FUNDAMENTALS OF FLUID MECHANICS

MMI103

Lesson 2 (course and exercices)

Thursday 18th september 2025

MASS CONSERVATION (reminder; Lesson 1)

- global equivalent formulations

$$\frac{d}{dt}\int_{\mathcal{D}(t)} \rho d\Omega = 0, \qquad \frac{\delta}{\delta t}\int_{\mathcal{D}} \rho(\underline{x},t) d\Omega = -\int_{\partial\mathcal{D}} \rho \underline{u}.\underline{n} da$$

- local equivalent formulations (where the flow is smooth)

$$\frac{d\rho}{dt} + \rho \mathrm{div}(\underline{u}) = 0, \qquad \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\underline{u}) = 0, \qquad \frac{dg}{dt} = \frac{\partial g}{\partial t} + \underline{grad}[g].\underline{u}$$

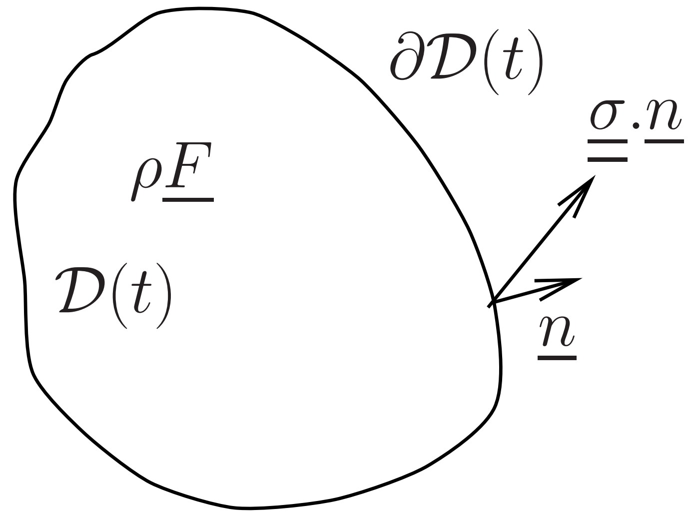
- Incompressible flow?

$$\frac{d\rho}{dt} = 0, \qquad \mathrm{div}(\underline{u}) = 0$$

- new useful identity for f a specific quantity and a smooth flow

$$F = \rho f, \qquad \frac{d}{dt}\int_{\mathcal{D}(t)} \rho f d\Omega = \int_{\mathcal{D}(t)} \rho [\frac{df}{dt}] d\Omega$$

GLOBAL MOMENTUM CONSERVATION LAW



$$\frac{d}{dt}\int_{\mathcal{D}(t)} \rho\underline{u}d\Omega = \int_{\mathcal{D}(t)} \rho\underline{F}d\Omega + \int_{\partial\mathcal{D}(t)} \underline{\underline{\sigma}}.\underline{n}da$$

- Valid also if there is an inert surface $\Sigma(t)$ of discontinuities of $\rho\underline{u}$ inside the domain $\mathcal{D}(t)$
- $\underline{F}$: specific (i.e. per unit mass) body force
- $\underline{F}$ must include the inertial body forces if the frame of reference is not Galilean (see later)
- Examples: $\underline{F} = \underline{g}$ for gravity, $\underline{F} = -\underline{grad}[\Phi]$ when conservative and deriving from the potential Φ
- $\underline{\underline{\sigma}}$ is the so-called Cauchy stress tensor. It is expressed in terms of $(\rho, p, \underline{u})$ by the fluid rheological law.

Its Cartesian components are σ_{ij} and $\underline{\underline{\sigma}}.\underline{n} = (\sigma_{ij}n_j)\underline{e}_i$

OBTAINED EQUIVALENT FORMS TAKEN BY A GLOBAL CONSERVATION LAW

General case of a moving domain

$$\frac{\delta}{\delta t_{\underline{V}}} \int_{\mathcal{D}(t)} F(\underline{x}, t) d\Omega = \int_{\mathcal{D}} P_F(\underline{x}, t) d\Omega - \int_{\partial \mathcal{D}} [F(\underline{u} - \underline{V}) - \underline{A}].\underline{n} da$$

Steady domain

$$\frac{\delta}{\delta t} \int_{\mathcal{D}} F(\underline{x}, t) d\Omega = \int_{\mathcal{D}} P_F(\underline{x}, t) d\Omega - \int_{\partial \mathcal{D}} (F\underline{u} - \underline{A}).\underline{n} da$$

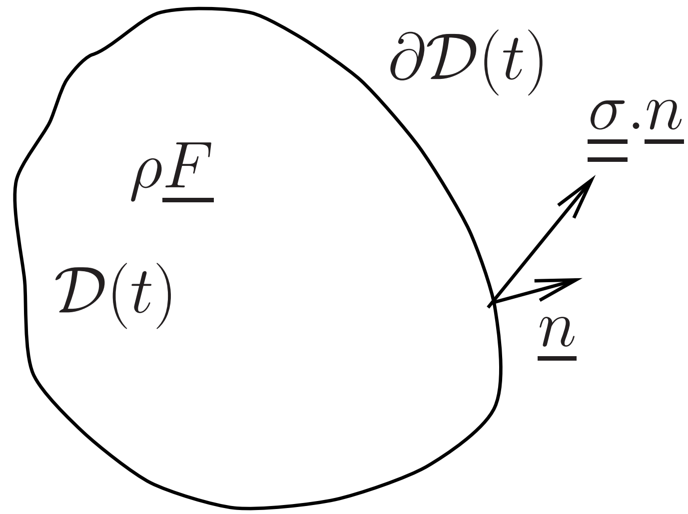
Material domain

$$\frac{d}{dt} \int_{\mathcal{D}(t)} F(\underline{x}, t) d\Omega = \int_{\mathcal{D}(t)} P_F(\underline{x}, t) d\Omega + \int_{\partial \mathcal{D}(t)} \underline{A}.\underline{n} da$$

Hold even in presence of a surface of discontinuities $\Sigma(t)$

INERT for the quantity $\mathcal{F}$

OTHER GLOBAL AND LOCAL MOMENTUM CONSERVATION FORMULATIONS



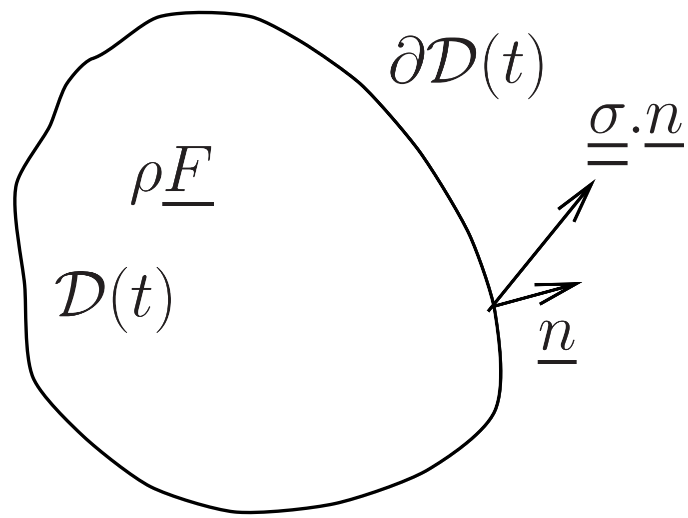
$$\frac{\delta}{\delta t}\int_{\mathcal{D}}\rho\underline{u}d\Omega = \int_{\mathcal{D}}\rho\underline{F}d\Omega + \int_{\partial\mathcal{D}}[\underline{\underline{\sigma}}.\underline{n} - \rho(\underline{u}.\underline{n})\underline{u}]da$$

$$\rho\frac{d\underline{u}}{dt} = \rho\underline{F} + \mathrm{div}(\underline{\underline{\sigma}}), \qquad \mathrm{div}(\underline{\underline{\sigma}}) = \sigma_{ij,j}\underline{e}_i$$

Recall that

$$\frac{d\underline{u}}{dt} = \frac{\partial\underline{u}}{\partial t} + \underline{grad}[u^2/2] + \underline{rot}(\underline{u}) \wedge \underline{u}, \qquad \sigma_{ij,j} = \frac{\partial\sigma_{ij}}{\partial x_j}$$

ANGULAR MOMENTUM CONSERVATION LAW?



$$\frac{d}{dt}\int_{\mathcal{D}(t)} \rho \underline{u} d\Omega = \int_{\mathcal{D}(t)} \rho \underline{F} d\Omega + \int_{\partial\mathcal{D}(t)} \underline{\underline{\sigma}}.\underline{n} da$$

$$\frac{d}{dt}\int_{\mathcal{D}(t)} \underline{x} \wedge (\rho \underline{u}) d\Omega = \int_{\mathcal{D}(t)} \underline{x} \wedge (\rho \underline{F}) d\Omega + \int_{\partial\mathcal{D}(t)} \underline{x} \wedge (\underline{\underline{\sigma}}.\underline{n}) da$$

For a symmetric stress tensor $\underline{\underline{\sigma}}$ (which means $\sigma_{ij} = \sigma_{ji}$) the second identity is induced by the momentum conservation law!

IN SUMMARY

- 5 unknown Eulerian fields: $\rho, p, \underline{u}$
- $\underline{\underline{\sigma}}$ symmetric. True for most encountered fluids
- $\underline{\underline{\sigma}}$ provided versus $(\rho, p, \underline{u})$ from the fluid nature. $\underline{F}$ is also supplied.
- T and other thermodynamic variables deduced from (ρ, p) using the fluid equation of state
- Mass and momentum law conservations provide 4 equations

$$\frac{d\rho}{dt} + \rho \mathrm{div}(\underline{u}) = 0, \qquad \rho \frac{d\underline{u}}{dt} = \rho \underline{F} + \mathrm{div}(\underline{\underline{\sigma}})$$

- In general one additional equation is still lacking at that stage. Provided by the energy conservation (due to thermodynamics, see later)
- In some cases mass and momentum conservation are sufficient. For instance: fluid at rest, homogeneous flowing fluid

DEFINITION OF A FLUID

- A fluid: medium which when at rest experiences no tangential stress
- For a fluid at rest

$$\underline{\underline{\sigma}}.\underline{n} = -p\underline{n}$$

with p the so-called pressure

NON-VISCOUS FLUID

- A fluid is non-viscous if even when flowing

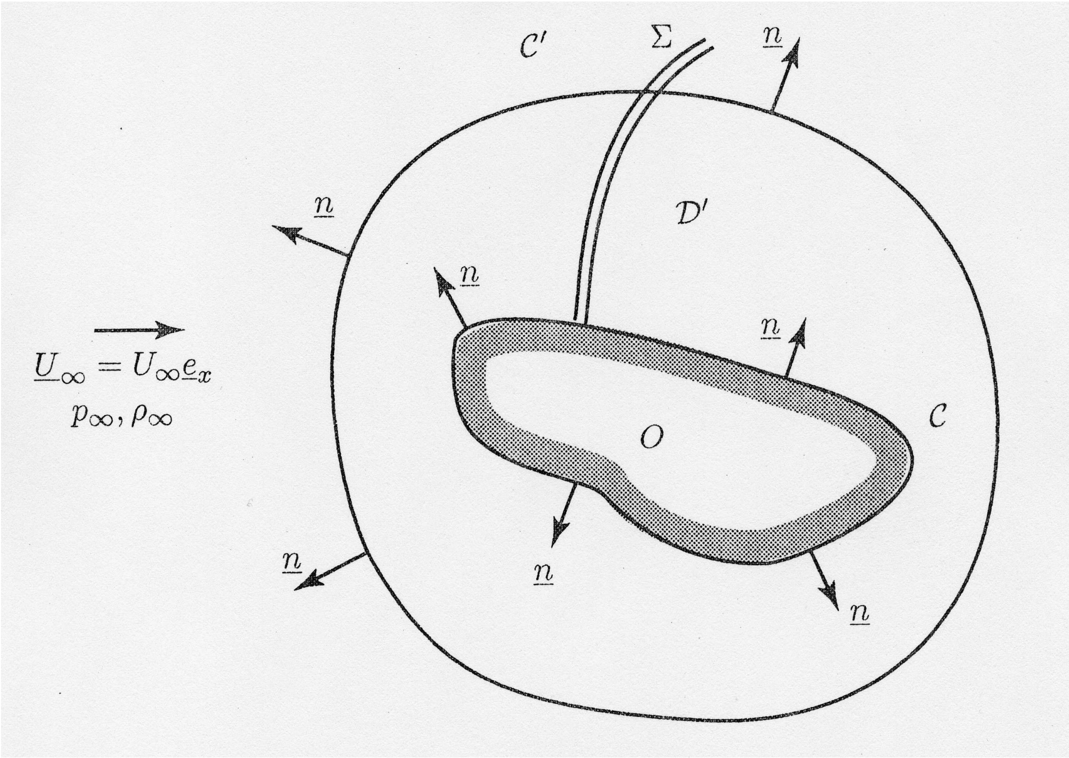
$$\underline{\underline{\sigma}}.\underline{n} = -p\underline{n}$$

- Example: a gas

ILLUSTRATING EXERCICES

- fluid at rest (stratified or homogeneous fluid)
- flowing non-viscous fluid (compressible or homogeneous)

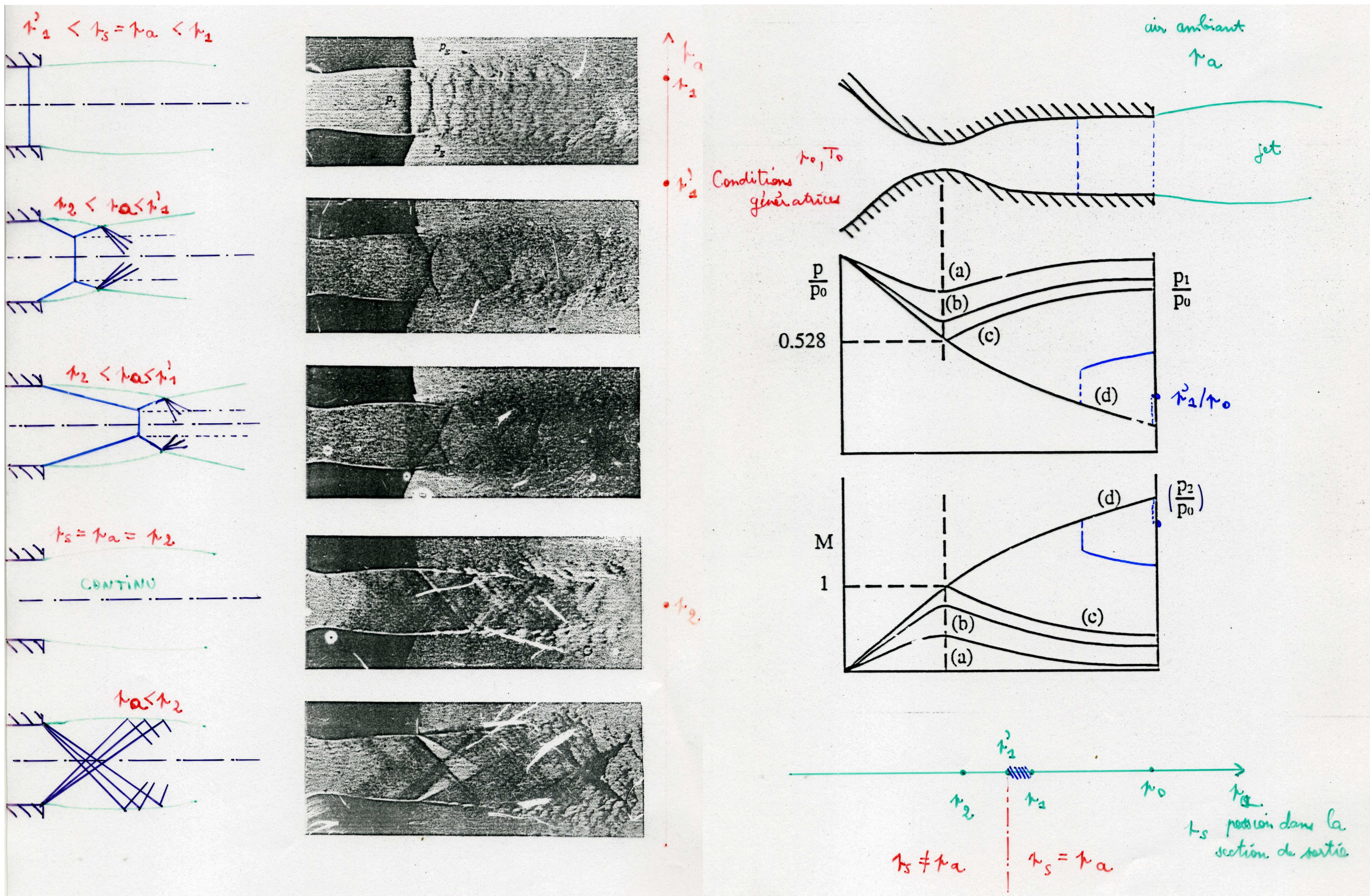
Force exerted by a steady flow of a non-viscous fluid on a motionless body



$$\int_{\mathcal{D}'} \{p\underline{n}' + \rho(\underline{u}.\underline{n}')\underline{U}\} da = \underline{0}$$

$$\underline{R} = -\int_{\mathcal{C}'} \{p\underline{n} + \rho(\underline{u}.\underline{n})\underline{u}\} da$$

Requires the far-field behaviour of $(\rho, p, \underline{u})$



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