Critical Behavior in the Relaminarization of Localized Turbulence in Pipe Flow

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The statistics of the relaminarization of localized turbulence in a pipe are examined by direct numerical simulation. As in recent experimental data [J. Peixinho and T. Mullin, Phys. Rev. Lett. 96, 094501 (2006)], the half-life for the decaying turbulence is consistent with the scaling (Re_c − Re)^{-1}, indicating a boundary crisis of the localized turbulent state familiar in low-dimensional dynamical systems. The crisis Reynolds number is estimated as Re_c = 1870, a value within 7% of the experimental value 1750. We argue that the frequently asked question, of which initial disturbances at a given Re trigger sustained turbulence in a pipe, is really two separate questions: the “local phase space” question (local to the laminar state) of what threshold disturbance at a given Re is needed to initially trigger turbulence, followed by the “global phase space” question of whether Re exceeds Re_c at which point the turbulent state becomes an attractor.

Understanding the behavior of fluid flow through a circular straight pipe remains one of the outstanding problems of classical physics and has continued to intrigue the physics community for more than 160 years [1–4]. Although all evidence indicates that the laminar parabolic flow is linearly stable, the flow can become turbulent even at modest flow rates. The exact transition point depends not only on the flow rate (measured by the Reynolds number Re = U/Dν, where U is the axial flow speed, D is the pipe diameter, and ν is the fluid’s kinematic viscosity) but also sensitively on the shape and amplitude of the disturbance(s) present [5–8]. When it occurs, transition is abrupt with the flow immediately becoming temporally and spatially complex. Given that most industrial pipe flows are turbulent and, hence, more costly to power than if laminar, a central issue is to understand the conditions which trigger sustained turbulence. The problem is, however, severely complicated by the fact that the threshold appears very sensitive to the exact form of the disturbance and turbulent transients can exist for long times. Of particular interest is the low-Re situation where the transition typically leads to a clearly localized turbulent structure called a “puff” within the laminar flow [3,9]. A puff has a typical length of about 20D along the pipe (see Fig. 1) and, despite appearing established, can relaminarize without warning at sufficiently low Re after traveling many hundreds of pipe diameters downstream.

There have been a number of contributions to this problem but so far no consensus on the minimum Reynolds number Re_c above which turbulence is sustained. Experimental studies have focused on plotting transition-threshold curves in disturbance amplitude-Re space for specific forms of applied perturbation. One well-studied perturbation having sixfold rotational symmetry gave rise to a threshold amplitude which scaled like Re^{-1} above Re = 2000 [6] but diverged at Re ≈ 1800 [7]; i.e., below this value, no sustained turbulence could be excited however hard the flow was disturbed. Subsequent experiments [8] studying the statistics of relaminarizations of puffs as Re is reduced have lowered this threshold value to Re_c = 1750 ± 10, close to a previous estimate of 1760 [5] but not to others of 1876 [10] and ≈ 2000 [9]. The only complementary numerical work performed so far has been in a short periodic pipe of 5D length [11], where it was demonstrated that the pipe-long turbulent state displays the transient characteristics of a chaotic repellor until Re_c = 2250, above which it becomes a chaotic attractor. Recent experiments using a very long pipe [12] in which the statistics on long transients are available, however, suggests that there is no critical behavior. Rather than the turbulent half-life scaling like τ ∼ (Re_c − Re)^{-1} [8,11], it is found to increase exponentially instead. Interestingly, re-interpretation of the 5D-pipe data (by accounting for an initial adjustment phase) seems to corroborate this alternative exponential lifetime behavior even though the pipe is too short to capture a turbulent puff.

In this Letter, we consider a much longer pipe of length 16πD (= 50D), in which turbulent puffs can be represented faithfully using direct numerical simulation [13], and examine the statistics of how they relaminarize. We find an exponential distribution of lifetimes and the critical scaling law τ ∼ (Re_c − Re)^{-1}, with a constant of proportionality and an estimate of Re_c = 1870 (see Fig. 4) both in good agreement with experimental data [8]. Surprisingly, given its long history, this represents the first time that a quantitative connection between theory and experiment has been established in the pipe flow problem.

The Navier-Stokes equations for an incompressible Newtonian fluid

\[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \]  

(1)

in a straight pipe with a circular cross section and for constant mass flux, were solved numerically in cylindrical coordinates (r, θ, z) using a mixed pseudospectral-finite difference formulation [14]. The code was found to accurately reproduce linear stability results for Hagen-
Poiseuille flow, instabilities of nonlinear traveling wave solutions, and the statistical properties of turbulent pipe flow [16] (as well as being cross-validated with another code [17]). A resolution of 40 radial points was adopted with grid points concentrated at the boundary, and Fourier modes were kept up to \( \pm 24 \) in \( \theta \) and to \( \pm 384 \) in \( z \) for a periodic pipe of length \( L = 16\pi D \). This ensured spectral dropoff of 6 orders of magnitude in the power of the coefficients when representing a puff velocity field at \( \text{Re} = 1900 \); see the inset in Fig. 2. The time step was dynamically controlled using information from a predictor-corrector method and was typically around 0.006\( D/U \). The initial conditions for the calculations were randomly selected velocity snapshots taken from a long puff simulation performed at \( \text{Re} = 1900 \). A body forcing applied over 10\( D \) of the pipe and for a time 10\( D/U \) was used to generate an “equilibrium” puff which remained stable in length and form for a time period of over 2000\( D/U \) (see Fig. 1). At a chosen \( \text{Re} < 1900 \), a series of more than 40 and up to 60 independent simulations was performed, each initiated with a different puff snapshot to generate a data set of relaminarization times. The signature of the relaminarization was a clear and sudden transition to exponential decay of the energy. The criterion for relaminarization was taken to be such that the energy of the axially dependent modes was less than \( 5 \times 10^{-4} \rho U^2 D^3 \), below which all solutions were well within the decaying regime. The range of measured \( \text{Re}_c \) discussed above indicates sensitivity to noise. Robustness of the relaminarization statistics was verified by comparing the half-lives of data sets obtained by varying different computational parameters of the simulation (see Fig. 2). All modifications produce half-life values within the 95% confidence interval about the default half-life prediction.

Decay probabilities for a range of Reynolds numbers are shown in Fig. 3 over an observation window of 1000\( D/U \). The linear dropoff of the probability on the log-plot strongly suggests the exponential distribution \( P(T - t_0) \sim \exp[-(T - t_0) \ln 2/\tau] \), where \( t_0 \) is the start of the decay process after an initial adjustment phase and \( \tau = \tau(\text{Re}) \) is the half-life of a puff (\( t_0 = 50D/U \) was selected by looking for the least sensitivity in the half-life prediction and typically meant excluding the first 5%-10% of the data). The results plotted in Fig. 4 are consistent with the relation \( \tau = \alpha (\text{Re}_c - \text{Re})^{-1} \), where \( \alpha \) is 2.4 \times 10^{-4} \) compared to 2.8 \times 10^{-4} obtained in Ref. [8] and there is a shift of 7% in \( \text{Re}_c \) up to 1870 in the numerical data. Also shown is the reinterpreted numerical data for the 5\( D \) pipe [11] and the recent half-life results from the long pipe experiments [12], which indicate that \( 1/\tau \) varies exponentially with \( \text{Re}_c \) rather than linearly. Although the data from Ref. [12] are for longer times, there is sufficient overlap to suggest that the data from Ref. [8] and our results are not consistent.
with being the earlier linear-looking part of this exponential. Rather, the results indicate qualitatively different behavior [18].

The exponential probability distribution $P(T)$ found here in Fig. 3 implies that puff relaminarization is a memoryless process—the probability that the puff will decay in a given interval of time is proportional to the length of the period but independent of previous events. This feature has been found previously in turbulent relaminarization experiments in pipe flow [8,12,19] as well as in plane Couette flow [20,21] and numerical calculations using models of this together with other linearly stable shear flows [22,23]. Faisst and Eckhardt [11] interpret this result as indicating that the transient turbulent state for $\text{Re} < \text{Re}_c$ represents a chaotic repellor in phase space. Our results indicate that this conclusion carries over to a localized turbulent puff in a long pipe. The building blocks for such a repellor are saddle points, and families of these in the form of traveling waves with discrete rotational symmetries are now known to exist down to $\text{Re} = 1251$ [24–26]. Tentative experimental evidence for their relevance to puffs has already been found [27], and corroborating numerical evidence is now emerging [17]. The entanglement of all of the stable and unstable manifolds associated with these saddles at some higher $\text{Re}$ presumably gives rise to sufficiently complicated phase dynamics to appear as a turbulent puff in real space. That this phase space structure is initially “leaky” ultimately allowing escape (relaminarization) is perhaps unsurprising, but what is less clear is how it suddenly becomes an attractor at $\text{Re}_c$. The power law scaling of the transient decay half-life $\tau \sim (\text{Re}_c - \text{Re})^{-1}$ strongly suggests a boundary crisis [28], while the precise value of the critical exponent hints at a simple dynamical systems explanation. One, of course, cannot rule out the possibility that the region never becomes an attractor with the exit probability becoming extremely small but staying finite as $\text{Re}$ increases [12] or, in fact, that there are a number of “leaks” which one by one seal up giving a half-life behavior which varies over a number of discrete time scales. Also, at some point, the effect of noise must surely become significant over long times. However, the fact that the numerical simulations and the experimental results [8] are quantitatively consistent despite being subject to different types of errors or disturbances indicates that noise is not important over time scales of $O(1000D/U)$ for the levels maintained here and in the experiments.

The simulations confirm that the puff characteristics are continuous as $\text{Re}$ crosses $\text{Re}_c$ and that a puff corresponds to a part of phase space disjoint from the laminar state (see Fig. 5 and inset). This observation naturally divides the usual question as to how to trigger sustained turbulence in pipe flow into two separate issues. First, what disturbance at a given $\text{Re}$ is needed to trigger turbulence initially—i.e., what initial conditions will cause the flow to leave the neighborhood of the laminar state to reach the puff region of phase space? Second, what $\text{Re}$ is needed so that, for a flow already in the turbulent region, the flow never leaves—i.e., the puff has become an attractor? The implications of this realization are that experimental curves in Refs. [5,7] showing a threshold curve on a disturbance amplitude-Re plot must, in fact, be two curves as shown in Fig. 6. Figure 5 shows how initially a threshold amplitude of disturbance is required to push the solution away from the laminar state and into the turbulent region. Once

![FIG. 4 (color online). The reciprocal of the puff half-life $\tau$ plotted against $\text{Re}$. Data plotted: WK—50D data (each data point is the result of 40–60 simulations); PM—experimental data from Ref. [8]; FE—reinterpreted 5D data [11]; H—experimental data from Ref. [12]. Inset: log-plot of $1/\tau$ vs $\text{Re}$.](image1)

![FIG. 5 (color online). Trace of perturbation energy versus additional pressure fraction required to maintain fixed mass flux, $1 + \beta = (\dot{a}_i, \dot{p})/d_i p_{\text{lam}}$ (the origin represents laminar flow), for the three cases of a sustained puff at $\text{Re} = 1900$ (solid line), a metastable puff at $\text{Re} = 1860$ with sudden relaminarization (dotted line), and the immediate decay of a perturbation (dashed line). The inset shows that the energy trace for the metastable puff essentially overlaps that of the sustained puff before it suddenly laminarizes.](image2)
FIG. 6 (color online). Sketch of the two (independent) thresholds associated with transition: one is amplitude-dependent (and highly form-dependent), indicating when a turbulent episode is triggered, and the other is a global Re-dependent threshold, indicating when the turbulence will be sustained.

there, the exit from the metastable state is sudden and unrelated to the entry as relaminarization is a memoryless process.

To summarize, numerical simulations described in this Letter have clarified the existence of two independent thresholds for sustained turbulence. Results probing the relaminarization threshold closely match a recent experimental investigation [8]. For time scales extending these experiments—\( t \approx 1000D/U \)—we confirm the presence of an exponential distribution for the probability of puff relaminarization and corroborate critical-type behavior in which the puff half-life diverges as \( (Re_c - Re)^{-1} \). Good quantitative agreement between the experimentally and theoretically estimated value of \( Re_c \) (less than 7% difference) is a rare triumph in this famous canonical problem.

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[14] Incompressibility was satisfied automatically by adopting a toroidal-poloidal potential formulation [15], further reformulated into five simple second order equations in \( r \). The numerical discretization was via a nonequispaced 9-point finite difference stencil in \( r \) and by Fourier modes in \( \theta \) and \( z \). At the pipe wall boundary conditions coupling the potentials were solved to numerical precision using an influence-matrix method, and axial symmetry properties imposed by the geometry on each Fourier mode were enforced implicitly in the finite difference weights.
[18] In Ref. [12], the flow is disturbed by a jet of injected fluid much as in earlier experiments [7] where a six-jet disturbance was used. This latter study found that results were sensitive to the exact flux fraction of the laminar flow injected, with a (large) value of 0.1 giving \( Re_c = 1710 \pm 10 \), whereas a (small) disturbance of 0.01 gave \( Re_c = 1830 \pm 10 \); [12] quotes injected flux rates of \( \approx 0.07 \).