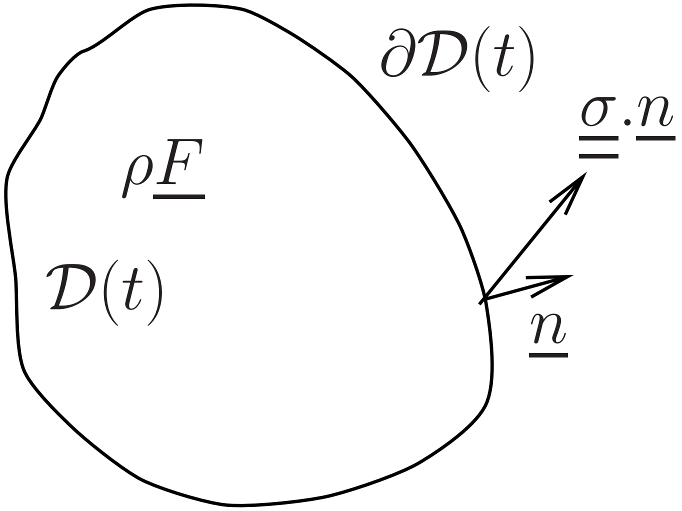
FUNDAMENTALS OF FLUID MECHANICS

MMI103

Lesson 3 (course and exercices)

Thursday 25th September 2025

MOMENTUM CONSERVATION LAW (reminder; Lesson 2)



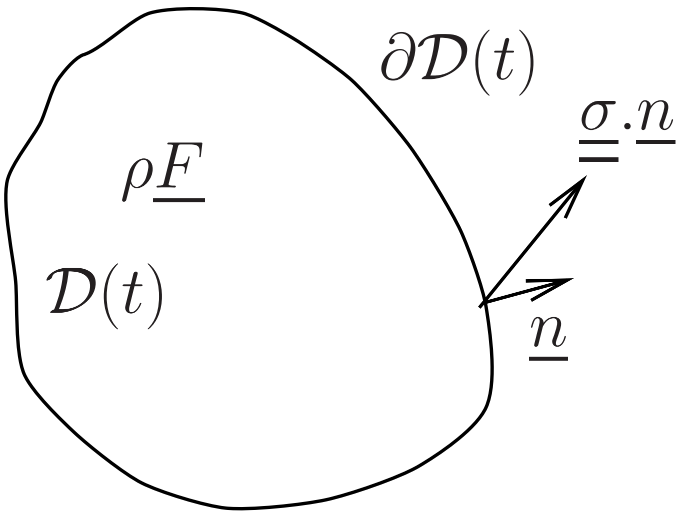
$$\frac{d}{dt}\int_{\mathcal{D}(t)}\rho\underline{u}d\Omega=\int_{\mathcal{D}(t)}\rho\underline{F}d\Omega+\int_{\partial\mathcal{D}(t)}\underline{\underline{\sigma}}.\underline{n}da$$

$$\frac{\delta}{\delta t}\int_{\mathcal{D}}\rho\underline{u}d\Omega=\int_{\mathcal{D}}\rho\underline{F}d\Omega+\int_{\partial\mathcal{D}}[\underline{\underline{\sigma}}.\underline{n}-\rho(\underline{u}.\underline{n})\underline{u}]da$$

$\underline{\underline{\sigma}}$ symmetric , $\rho\frac{d\underline{u}}{dt}=\rho\underline{F}+\mathrm{div}(\underline{\underline{\sigma}}),\quad \mathrm{div}(\underline{\underline{\sigma}})=\sigma_{ij,j}\underline{e}_i,\quad \sigma_{ij,j}=\frac{\partial\sigma_{ij}}{\partial x_j}$

$$\frac{d\underline{u}}{dt}=\frac{\partial\underline{u}}{\partial t}+(u_{i,j}u_j)\underline{e}_i,\quad \frac{d\underline{u}}{dt}=\frac{\partial\underline{u}}{\partial t}+\underline{grad}[\frac{u^2}{2}]+\underline{rot}(\underline{u})\wedge\underline{u}$$

FLOWING NON-VISCOUS FLUID: EULER EQUATIONS



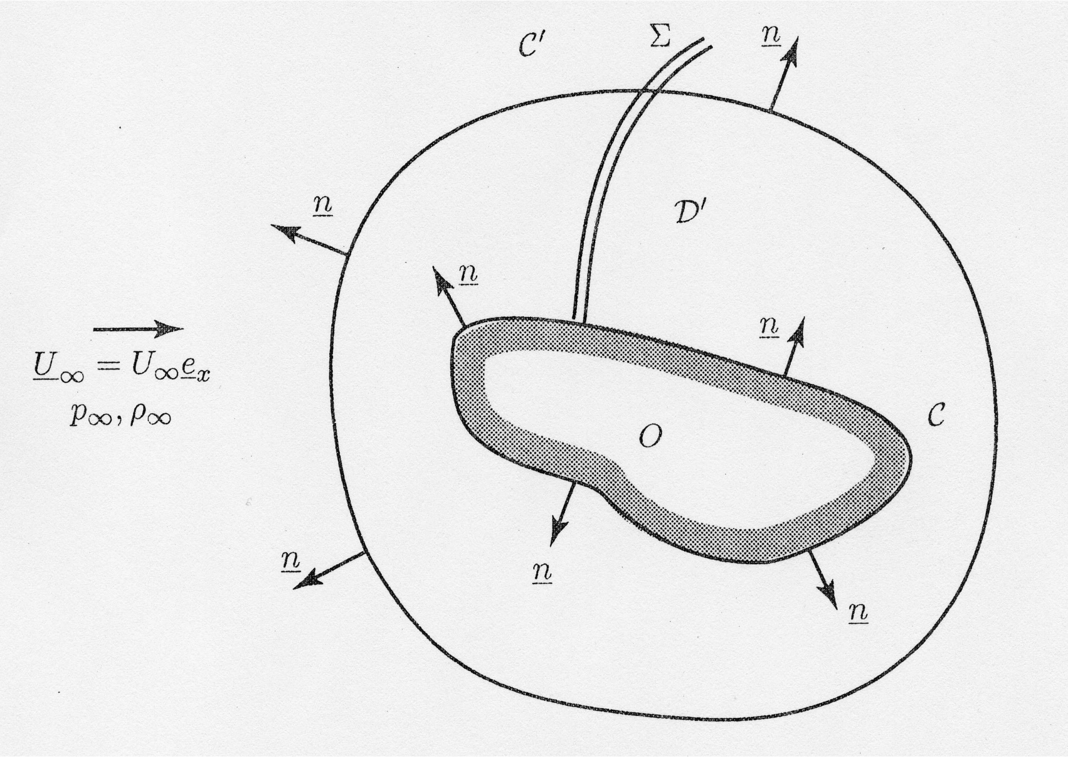
$$\frac{d}{dt}\int_{\mathcal{D}(t)} \rho\underline{u}\,d\Omega = \int_{\mathcal{D}(t)} \rho\underline{F}\,d\Omega - \int_{\partial\mathcal{D}(t)} p\underline{n}\,da$$

$$\frac{\delta}{\delta t}\int_{\mathcal{D}} \rho\underline{u}\,d\Omega = \int_{\mathcal{D}} \rho\underline{F}\,d\Omega - \int_{\partial\mathcal{D}} [p\underline{n} + \rho(\underline{u}.\underline{n})\underline{u}]\,da$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{1}}\text{(symmetric)}, \quad \frac{d\rho}{dt} + \rho\,\mathrm{div}(\underline{u}) = 0, \quad \rho\frac{d\underline{u}}{dt} = \rho\underline{F} - \underline{grad}[p],$$

Different cases: incompressible homogenous fluid: $\rho = cste$, compressible barotropic flow: $\rho = \rho(p)$, general $\rho = \rho(p, T)$

Force exerted by a steady flow of a non-viscous fluid on a motionless body

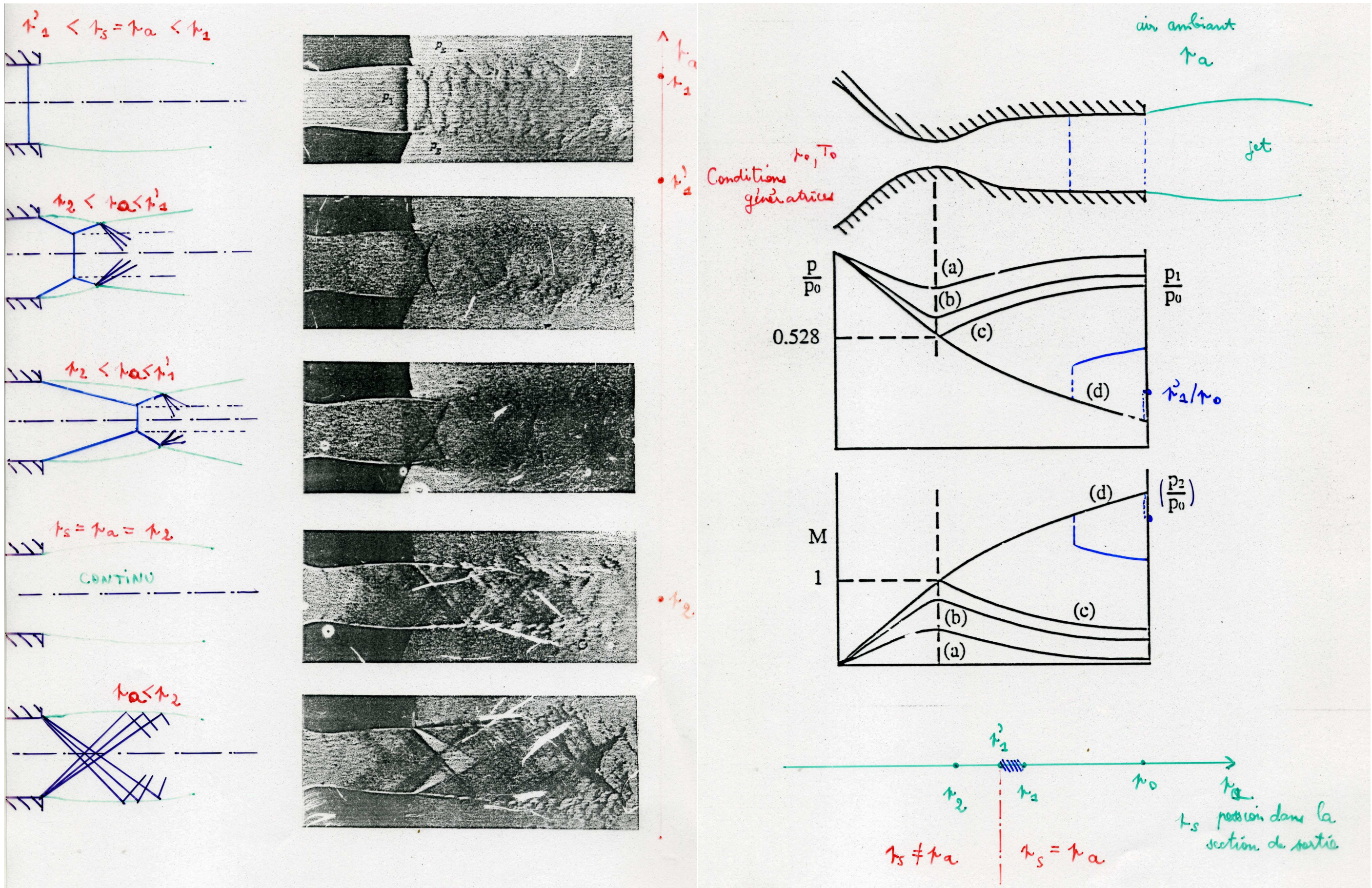


$$\int_{\mathcal{D}'} \{p\underline{n}' + \rho(\underline{u}.\underline{n}')\underline{U}\} da = \underline{0}$$

$$\underline{R} = -\int_{\mathcal{C}'} \{p\underline{n} + \rho(\underline{u}.\underline{n})\underline{u}\} da$$

Requires the far-field behaviour of $(\rho, p, \underline{u})$

Momentum conservation for the one-dimensional approximation of a flowing non-viscous fluid



FLOWING NON-VISCOUS FLUID: BERNOULLI THEOREMS

First case

- potential flow: $\underline{u} = \underline{grad}[\phi]$ (then $\underline{rot}(\underline{u}) = \underline{0}$)
- conservative body force: $\underline{F} = -\underline{grad}[\Phi]$ (for example for the gravity, $\Phi = -\underline{g}.\underline{x}$)
- non-viscous fluid

$$\frac{\partial \underline{u}}{\partial t} + \underline{grad}[\frac{u^2}{2}] + \underline{rot}(\underline{u}) \wedge \underline{u} = \underline{F} - [\frac{1}{\rho}]\underline{grad}[p]$$

which, under the assumptions, becomes

$$\underline{grad}[\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \Phi] + [\frac{1}{\rho}]\underline{grad}[p] = \underline{0}$$

(i) homogeneous fluid (ρ uniform). Then the flow is incompressible so that $\Delta\phi = 0$ and

$$\underline{grad}[\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \Phi + p/\rho] = \underline{0}, \quad \frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \Phi + p/\rho = C(t) \text{ Bernoulli theorem}$$

(ii) compressible (ρ non-uniform) and barotropic flow

$$\rho = \rho(p), \quad \frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \Phi + \int dp/\rho = C(t) \text{ Bernoulli theorem}$$

FLOWING NON-VISCOUS FLUID: BERNOULLI THEOREMS

Second case

- steady flow
- conservative body force: $\underline{F} = -\underline{grad}[\Phi]$
- barotropic flow, i.e. $\rho = \rho(p)$

Then

- trajectories=streamlines. On such a curve, the elementary displacement $d\underline{x}$ is parallel with $\underline{u}$ and

$$\underline{grad}[\frac{u^2}{2} + \Phi].d\underline{x} + [\frac{1}{\rho}]\underline{grad}[p].d\underline{x} = 0$$

(i) homogeneous fluid (ρ uniform). Then the flow is incompressible and

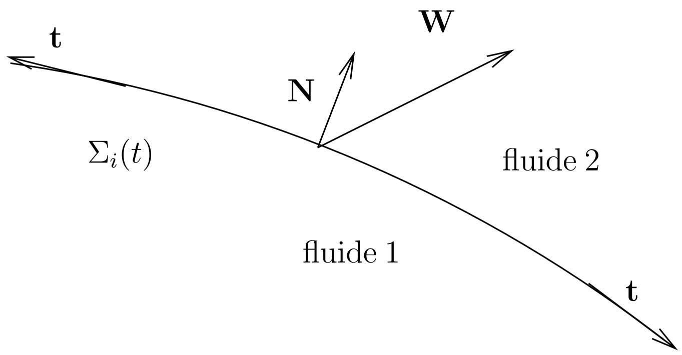
$$\underline{grad}[\frac{u^2}{2} + \Phi + p/\rho] = \underline{0}, \quad \frac{u^2}{2} + \Phi + p/\rho = C_{streamline}, \quad \text{Bernoulli theorem}$$

(ii) compressible (ρ non-uniform) and barotropic flow

$$\underline{grad}[\frac{u^2}{2} + \Phi + \int dp/\rho] = \underline{0}, \quad \frac{u^2}{2} + \Phi + \int dp/\rho = C_{streamline}, \quad \text{Bernoulli theorem}$$

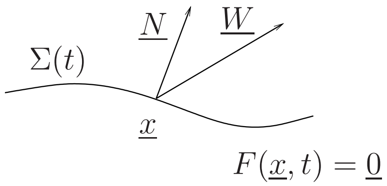
Two illustrating applications: the Venturi tube and the Torricelli formula

Fluid-fluid interface (liquid-gas or liquid-liquid interface)

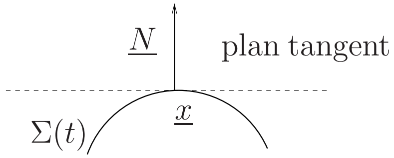


- immiscible fluids
- $\Sigma_i(t)$ has its own velocity $\underline{W}$
- Henceforth, $[\![g]\!] = g(2) - g(1)$ on the interface
- $\underline{N}$ unit normal adequately directed depending upon the interface local concavity and directed from fluid 1 to fluid 2

(a)



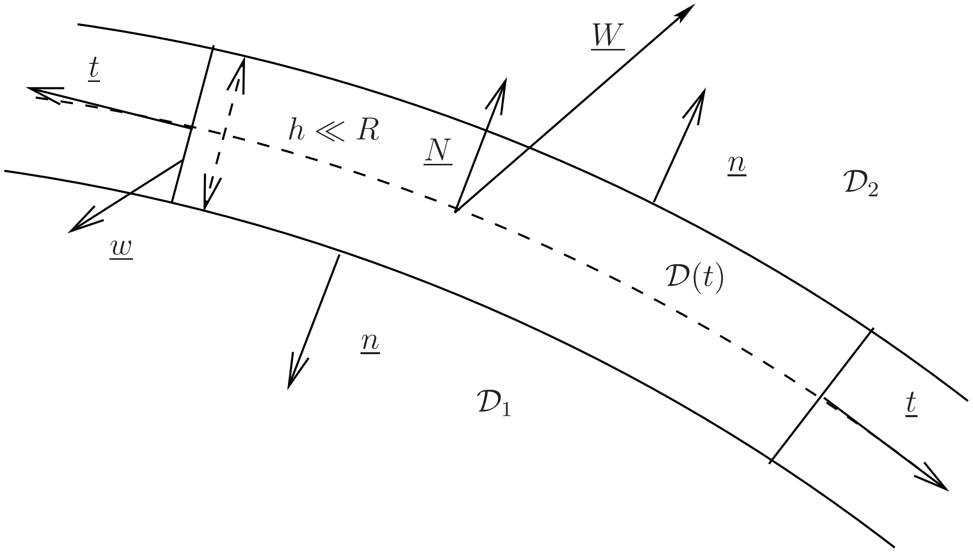
(b)



- Local mean curvature H

$$\mathrm{div}_S(\underline{N}) = 2H, \qquad 2H = 1/R_1 + 1/R_2$$

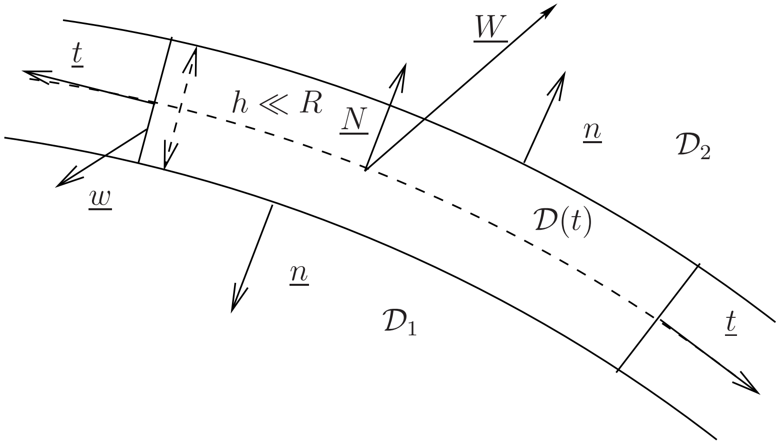
Mass conservation



- Vanishing thickness $h \ll R$
- The interface density ρ_{int} negligible
- Immiscible fluids (gas-liquid, liquid-liquid)
- Mass conservation

$$\underline{u}_1.\underline{N} = \underline{u}_2.\underline{N} = \underline{W}.\underline{N} \quad \text{on } \Sigma_i$$

Momentum conservation

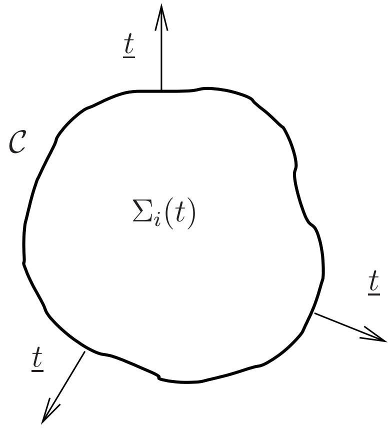


$$\frac{\partial}{\partial t_{\underline{w}}} \int_{\mathcal{D}(t)} \rho_{int} \underline{u} d\Omega - \int_{\mathcal{D}(t)} \rho_{int} \underline{F} d\Omega = \int_{\partial\mathcal{D}_1 \cup \partial\mathcal{D}_2} \underline{\underline{\sigma}}.\underline{n} dS + \int_{\partial\mathcal{D}_l} \underline{\underline{\sigma}}.\underline{n} dS - \int_{\partial\mathcal{D}(t)} [\rho(\underline{u} - \underline{w}).\underline{n}] \underline{u} dS$$

- The interface density is negligible ($\rho_{int} \sim 0$) so that

$$\int_{\Sigma_i} [\![\underline{\underline{\sigma}}]\!].\underline{N} dS + \lim_{h \to 0} \int_{\partial\mathcal{D}_l} \underline{\underline{\sigma}}.\underline{n} dS = \underline{0}$$

Surface tension



$$\lim_{h\to 0}\int_{\partial\mathcal{D}_l}\underline{\underline{\sigma}}.\underline{n}dS = \oint_{\mathcal{C}}\gamma\underline{t}dl$$

- $\gamma \geq 0$ is the interface surface tension (given in N/m)
- Auxiliary mathematical identity

$$\oint_{\mathcal{C}}\gamma\underline{t}dl = \int_{\Sigma_i}\{\underline{grad}_S[\gamma] - \gamma\mathrm{div}_S(\underline{N})\underline{N}\}dS$$

Global and local results

$$\int_{\Sigma_i}\{[\![\underline{\underline{\sigma}}]\!].\underline{N}+\underline{grad}_S[\gamma]-\gamma\mathrm{div}_S(\underline{N})\underline{N}\}dS=\underline{0}$$

$$[\![\underline{\underline{\sigma}}]\!].\underline{N}+\underline{grad}_S[\gamma]-2\gamma H\underline{N}=\underline{0}\quad \text{sur } \Sigma_i$$

In summary, at each point on the interface

$$\underline{N}.[\![\underline{\underline{\sigma}}]\!].\underline{N}=2\gamma H\qquad \text{sur } \Sigma_i$$

$$([\![\underline{\underline{\sigma}}]\!].\underline{N}+\underline{grad}_S[\gamma])\wedge\underline{N}=\underline{0}\quad \text{on } \Sigma_i$$

$$\underline{u}_1.\underline{N}=\underline{u}_2.\underline{N}=\underline{W}.\mathbf{N}\quad \text{on } \Sigma_i$$

- With $H\geq 0, \underline{N}$ directed from fluid 1 to fluid 2 and $[\![g]\!]=g(2)-g(1)$
- Comments

Experimental values of the surface tension versus the temperature

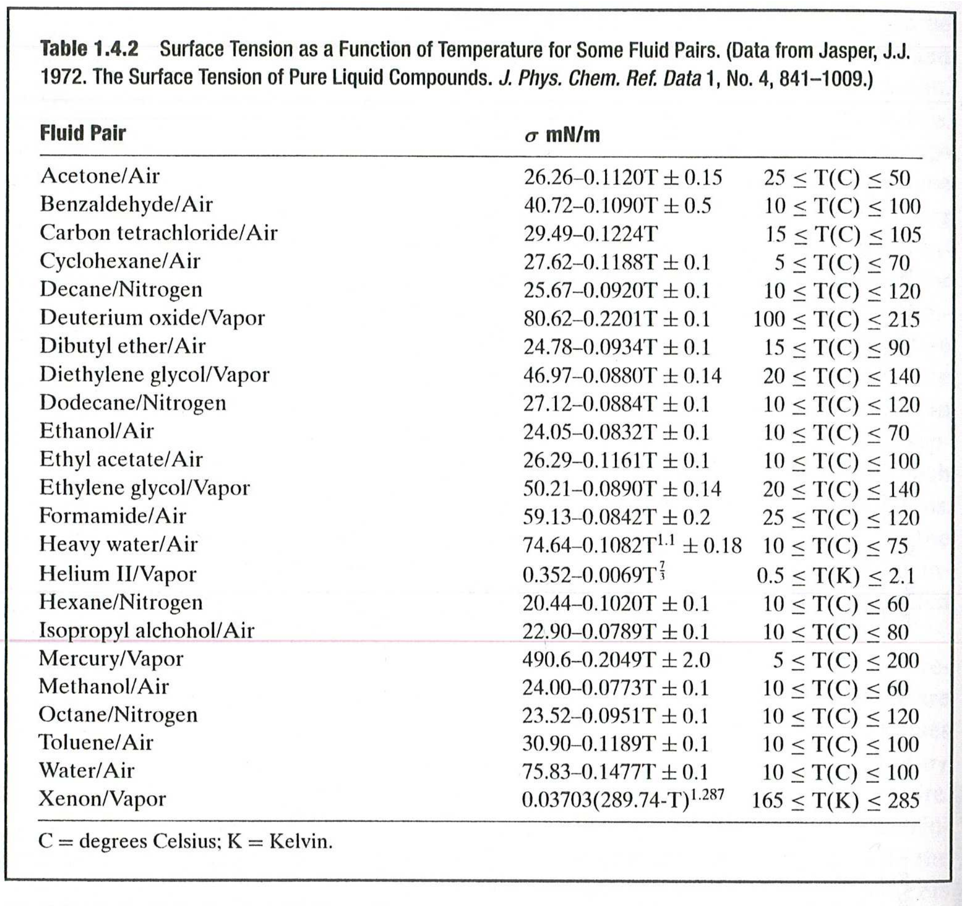
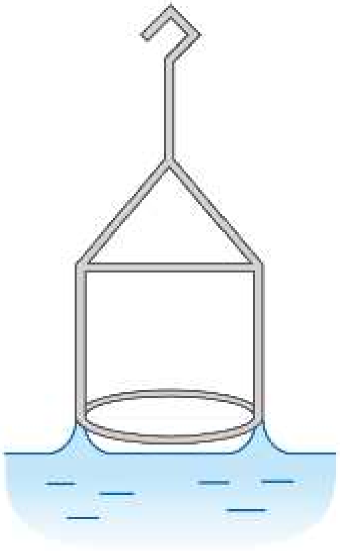


Table 1.4.2 Surface Tension as a Function of Temperature for Some Fluid Pairs. (Data from Jasper, J.J. 1972. The Surface Tension of Pure Liquid Compounds. *J. Phys. Chem. Ref. Data* 1, No. 4, 841–1009.)

Fluid Pair	**σ mN/m**	
Acetone/Air	$26.26–0.1120T \pm 0.15$	$25 \le T(C) \le 50$
Benzaldehyde/Air	$40.72–0.1090T \pm 0.5$	$10 \le T(C) \le 100$
Carbon tetrachloride/Air	$29.49–0.1224T$	$15 \le T(C) \le 105$
Cyclohexane/Air	$27.62–0.1188T \pm 0.1$	$5 \le T(C) \le 70$
Decane/Nitrogen	$25.67–0.0920T \pm 0.1$	$10 \le T(C) \le 120$
Deuterium oxide/Vapor	$80.62–0.2201T \pm 0.1$	$100 \le T(C) \le 215$
Dibutyl ether/Air	$24.78–0.0934T \pm 0.1$	$15 \le T(C) \le 90$
Diethylene glycol/Vapor	$46.97–0.0880T \pm 0.14$	$20 \le T(C) \le 140$
Dodecane/Nitrogen	$27.12–0.0884T \pm 0.1$	$10 \le T(C) \le 120$
Ethanol/Air	$24.05–0.0832T \pm 0.1$	$10 \le T(C) \le 70$
Ethyl acetate/Air	$26.29–0.1161T \pm 0.1$	$10 \le T(C) \le 100$
Ethylene glycol/Vapor	$50.21–0.0890T \pm 0.14$	$20 \le T(C) \le 140$
Formamide/Air	$59.13–0.0842T \pm 0.2$	$25 \le T(C) \le 120$
Heavy water/Air	$74.64–0.1082T^{1.1} \pm 0.18$	$10 \le T(C) \le 75$
Helium II/Vapor	$0.352–0.0069T^{\frac{7}{3}}$	$0.5 \le T(K) \le 2.1$
Hexane/Nitrogen	$20.44–0.1020T \pm 0.1$	$10 \le T(C) \le 60$
Isopropyl alchohol/Air	$22.90–0.0789T \pm 0.1$	$10 \le T(C) \le 80$
Mercury/Vapor	$490.6–0.2049T \pm 2.0$	$5 \le T(C) \le 200$
Methanol/Air	$24.00–0.0773T \pm 0.1$	$10 \le T(C) \le 60$
Octane/Nitrogen	$23.52–0.0951T \pm 0.1$	$10 \le T(C) \le 120$
Toluene/Air	$30.90–0.1189T \pm 0.1$	$10 \le T(C) \le 100$
Water/Air	$75.83–0.1477T \pm 0.1$	$10 \le T(C) \le 100$
Xenon/Vapor	$0.03703(289.74\text{-}T)^{1.287}$	$165 \le T(K) \le 285$

C = degrees Celsius; K = Kelvin.

How to experimentally measure γ?
Examples: Jurin's law, Wilhelmy's ring



$$\gamma = \frac{F}{4\pi R} C$$

Where:

γ : Surface tension

F : Force acting on the ring

R : Center diameter of the ring

C : Correction factor