^{1/2}\mathbf{n}_\pm = \mathbf{e}_r - \partial_\theta\zeta_\pm \mathbf{e}_\theta,$$

so that

$$\frac{ds_\pm}{\zeta_\pm \, d\theta} = [1+|\nabla\zeta_\pm|^2]^{1/2}.$$

The set of equations (3) and (2) implemented with the boundary conditions (4) admits a stationary solution which is characterized by a constant specific angular momentum distribution, that is:

$$\mathbf{u} = (s_0/r)\mathbf{e}_\theta,$$

$$P(r) = C + (GM/r) - \tfrac{1}{2}(s_0/r)^2, \tag{5}$$

corresponding to a velocity potential $\Phi = s_0\theta$. The inner and outer surfaces are located at radius r_- and r_+ such that:

$$C = \tfrac{1}{2}(r_0/r_\pm)^2 - (r_0/r_\pm), \tag{6}$$

where r_0 is the radius at which the pressure is maximum. The single parameter ε $(0 \leqslant \varepsilon \leqslant 1)$, defined by:

$$r_\pm/r_0 = 1/(1 \mp \varepsilon), \tag{7}$$

characterizes then the basic solution. The two limiting cases $\varepsilon=0$ and $\varepsilon=1$ correspond respectively to infinitely slender and infinitely fat tori. In the sequel, we shall take ε as the bifurcation parameter.

2.2 *Linear stability analysis*

The linear stability analysis of the model can be performed using a Fourier expansion for the surface deformations $\delta_\pm = \zeta_\pm r_\pm$ and an harmonic expansion for the velocity

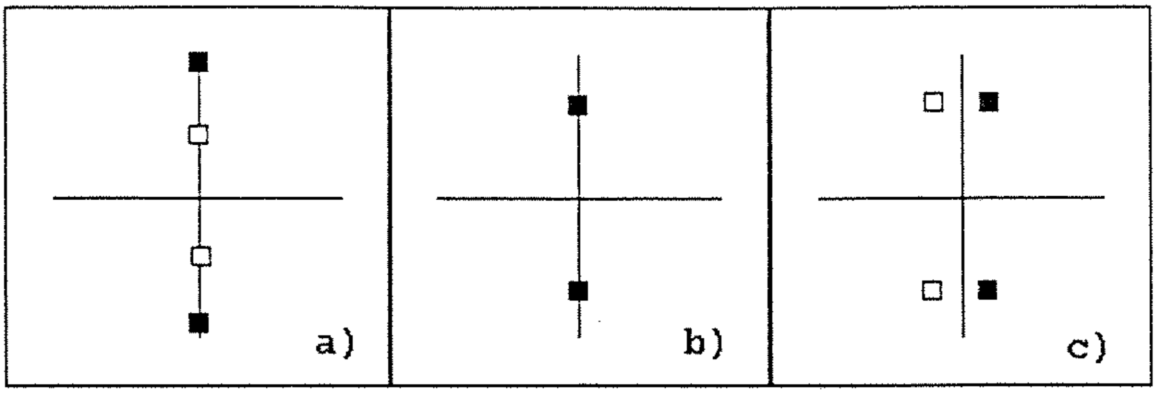

Figure 2 The 1:1 non-semisimple resonance. Before the bifurcation (2a), the spectrum consists of two pairs of imaginary conjugated eigenvalues. At the bifurcation (2b), the two pairs collapse onto the imaginary axes. After the bifurcation (2c), the two pairs of eigen values escape orthogonally from the imaginary axes. The spectrum is then made of quaternions.

potential (see e.g. Blaes and Glatzel, 1986). It is then found that when the parameter ε is smaller than the critical value $\varepsilon_c = 0.52$, the annulus becomes unstable with respect to non-axisymmetric perturbations. The first mode to be destabilized is the fundamental ($m = 1$) mode. The instability occurs when two pairs of pure imaginary eigenvalues, moving as a function of ε, meet together at $\varepsilon = \varepsilon_c$ and escape orthogonally from the imaginary axis (see Figure 2). Our purpose is now to study the non-linear evolution of the instability in the vicinity of the bifurcation.

2.3 Description of the method

Since we are working on an inviscid model, the total energy is conserved and the system is Hamiltonian. We will therefore use the Hamiltonian Hopf bifurcation method. In this section, we outline the procedure to be used. A more detailed account of this can be found in van der Meer (1982).

Let us decompose the Hamiltonian of the system, H as

$$H = H_2 + H_3 + H_4 + \dots, \tag{8}$$

where $H_2, H_3, \dots$ correspond respectively to quadratic, ternary ... interactions. These components may for example correspond to successive terms in a Taylor expansion of the original Hamiltonian. Because of the Hamiltonian nature of the system, the eigenvalues of the matrix associated with H_2 must obey the conditions ensuring conservation of area in phase space: if λ is an eigenvalue, so is its opposite, $-\lambda$. Moreover, since the initial problem is real, if an eigenvalue λ is complex, its conjugate, λ^* is also an eigenvalue. In other words, the spectrum is composed only of pair of conjugated imaginary eigenvalues (Figure 2a), or quaternions ($\lambda = \pm\lambda_r \pm i\lambda_i$) (Figure 2c), where λ_r and λ_i are the real and imaginary parts of λ.

Let us focus on the case of the purely imaginary eigenvalues which occur at the bifurcation. By a suitable change of variable, one can show that they arise from only two different generic quadratic interactions; the first one has terms like

$$\lambda_k(x_k^2+y_k^2)+\lambda_l(x_l^2+y_l^2), \tag{9}$$

while the second has terms like

$$\alpha(x_k y_l - x_l y_k) - \tfrac{1}{2}\eta(x_k^2+x_l^2), \tag{10}$$

where x and y are the canonical variables, and $\eta=\pm 1$. For both interactions, one can investigate the stability of the system by writing down the Hamilton equations associated with the generic Hamiltonians (9) and (10) and compute the eigenvalues of the corresponding matrix. In the first case (9), the eigenvalues are $\pm i\lambda_k$ and $\pm i\lambda_l$. In the second case (10), only two imaginary eigenvalues exist, $\pm i\alpha$, with multiplicity two. The corresponding eigenspaces is of dimension one, and the matrix cannot be put under a pure diagonal form. Therefore, the simplest expression for the matrix corresponding to the first case (9) would be

$$\begin{bmatrix} \lambda_k & 0 & 0 & 0 \\ 0 & \lambda_l & 0 & 0 \\ 0 & 0 & -\lambda_k & 0 \\ 0 & 0 & 0 & -\lambda_l \end{bmatrix} \tag{11}$$

while in the second case (10), it would be

$$\begin{bmatrix} \alpha & \eta & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & \eta \\ 0 & 0 & 0 & \alpha \end{bmatrix}. \tag{12}$$

In Hamiltonian systems, instabilities occur via resonances between the eigenfrequencies. There are two kinds of resonances, according to whether the eigenvalues arise from quadratic interaction of the type (9) or (10).

In the case (9), a resonance occurs whenever two eigenfrequencies are linked through the relation

$$q\lambda_k - p\lambda_l = 0,$$

where p and q are integer. In the Hamiltonian theory terminology, this case is known as a semisimple $p{:}q$ resonance. In the case (10), one speaks of a non-semisimple $1:-1$ resonance.

The Hamiltonian Hopf bifurcation occurs when an Hamiltonian system goes through a non-semisimple resonance by a suitable continuous tuning of one physical parameter of the system (in our case, the width of the torus, ε). This bifurcation is achieved for example in the Hamiltonian H_ν:

$$H_\nu = \alpha(x_1 y_2 - x_2 y_1) - \tfrac{1}{2}\eta(x_1^2 + x_2^2) + \tfrac{1}{2}\nu(y_1^2 + y_2^2). \qquad (13)$$

The eigenvalues of (13) are given in Figure 2, according to the sign of $\eta\nu$. The behavior of the system in the vicinity of an Hamiltonian Hopf bifurcation is found by transforming $H_{\nu=0}$ into its normal form:

$$H_{\nu=0} = \alpha(x_1 y_2 - x_2 y_1) - \tfrac{1}{2}\eta(x_1^2 + x_2^2) + a_1(x_1 y_2 - x_2 y_1)^2$$
$$a_2(y_1^2 + y_2^2)^2 + a_3(y_1^2 + y_2^2)(x_1 y_2 - x_2 y_1). \qquad (14)$$

Before the bifurcation, that is when $\eta\nu \leqslant 0$ (Figure 2a), the origin is a Liapounov-stable stationary point for the system. The behavior of the solutions after the bifurcation ($\eta\nu > 0$, Figure 2c) depends only on the sign of the coefficient $-\eta a_2$:

for $-\eta a_2 > 0$ solutions starting near zero are bounded: the non-linear interactions stabilize the system.
for $-\eta a_2 < 0$, solutions run off to infinity: the non-linear interactions have a destabilizing effect.

The dependence of the non-linear evolution on the parameters can be understood using a simple argument: in the vicinity of the bifurcation (small ν), the equations giving the evolution of x_1, x_2, y_1 and y_2 are

$$\dot{x}_1 = -\alpha x_2 + \nu y_1 + 4a_2 y_1^3,$$
$$\dot{x}_2 = \alpha x_1 + \nu y_2 + 4a_2 y_2^3,$$
$$\dot{y}_1 = -\alpha y_2 + \eta x_1,$$
$$\dot{y}_2 = \alpha y_1 + \eta x_2, \qquad (15)$$

where we have used (14) and (13), and neglected the terms in a_1 and a_3. These equations can be combined to give

$$\ddot{y}_1 = (\eta\nu - \alpha^2) y_1 + 4\eta a_2 y_1^3 - 2\alpha\eta x_2. \qquad (16)$$

Let us neglect the last term of (16). The resulting equation bears some strong resemblance with the normal form of Hopf bifurcation (1). It can be seen immediately that linear destabilization occurs only for $\nu\eta > 0$. This destabilization is either amplified or weakened by nonlinear terms according to the sign of ηa_2. Only when $\eta a_2 < 0$ does the solution saturate at an amplitude of the order $\sqrt{\nu/4a_2}$.

The purpose of the next section is to put the two-dimensional disk stability analysis into the Hamiltonian formalism and to compute the coefficient a_2.

2.4 Finding a Hamiltonian for the system

The first step of the computation is to find an Hamiltonian formulation of the system described in Section 2.1. Formally, our system is analog to the problem of surface gravity waves on sea. In that case, it has been shown by Miles (1977) that an Hamiltonian formulation was available provided one uses canonical coordinate the surface elevation and the value of the potential at the surface, both regarded as a function only of time and of the lateral coordinate (with respect to the surface). The physical reason for this, is that, when the bulk fluid is irrotational, the whole velocity field is determined solely by the movements of the fluid surface. The equation of motion in the fluid interior [namely, the Bernoulli equation (3)] can be viewed as the equation enabling the computation of the pressure throughout the flow.

The model described in Section 2.1 can then be reformulated as follows: let us call $\phi_\pm$ the values of Φ at the inner and outer surfaces. But construction, $\phi_\pm$ depends only on θ and t:

$$\phi_\pm(\theta, t) = \Phi(r, \theta, t)|_{r=\zeta_\pm(\theta, t)}. \tag{17}$$

From now on, we drop the subscript $\pm$ on the functions except in situations where a mixing of the + and the − component could occur in the same equation.

The values of the derivative $\nabla_\theta \phi$ and $\nabla_\theta \Phi$ at the surface, where $\nabla_\theta = \zeta^{-1}\partial_\theta$, are not the same, but are linked by

$$(\nabla_\theta \Phi)_{r=\zeta} = \nabla_\theta \phi - (\nabla_\theta \zeta) v_r, \tag{18}$$

where v_r is the value of the radial velocity at the surface. In the same way, the two normal derivatives of Φ and ϕ are related by (Milder, 1990, Henyey *et al.* 1988))

$$\gamma(\nabla_n \Phi)_{r=\zeta} = v_r - \nabla_\theta \phi - (\nabla_\theta \zeta)^2 v_r, \tag{19}$$

where γ and ∇_n, the normal derivatives have been defined in Section 2.1. Using (18) and (19), one can then transform the kinematic and the dynamic conditions at the surface [(4) and (3)] into

$$\partial_t \zeta = -(\nabla_\theta \phi)(\nabla_\theta \zeta) + v_r[1 + (\nabla_\theta \zeta)^2],$$

$$\partial_t \phi = C + \tfrac{1}{2}(v_r^2[1 + (\nabla_\theta \zeta)^2] - (\nabla_\theta \phi)^2) + V(\zeta). \tag{20}$$

The equations are self-contained when the radial velocity at the sufrace v_r is expressed in term of ϕ and ζ. In general, this can be done only via an expansion procedure, where v_r is developed in a power series in $\zeta_\pm - r_\pm$:

$$v_{r,\pm} = \sum_{n=0}^{\infty} a_{n,\pm}(\phi_\pm)(\zeta_\pm - r_\pm)^n.$$

For moderate surface slope, a good approximation to v_r can be obtained by truncating the series at finite order m:

$$v_{r,\pm}^{(m)} = \sum_{n=0}^{m} a_{n,\pm}(\phi_\pm)(\zeta_\pm - r_\pm)^n.$$

One can then substitute this quantity into the exact equations (20) to approximate the dynamics, but the resulting ystem is no longer canonical (Milder, 1990). That is why we are going to use another procedure: we shall truncate the Hamiltonian to a given order and let the truncated Hamiltonian define the approximate field equations. The meaning and the relevance of such a procedure will be discussed in the third section. For the moment, we proceed to find the Hamiltonian of the system.

We start by decomposing the velocity potential Φ and the surface elevation ζ into a mean part corresponding to the basic solution and a perturbed quantity. Using the radius of maximum pressure as characteristic length [see (6)] and relabelling the adimensional quantity as r, we note that

$$\Phi = s_0\theta + \Psi$$

$$\zeta_\pm = r_\pm + \delta_\pm, \tag{21}$$

where the equilibrium values $r_\pm$ are given by (7). The values of the perturbed velocity potential Ψ at the perturbed surface ζ will be denoted ψ.

Following Miles (1977), we write the Hamiltonian H as the sum of the kinetic and potential energy of the system:

$$H = T + V, \tag{22}$$

with

$$T = \int_0^{2\pi} d\theta \int_{\zeta_-}^{\zeta_+} \tfrac{1}{2}(\nabla\phi)^2 r\, dr,$$

$$V = \int_0^{2\pi} d\theta \sum_{n=2}^{\infty} g^{(n)}(r_+)(\zeta_+ - r_+)^n - g^{(n)}(r_-)(\zeta_- - r_-)^n,$$

$$g^{(n)}(x) = (1/n!)[\partial_r^n(r^{-1})]_{r=x}. \tag{23}$$

Using the Green's formula and the incompressibility (2), one sees that

$$\int_0^{2\pi} d\theta \int_{\zeta_-}^{\zeta_+} \tfrac{1}{2}(\nabla\phi)^2 r\, dr = \int \tfrac{1}{2}\phi\nabla_n\Phi\, ds,$$

$$= \int \tfrac{1}{2}(r_+\phi_+\partial_t\zeta_+ - r_-\phi_-\partial_t\zeta_-)\, d\theta. \tag{24}$$

One can then check that $\phi_{\pm}$ and $h_{\pm} = \pm r_{\pm}\delta_{\pm}$ are the two pairs of canonical variables for which the perturbed field equations can be derived from the Hamiltonian equations

$$\partial_t h_{\pm} = \frac{\delta H}{\delta \psi_{\pm}}$$

$$\partial_t \psi_{\pm} = -\frac{\delta H}{\delta h_{\pm}}. \tag{25}$$

To be able to use the Hamiltonian bifurcation method described in Section 2.2, we need now to convert our continuous Hamiltonian into a discrete version. This can be achieved using a modal expansion in orthogonal functions. Then, by virtue of Parseval's theorem, the mode coefficients will obey the discrete canonical equations.

Since Ψ satisfies (2), it is natural to expand it in harmonical functions:

$$\Psi(r, \theta) = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \exp(in\theta) + c.c, \tag{26}$$

where A_n and B_n are complex coefficient and $c.c$ means "complex conjugated of the former expression". It is then logical to expand both $\psi_{\pm}$ and $h_{\pm}$ as Fourier series

$$h_{\pm} = \sum_{n=1}^{\infty} d_{\pm,n}(t) \exp(in\theta) + c.c,$$

$$\psi_{\pm} = \sum_{n=1}^{\infty} f_{\pm,n}(t) \exp(in\theta) + c.c. \tag{27}$$

The expansions (26) and (27) enable the computation of the radial velocity $v_{r,\pm}$ in function of the coefficients $f_{\pm,n}$ and $d_{\pm,n}$ via an iterative procedure. Schematically (see West *et al.*, 1987 for further details), one writes the Taylor expansion:

$$\psi_{\pm} = \Psi(r_{\pm}) - \delta_{\pm} \partial_r \Psi|_{r=r_{\pm}} - \dots \tag{28}$$

Expanding the quantities in Fourier series via (26) and (27) and using the quantity $d_{\pm,n}$ as small parameters, one can then solve (28) recursively and express the coefficient A_n and B_n in term of the $f_{\pm,n}$ and the $d_{\pm,n}$. This expression can then be used to compute the expansion of $v_{r,\pm}$ via:

$$v_{r,\pm} = \sum_{n=1}^{\infty} n[A_n(r_{\pm} + \delta_{\pm})^{n-1} - B_n(r_{\pm} + \delta_{\pm})^{-(n+1)}] \exp(ni\theta) + c.c. \tag{29}$$

2.5 Results

The procedure described in the previous section was used to compute the discrete (modal) Hamiltonian of the system as a function of $f_{\pm,n}$, $d_{\pm,n}$ and their complex conjugated values. At the critical value $\varepsilon=0.52$, the only resonant mode is the mode $m=1$. We shall therefore restrict our attention to this peculiar mode. From now on, the quantities $f_{\pm}$ and $d_{\pm}$ refer to $f_{\pm,1}$ and $d_{\pm,1}$.

The $m=1$ mode can only react onto itself via mono, termary ... ("odd-interactions") so in our case, the only terms appearing in the decomposition (8) are H_2 and H_4. The expression of the former is given in the Appendix. As for H_4, it is made of more than forty terms and we judged it unnecessary to include it in the Appendix. The equations of motion corresponding to this truncated level can be found via the canonical equations:

$$\dot{d}_{\pm}=\partial(H_2+H_4)/\partial f^*_{\pm},$$

$$\dot{f}_{\pm}=-\partial(H_2+H_4)/\partial d^*_{\pm}.$$

In this equation, $d^*_{\pm}$ and $f^*_{\pm}$ stands for the complex conjugate of $d_{\pm}$ and $f_{\pm}$. They obey the corresponding complex equations. Note also that since $d_{\pm}$ and $f_{\pm}$ are complex, the system has still 8 degrees of freedom. The matrix of the linearized system [only the contribution of H_2 in (30)] is

$$\mathbf{A}=\begin{pmatrix}\mathbf{M} & 0\\ 0 & \mathbf{M}^*\end{pmatrix}, \tag{31}$$

where $\mathbf{M}$ is the 4 by 4 complex matrix

$$\mathbf{M}=\begin{bmatrix} -\frac{i}{r_+^2} & 0 & \frac{r_-^2+r_+^2}{-r_-^2+r_+^2} & \frac{-2r_-r_+}{-r_-^2+r_+^2}\\ 0 & -\frac{i}{r_-^2} & \frac{-2r_-r_+}{-r_-^2+r_+^2} & \frac{r_-^2+r_+^2}{-r_-^2+r_+^2}\\ -\frac{-1+r_+}{r_+^4} & 0 & -\frac{i}{r_+^2} & 0\\ 0 & \frac{-1+r_-}{r_-^4} & 0 & -\frac{i}{r_-^2}\end{bmatrix}, \tag{32}$$

r_+ and r_- being given by equation (6) with $r_0=1$. Before the bifurcation, $\mathbf{M}$ has four distinct purely imaginary eigenvalues (Figure 2a). At the critical value, $\varepsilon_c=1-1/(2.067946097672506)$, two imaginary eigenvalues of $\mathbf{M}$ collapse onto the value $\lambda_c=-0.67071i$, corresponding to the nonsemisimple resonance (Figure 2b). We then use the linear change of variable described in Burgoyne and Cushman (1974) to transform the Hamiltonian H_2 into its normal form. In the new coordinates, denoted

$(x_1, y_1, x_2, y_2, \ldots, x_4, y_4)$, H_2 becomes

$$H_2 = 0.0308642(x_1^2 + y_1^2) + 1.89355(x_2^2 + y_2^2) + 0.67071(x_3 y_4 - x_4 y_3) + 0.5(x_3^2 + x_4^2). \quad (33)$$

We see from this expression that the coordinates x_1, y_1 and x_2, y_2, which have been normalized so that the numer appearing in front of their square is half the modulus of the corresponding (imaginary) eigenvalue, decouple from the dynamics of the resonance. They just correspond to regular diagonal blocs in the diagonal form of the matrix **M**, which translates into regular oscillatory motions in the physical space. In that sense, they do not participate into and do not contribute to the dynamics of the resonance and can be subsequently ignored.

Following van der Meer (1982) we then proceed to compute a_2, the coefficient of the polynomia $(y_3^2 + y_4^2)^2$, which determines the behaviour of the system near the resonance. The final result is

$$a_2 = 0.580036 > 0.$$

Note that all the computations implied in the procedure we used involved a great deal of formal algebric manipulations using enormous expressions (e.g. because of the complexity of H_4). We therefore resorted to a formal algebric manipulator software (Mathematica) to handle the computation.

3. DISCUSSION

Before discussing the astrophysical implications of the result, we shall discuss the approximations made at various stages of the computation.

The first approximation we made was to model a compressible, slender, three-dimensional accretion torus by an incompressible, two-dimensional disk. Justification of such an approximation was already given in the introduction: since neither compressibility, not three-dimensionality appear to be crucial in the occurrence of the instability, such a model probably keeps the main feature of the bifurcation. The advantage of the simplified model is that it can be used for tractable analytical computations.

The second approximation performed was to use an inviscid model. This is because the PP instability mechanism does not appear to be a viscous one. Moreover, since the actual viscosity in an astrophysical accretion disk is usually very small, one can consider viscous mechanisms as slight perturbations, which do modify slightly the final result, but not its generic features.

The Hamiltonian of the system was found by an expansion procedure using the relative deformation of the equilibrium surface, $\delta_\pm / r_\pm$. It is clear that this expansion will be rapidly converging only for small values of this parameter, that is near the linear regime. Moreover, another level of approximation was introduced at this stage by describing the approximate dynamics of the solution through this truncated Hamiltonian rather than via the truncation of the expansion of the equations of motion (see Section 2.2). The effect of such a truncature procedure has been discussed recently by Milder (1990), in the case of surface waves. It was found that this expansion procedure was able to generate very nonlinear, realistic-looking solutions, even when the Hamiltonian is truncated to low order-two or three terms in its slope expansion. However, an unphysical behavior was found at wavenumber greater than g/w^2, where g and w are respectively the gravity and the local normal velocity. The introduction of one more term in the Hamiltonian was sufficient to mitigate the numerical instability, extending the regime of validity of the procedure to smaller wavenumbers and higher slopes. In our expansion procedure, we retain terms up to fourth order, which means that we are working in this extended regime. In our case, the normal velocity is of the order of $|\lambda_c|\delta_{\pm}$, where λ_c is the critical eigenvalue at the bifurcation ($\lambda_c = -0.607i$) and g/k (k being the wave number) is of the order of the gravitational potential $1/r_{\pm} - 1/(2r_{\pm}^2)$. The regime of validity extends then at least up to relative surface elevation of the order of $[\lambda_c^2(r_{\pm} - 0.5)]^{-1} \sim 1.4$ at r_+ and 14 at r_-. This regime extends therefore well into the non-linear regime that we are studying.

In restricting our investigation to the dynamics of the linearly resonant mode $m = 1$, we have neglected possible nonlinear resonances. However, as pointed out by Chirikov (1979), oscillations induced by nonlinear resonances are always bounded. The possible explosive instabilities can therefore only come from linear resonances. This justifies our approximation, since we are only concerned with the answer to the question: does the instability found in accretion tori saturate?

4. ASTROPHYSICAL IMPLICATIONS

The main result of this paper is that nonlinear interactions stabilize a system subject to PP instability. Our result holds for perturbations of amplitude of order unity and for configurations in which the specific angular momentum is conserved. Our finding that nonlinear interaction stabilizes the system suggests the following scenario for the nonlinear regime of PP instability: unstable modes undergo a regime of linear growth which is limited by non-linear interactions. The system then sets up onto a stable homoclinic orbit around the origin, with an amplitude modulation of order unity. In the physical space, this translates into a single modulation of the surface (a "planet"), with amplitude variations of order unity. This configuration can persist for several dynamical times. However, because the number of degrees of freedom (relevant modes) is not too large, it is possible that the homoclinic orbit returns to the origin in a finite time. One would then observe the disappearance of the planet and resurgence of the basic, unperturbed stage. This would be again followed by the reappearance of the planet, as the system drifts away from the origin along the homoclinic orbit. Such behaviour has been predicted for Kelvin-Helmholtz waves in a tilting tank, which undergoes a similar resonance (Weissman, 1979). If we now allow the viscosity to operate as a small perturbation, the scenario can change slightly. In that case, the specific angular momentum is not strictly conserved and can change moderately along the evolution. In that case, one would still be able to observe the appearance and disappearance of the planet, but the state of the system "without planet" will be now characterized by a slightly different angular momentum distribution, as it has changed in the evolution.

Such a scenario, applied to a system in which the mode $m = m_0$ becomes unstable first, could therefore explain the appearance of m_0 planets, which would persist on several dynamical times, and can disappear to leave room for a new, axisymmetric structure, with a rotation profile slightly changed from the original one (due to real or numerical viscosity). Such a scenario could for example explain the evolution of model 1 of Christodoulou and Narayan (1992), in which a $m = 3$ PP instability is excited.

Note however that for $m > 1$, the planet configuration can be subject to secondary, pairing instability in which all the planets merge into a single large planet. This evolution would however be definitely different from the one described above, for one would not see that disappearance of all the planets together, but a slow decrease of the number of planets present in the system.

However, the astrophysically interesting case is that of giant tori with a ratio of the outer radius to the inner radius of order 10^3. For two-dimensional models with such a ratio, PP instability is not relevant anymore and compressibility effects become predominant. Glatzel (1987a, b), studying shear flows in compressible cylinders, showed that at the critical width at which PP instability disappears, compressible modes take over via probably a simple mode crossing. This behaviour cannot of course be captured by our incompressible treatment. Some numerical investigation of the behaviour of this instability has been undertaken by Frank and Robertson (1988) and Kojima (1986) in the three-dimensional case. The computations were limited to an aspect ratio of the order of 40, still far away from interesting configurations. However, these computations have shown that the growth rate of the instability was decreasing with the size of the torus. A similar behaviour is found for shear flows in compressible cylinders (Glatzel, 1987a, b) but the solution of the eigenvalue problem for a cylinder of infinite radial extent shows that the growth rate is small, but finite. If giant accretion tori tend towards cylindrical flows, the same behaviour is likely to be expected. The study of the nonlinear regime is therefore of prime importance to determine the possible viability of models of fat accretion tori. Interesting configurations are still beyond computational possibilities but it might be possible to get some results via an analysis of the type used here. The major conceptual difficulties involve establishing the bifurcation framework and finding an Hamiltonian for the system. The fact that the growth rate of unstable modes tends to a finite value when increasing the width of the torus, precludes the use of the latter as a bifurcation parameter, as we did in this paper. However, the work of Glatzel suggests that to take q, the rotation indice as the bifurcation parameter, for its tuning enables us to stabilize modes one by one. The problem of finding an Hamiltonian for such a framework however still remains.

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APPENDIX: EXPRESSION OF THE HAMILTONIAN

The expression for the second order term H_2 in the expansion of the Hamiltonian of the system is given by

$$\begin{aligned} NH_2 = {} & d_+d_+^*r_-^6 - d_+d_+^*r_-^6r_+ - d_+d_+^*r_-^4r_+^2 + -id_+^*f_+r_-^6r_+^2 + id_+f_+^*r_-^6r_+^2 \\ & + d_+d_+^*r_-^4r_+^3 - d_-d_-^*r_-^2r_+^4 + d_-d_-^*r_-^3r_+^4 - id_-^*f_-r_-^4r_+^4 \\ & + id_-f_-^*r_-^4r_+^4 + id_+^*f_+r_-^4r_+^4 - id_+f_+^*r_-^4r_+^4 + f_-f_-^*r_-^6r_+^4 \\ & + f_+f_+^*r_-^6r_+^4 - 2f_-^*f_+r_-^5r_+^5 - 2f_-f_+^*r_-^5r_+^5 + d_-d_-^*r_+^6 - d_-d_-^*r_-r_+^6 \\ & + id_-^*f_-r_-^2r_+^6 - id_-f_-^*r_-^2r_+^6 + f_-f_-^*r_-^4r_+^6 + f_+f_+^*r_-^4r_+^6, \end{aligned} \tag{A.1}$$

where

$$N = -(r_-^6r_+^4) + r_-^4r_+^6. \tag{A.2}$$