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Fundamental MHD creeping flow bounded by a motionless plane solid wall

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ABSTRACT
This work determines the three-dimensional (3D) fundamental MHD creeping flow and associated electric potential produced by a concentrated source point, with given unit strength $\mathbf{e}$, located in a conducting Newtonian liquid bounded by a plane solid and motionless wall and subject to a given uniform magnetic field *normal* to the wall. The wall is no-slip but may be either perfectly conducting or insulating. By linearity, the analysis is confined to the cases of $\mathbf{e}$ either normal or parallel to the wall. Such different wall natures and force orientations result in different flows and electric potential functions which are obtained using direct and inverse two-dimensional Fourier transforms. As a result, it has been possible to analytically express in closed-form each resulting fundamental flow and potential.

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MagnetoHydrodynamics; stokes flow; fundamental solution; wall; fourier transform

1. Introduction

It is of basic importance for some applications to calculate the flow produced by a solid body moving in a quiescent conducting Newtonian liquid subject to a prescribed ambient uniform and steady magnetic field $\mathbf{B}_0$. In general, such a problem is tremendously involved because one has to gain four different fields in the conducting liquid: the liquid pressure field p, the liquid velocity field $\mathbf{u}$ and also the induced electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The flow $(\mathbf{u}, p)$ is driven by the Lorentz body force $\mathbf{f}_L = \mathbf{j} \wedge \mathbf{B}$ with $\mathbf{j}$, the current density given in practice by the Ohm's law $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B})$ where $\sigma > 0$ designates the liquid uniform conductivity. One has then to solve coupled unsteady Maxwell and non-linear incompressible Navier–Stokes equations (Branover & Tsinober, 1970; Moreau, 1990, Tsinober, 1970) to get $(\mathbf{B}, \mathbf{E}, \mathbf{u}, p)$.

For a body with typical length scale a and a MHD flow $(\mathbf{u}, p)$ with velocity scale $V > 0$, one introduces the magnetic Reynolds number $\mathrm{Re}_m = \mu_m \sigma V a$ with $\mu_m > 0$ the fluid uniform magnetic permeability. Assuming that $\mathrm{Re}_m \ll 1$ (a very good assumption in practice) and that the liquid domain boundary (made of the body surface and eventually solid boundaries) has the same permeability as the fluid then yields $\mathbf{B} = \mathbf{B}_0$ in the entire liquid. In other words, we shall

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consider that the magnetic field $\mathbf{B}$ is *steady*, uniform and prescribed in the liquid. Consequently, $\nabla \wedge \mathbf{E} = \mathbf{0}$ and therefore $\mathbf{E} = -\nabla\phi$ with ϕ one unknown electric potential satisying, from the charge conservation $\nabla.\mathbf{j} = 0$, the equation $\Delta\phi = \nabla.(\mathbf{u} \wedge \mathbf{B})$ which in general couples ϕ and $\mathbf{u}$. Accordingly, one ends up with three unknown fields $(\mathbf{u}, p, \phi)$. Those fields deeply depend upon the flow Reynolds number $\mathrm{Re} = \rho_l Va/\mu$ and the Hartmann number $M = a/d$ where ρ_l and μ denotes the liquid uniform density and viscosity, whereas $d = (\sqrt{\mu/\sigma})/|\mathbf{B}|$ is the so-called Hartmann layer thickness (Hartmann, 1937). Unfortunately, determining $p, \mathbf{u}$ and ϕ remains a very challenging task still because the Navier–Stokes equations are non-linear ones!

For some applications (small particles and/or viscous liquid), it turns out that $\mathrm{Re} \ll 1$. Since usually $\mathrm{Re}_m \ll \mathrm{Re}$, the MHD flow $(\mathbf{u}, p)$ is then governed by the quasi-steady Stokes equations driven by the Lorentz body force $\mathbf{f}_L = \sigma(\mathbf{u} \wedge \mathbf{B} - \nabla\phi) \wedge \mathbf{B}$ in which $\mathbf{B}$ is *uniform* in the liquid. Since the Stokes equations are linear, one gets, within this Low-Reynolds-Number framework, a more tractable MHD problem for $(\mathbf{u}, p, \phi)$. In the absence of symmetries (for the body shape or motion), this problem is fully three-dimensional and this explains why, to the author's very best knowledge, no solution has been yet given for the MHD flow $(\mathbf{u}, p)$ and electrostatic potential ϕ about a solid body with arbitrary shape and rigid-body motion. The only three-dimensional solution is the nice one derived in Priede (2013) for the fundamental MHD Stokes flow and electrostatic potential produced by a concentrated force. When the Stokes flow is *without swirl* and axisymmetric about an axis parallel with $\mathbf{B}$, it fortunately turns out that $\mathbf{E} = \mathbf{0}$ (Chester, 1957; Gotoh, 1960a, 1960b)! In such pleasant circumstances, one has only to determine the axisymmetric flow $(\mathbf{u}, p)$. A very simple example is the one of a sphere with radius a, translating parallel with $\mathbf{B}$. It has been addressed for small Hartmann number $M = a/d$ in Chester (1957), for large M in Chester (1961) and recently for arbitrary M in Sellier and Aydin (in press), (2016a). Actually, Sellier and Aydin (in press), (2016a) develop a new boundary approach of the problem involving two basic fundamental axisymmetric MHD Stokes flows *without swirl* produced by a ring distribution of axial or radial forces and obtained earlier in Sellier and Aydin (2016b). The determination of those two key fundamental axisymmetric MHD Stokes flows appeals to the fundamental three-dimensional solutions derived in Priede (2013).

Since boundaries are also encountered in applications, it is worth examining to which extent a general (creeping or not) MHD flow may be affected when bounded. This issue has been investigated in Tsinober (1973a) for different low Reynolds Number steady *axisymmetric* MHD flow bounded by a plane solid wall Σ normal to a uniform magnetic field $\mathbf{B}$ prevailing in the entire liquid domain. One should note that Tsinober (1973a) handles different boundary conditions at the motionless plane wall both for $\mathbf{u}$ (not necessarily a no-slip condition) and when needed (case of a swirling flow) for ϕ. Moreover, it considers two types of axisymmetric flows: flows without swirl (for which $\mathbf{E} = \mathbf{0}$, as mentioned above) and also pure swirling Stokes flow (with velocity having only a swirl component) for which $\mathbf{E} = -\nabla\phi \neq \mathbf{0}$ with additional boundary conditions prescribed for ϕ on Σ.

In a second paper, Tsinober (1973b) the Green function for a Stokes axisymmetric MHD flow without swirl produced near a plane motionless wall by a point force *oriented normal* to the wall has been investigated. Unfortunately, formulae for the associated velocity components and pressure are not given in Tsinober (1973b). In the present work, we look at the general three-dimensional MHD Stokes flow $(\mathbf{u}, p, \phi)$ produced by a point force with *arbitrary* unit strength $\mathbf{e}$ near a plane motionless and no-slip wall Σ which may be either perfectly conducting or insulating. The paper is organised as follows. The addressed fundamental and resulting auxiliary MHD problems due to a concentrated point force with unit strength $\mathbf{e}$ and located in a conducting liquid bounded by a motionless plane solid (either perfectly conducting or insulating) wall are given Section 2. The solution is then derived, using direct and inverse two-dimensional Fourier transforms, in Sections 3 or 4 for $\mathbf{e}$ normal or parallel to the wall, respectively. Finally, some concluding remarks close the paper in Section 5.

2. Addressed fundamental and auxiliary MHD problems for a Stokes flow in a bounded domain

This section introduces the considered fundamental three-dimensional MHD problem in a conducting liquid bounded by a plane wall and shows how to reduce the task to the treatment of another auxiliary MHD problem.

2.1. Fundamental MHD problem for a Stokes flow in a bounded domain

As sketched in Figure 1, we put a concentrated force with arbitrary unit strength $\mathbf{e}$ at point $\mathbf{x}_0$ in a conducting Newtonian domain with uniform density ρ_l and viscosity μ occupying the $z > 0$ domain $\mathcal{D}$ bounded by the solid and motionless $z = 0$ plane wall Σ.

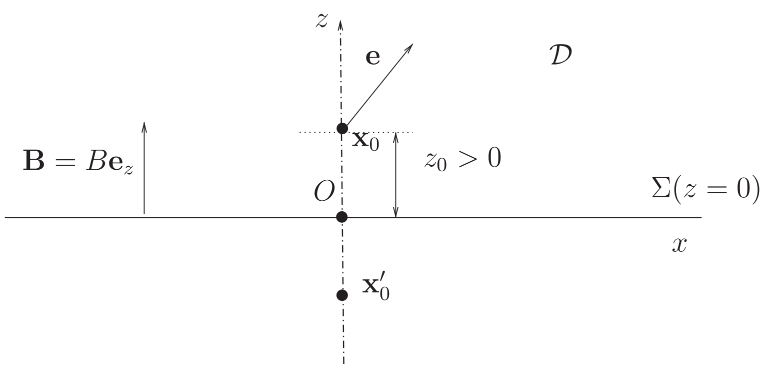

Figure 1. A concentrated force with unit strength $\mathbf{e}$ located at point $\mathbf{x}_0$ in the $z > 0$ liquid domain $\mathcal{D}$ bounded by the $z = 0$ plane motionless wall Σ.
Note: The point $\mathbf{x}'_0$ is the symmetric of $\mathbf{x}_0$ with respect to the wall.

The liquid has uniform conductivity $\sigma > 0$, uniform magnetic permeability $\mu_m > 0$ and is subject to a uniform (steady) magnetic field $\mathbf{B} = B\mathbf{e}_z$ *normal* to the wall Σ. For convenience, we introduce Cartesian coordinates (O, x, y, z) with associated unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ so that for point M having Cartesian coordinates (x, y, z), one has $\mathbf{x} = \mathbf{OM} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. The point force located at $\mathbf{x}_0$ produces a *steady* three-dimensional MHD flow with velocity field $\mathbf{u}$ (with respect to the motionless wall Σ) having typical magnitude $V > 0$, pressure field p and electrostatic potential field ϕ to be determined. Setting $h = z_0 = \mathbf{x}_0.\mathbf{e}_z > 0$ (see Figure 1), we assume vanishing magnetic Reynolds number $\mathrm{Re}_m = \mu_m \sigma V h$ and Reynolds number $\mathrm{Re} = \rho_l V h/\mu$. Consequently, the magnetic field is $\mathbf{B} = B\mathbf{e}_z$ in the liquid while the Stokes flow $(\mathbf{u}, p)$ is driven by the point force located at $\mathbf{x}_0$ and, in the entire liquid domain $\mathcal{D}$, by the Lorentz body force $\mathbf{f}_L = \sigma \mathbf{j} \wedge \mathbf{B}$ with $\mathbf{j}$ the current density. Adopting the Ohm's law $\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u} \wedge \mathbf{B})$ and requiring the charge conservation $\nabla.\mathbf{j} = 0$ in the liquid, the MHD flow $(\mathbf{u}, p, \phi)$ then obeys in the $z > 0$ liquid domain $\mathcal{D}$ the coupled equations

$$\mu\nabla^2\mathbf{u} = \nabla p + \sigma B\nabla\phi \wedge \mathbf{e}_z - \sigma B^2(\mathbf{u} \wedge \mathbf{e}_z) \wedge \mathbf{e}_z - \delta(\mathbf{x} - \mathbf{x}_0)\mathbf{e} \text{ for } \mathbf{x} \neq \mathbf{x}_0 \text{ in } \mathcal{D}, \tag{1}$$

$$\nabla.\mathbf{u} = 0 \text{ and } \Delta\phi = B\nabla.(\mathbf{u} \wedge \mathbf{e}_z) \text{ for } \mathbf{x} \neq \mathbf{x}_0 \text{ in } \mathcal{D} \tag{2}$$

with Δ and δ the three-dimensional Laplacian operator and the Dirac delta pseudo-function, respectively. Of course, (1)–(2) must be supplemented with boundary conditions far from the source $\mathbf{x}_0$ and on the wall. Since the wall is motionless and no-slipping, we required $\mathbf{u} = \mathbf{0}$ there. Hence, on the wall with unit normal $\mathbf{n} = \mathbf{e}_z$ one has $\mathbf{j} = -\sigma\nabla\phi$. Henceforth, attention is restricted to a perfectly conducting wall where $\mathbf{j} \wedge \mathbf{n} = \mathbf{0}$ (see Moreau, 1990) or an insulating wall where $\mathbf{j}.\mathbf{n} = 0$. In summary, the additional far-field behaviours and boundary conditions read

$$(\mathbf{u}, \nabla\phi, p) \to (\mathbf{0}, \mathbf{0}, 0) \text{ far from } \mathbf{x}_0 \text{ in } \mathcal{D}, \tag{3}$$

$$\mathbf{u} = \mathbf{0} \text{ on } \Sigma(z = 0), \tag{4}$$

$$\phi = 0 \text{ (conducting wall) or } \nabla\phi.\mathbf{e}_z = 0 \text{ (insulating wall) on } \Sigma(z = 0). \tag{5}$$

For the considered types of walls, the problem (1)–(5) is linear in both $(\mathbf{u}, p)$ and ϕ. However, it is not easy to solve since $(\mathbf{u}, p)$ and ϕ are coupled through (1) and the second Equation (2). For symmetry reasons and by superposition, the attention is restricted to the choices $\mathbf{e} = \mathbf{e}_z$ (Case 1) or $\mathbf{e} = \mathbf{e}_x$ (Case 2).

2.2. Analytical fundamental solution for the unbounded case

It is possible to analytically get the fundamental flow and electrostatic potential in absence of wall, i.e. for an unbounded liquid. As shown in Priede (2013), at $\mathbf{x} \neq \mathbf{x}_0$, the resulting fundamental MHD velocity $\mathbf{v}(\mathbf{x}_0, \mathbf{x})$, pressure $q(\mathbf{x}_0, \mathbf{x})$ and electrostatic potential $\psi(\mathbf{x}_0, \mathbf{x})$ produced by the unit force with strength $\mathbf{e}$ located at $\mathbf{x}_0$ are expressed, for $\mathbf{B} = B\mathbf{e}_z$, as

$$\mathbf{v}(\mathbf{x}_0,\mathbf{x}) = \frac{1}{\mu}\{\nabla \wedge (\nabla \wedge [H\mathbf{e}])\}, q(\mathbf{x}_0,\mathbf{x}) = \Delta[\nabla.(H\mathbf{e})] - \frac{1}{d^2}\frac{\partial}{\partial z}[H(\mathbf{e}.\mathbf{e}_z)] \quad (6)$$

$$\psi(\mathbf{x}_0,\mathbf{x}) = \frac{B}{\mu}\nabla.[H(\mathbf{e}_z \wedge \mathbf{e})] \quad (7)$$

with $H(\mathbf{x}_0,\mathbf{x})$ a 'generating' function. This function H vanishes as $R = |\mathbf{x} - \mathbf{x}_0|$ becomes large and obeys

$$\Delta(\Delta H) - \frac{1}{d^2}\frac{\partial^2 H}{\partial z^2} = \delta(\mathbf{x} - \mathbf{x}_0) \text{ for } \mathbf{x} \neq \mathbf{x}_0 \quad (8)$$

where $d = (\sqrt{\mu/\sigma})/|B|$ is the so-called Hartmann layer thickness (Hartmann, 1937). The function H is given in Priede (2013). Using Cartesian coordinates (x_0, y_0, z_0) for the source point $\mathbf{x}_0$ and (x, y, z) for the observation point $\mathbf{x}$, we here content ourselves with the useful relations (obtained in Sellier & Aydin, 2016b)

$$-4\pi\,\Delta H = \cosh\left(\frac{z-z_0}{2d}\right)\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_0|/(2d)}}{|\mathbf{x}-\mathbf{x}_0|},$$
$$-4\pi\frac{\partial H}{\partial z} = d\sinh\left(\frac{z-z_0}{2d}\right)\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_0|/(2d)}}{|\mathbf{x}-\mathbf{x}_0|}, \quad (9)$$

$$-8\pi\frac{\partial H}{\partial x} = d(x-x_0)\left\{\frac{2}{(x-x_0)^2+(y-y_0)^2} - \frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_0|/(2d)}}{|\mathbf{x}-\mathbf{x}_0|}\left[\frac{\mathrm{e}^{(z-z_0)/(2d)}}{|\mathbf{x}-\mathbf{x}_0|-(z-z_0)} + \frac{\mathrm{e}^{-(z-z_0)/(2d)}}{|\mathbf{x}-\mathbf{x}_0|+z-z_0}\right]\right\}. \quad (10)$$

Henceforth, we set $\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z$ and $R = |\mathbf{x} - \mathbf{x}_0|$. As the reader may easily check, applying (6)–(7) in conjunction with (9)–(10) then yields the following fundamental free-space solutions :

(i) Case 1: $\mathbf{e} = \mathbf{e}_z$. Then, $\psi(\mathbf{x}_0,\mathbf{x}) = 0$ and the flow $(\mathbf{v}, q)$ is axisymmetric about the axis $(M_0, \mathbf{e}_z)$ with point M_0 such that $\mathbf{OM}_0 = \mathbf{x}_0$. One gets

$$v_x(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z-z_0}{2d}\right)\left[1+\frac{2d}{R}\right]\left[\frac{x-x_0}{R}\right]\frac{\mathrm{e}^{-R/(2d)}}{8\pi\mu R}, \quad (11)$$

$$v_y(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z-z_0}{2d}\right)\left[1+\frac{2d}{R}\right]\left[\frac{y-y_0}{R}\right]\frac{\mathrm{e}^{-R/(2d)}}{8\pi\mu R}, \quad (12)$$

$$v_z(\mathbf{x}_0,\mathbf{x}) = \left\{\cosh\left(\frac{z-z_0}{2d}\right) + \sinh\left(\frac{z-z_0}{2d}\right)\left[1+\frac{2d}{R}\right]\left[\frac{z-z_0}{R}\right]\right\}\frac{\mathrm{e}^{-R/(2d)}}{8\pi\mu R}, \quad (13)$$

$$q(\mathbf{x}_0,\mathbf{x}) = \frac{1}{d}\left\{\sinh\left(\frac{z-z_0}{2d}\right) + \cosh\left(\frac{z-z_0}{2d}\right)\left[1+\frac{2d}{R}\right]\left[\frac{z-z_0}{R}\right]\right\}$$

$$\times \frac{e^{-R/(2d)}}{8\pi R}. \tag{14}$$

(ii) Case 2 $\mathbf{e} = \mathbf{e}_x$. In that case, the flow $(\mathbf{v}, q)$ is given by

$$v_x(\mathbf{x}_0, \mathbf{x}) = 2\cosh\left(\frac{z - z_0}{2d}\right)\frac{e^{-R/(2d)}}{8\pi\mu R} + \frac{d[T_1(\mathbf{x}_0, \mathbf{x}) - (x - x_0)^2 T_2(\mathbf{x}_0, \mathbf{x})]}{8\pi\mu}, \tag{15}$$

$$v_y(\mathbf{x}_0, \mathbf{x}) = -\left[\frac{d(x - x_0)(y - y_0)}{8\pi\mu}\right] T_2(\mathbf{x}_0, \mathbf{x}), \tag{16}$$

$$v_z(\mathbf{x}_0, \mathbf{x}) = \sinh\left(\frac{z - z_0}{2d}\right)\left[1 + \frac{2d}{R}\right]\left[\frac{x - x_0}{R}\right]\frac{e^{-R/(2d)}}{8\pi\mu R}, \tag{17}$$

$$q(\mathbf{x}_0, \mathbf{x}) = \frac{1}{d}\cosh\left(\frac{z - z_0}{2d}\right)\left[1 + \frac{2d}{R}\right]\left[\frac{x - x_0}{R}\right]\frac{e^{-R/(2d)}}{8\pi R}, \tag{18}$$

with occurring functions $T_1(\mathbf{x}_0, \mathbf{x})$ and $T_2(\mathbf{x}_0, \mathbf{x})$ obtained in Sellier and Aydin (2016b) and recalled in Appendix 1. The electrostatic potential ψ is non-zero and satisfies

$$\psi(\mathbf{x}_0, \mathbf{x}) = -\frac{Bd}{8\pi\mu}(y - y_0)\left\{\frac{2}{(x - x_0)^2 + (y - y_0)^2} - \frac{e^{-|\mathbf{x}-\mathbf{x}_0|/(2d)}}{|\mathbf{x} - \mathbf{x}_0|}\left[\frac{e^{(z-z_0)/(2d)}}{|\mathbf{x} - \mathbf{x}_0| - (z - z_0)} + \frac{e^{-(z-z_0)/(2d)}}{|\mathbf{x} - \mathbf{x}_0| + z - z_0}\right]\right\}, \tag{19}$$

$$\frac{\partial\psi}{\partial z}(\mathbf{x}_0, \mathbf{x}) = B\sinh\left(\frac{z - z_0}{2d}\right)\left[1 + \frac{2d}{R}\right]\left[\frac{y - y_0}{R}\right]\frac{e^{-R/(2d)}}{8\pi\mu R}. \tag{20}$$

In absence of magnetic field, one retrieves by letting d vanish for each previous case the free-space fundamental Stokeslet (Happel & Brenner, 1983; Kim & Karrila, 1983) $(\mathbf{v}_S, q_S)$ given by

$$\mathbf{v}_S(\mathbf{x}_0, \mathbf{x}) = \frac{1}{8\pi\mu R}\left\{\mathbf{e} + \frac{[\mathbf{e}.(\mathbf{x} - \mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0)}{R^2}\right\}, q_S(\mathbf{x}_0, \mathbf{x}) = \frac{1}{4\pi}\left[\frac{\mathbf{e}.(\mathbf{x} - \mathbf{x}_0)}{R^3}\right]. \tag{21}$$

2.3. Advocated decomposition and resulting auxiliary MHD problem for the bounded domain

The fundamental Stokes flow (take $\mathbf{B} = \mathbf{0}$) produced by a Stokeslet in presence of a no-slip wall has been obtained (Blake, 1971; Pozrikidis, 1992) by superposing the flows due to the Stokeslet with strength $\mathbf{e}$, the image Stokeslet with strength $-\mathbf{e}$ located at the symmetric $\mathbf{x}_0'$ of the source point $\mathbf{x}_0$ with respect to Σ (see Figure 1) and another regular Stokes flow. In a similar fashion, we seek the fundamental MHD flow $(\mathbf{u}, p)$ and electric potential ϕ governed by (1)–(5) as

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}_0, \mathbf{x}) - \mathbf{v}(\mathbf{x}'_0, \mathbf{x}) + \mathbf{U}(\mathbf{x}), p(\mathbf{x}) = q(\mathbf{x}_0, \mathbf{x}) - q(\mathbf{x}'_0, \mathbf{x}) + P(\mathbf{x}), \quad (22)$$

$$\phi(\mathbf{x}) = \psi(\mathbf{x}_0, \mathbf{x}) - \psi(\mathbf{x}'_0, \mathbf{x}) + \Phi(\mathbf{x}) \quad (23)$$

where the dependence in $\mathbf{x}_0$ for $(\mathbf{u}, p, \phi)$ and $(\mathbf{U}, P, \Phi)$ is dropped. Clearly, the MHD quantities $(\mathbf{U}, P, \Phi)$ are regular in the entire $z > 0$ domain and satisfy

$$\mu \nabla^2 \mathbf{U} = \nabla P + \sigma B \nabla \Phi \wedge \mathbf{e}_z - \sigma B^2 (\mathbf{U} \wedge \mathbf{e}_z) \wedge \mathbf{e}_z \text{ for } \mathbf{x} \text{ in } \mathcal{D}, \quad (24)$$

$$\nabla . \mathbf{U} = 0 \text{ and } \Delta \Phi = B \nabla . (\mathbf{U} \wedge \mathbf{e}_z) \text{ for } \mathbf{x} \text{ in } \mathcal{D}. \quad (25)$$

The far-field behaviour and conditions on the wall for $(\mathbf{U}, P, \Phi)$ are immediately deduced from (3)–(5). Recalling that $R = |\mathbf{x} - \mathbf{x}_0|$ and setting $R' = |\mathbf{x} - \mathbf{x}'_0|$, it appears that on the wall $R = R'$ and also, for previous Case 2 ($\mathbf{e} = \mathbf{e}_x$), that $L(\mathbf{x}_0, \mathbf{x}) = L(\mathbf{x}'_0, \mathbf{x})$ for $L = \psi, T_1, T_2$ (use the definitions of T_1 and T_2 in Appendix 1). From (11)–(13), (15)–(17) and (19)–(20), the required far-field and boundary conditions then read

$$(\mathbf{U}, \nabla \Phi, P) \to (\mathbf{0}, \mathbf{0}, 0) \text{ as } |\mathbf{x} - \mathbf{x}_0| \to \infty, \quad (26)$$

$$U_x(\mathbf{x}) = -2 v_x(\mathbf{x}_0, \mathbf{x}), U_y(\mathbf{x}) = -2 v_y(\mathbf{x}_0, \mathbf{x}), U_z(\mathbf{x}) = 0 \text{ on } \Sigma \text{ in Case 1}, \quad (27)$$

$$\Phi(\mathbf{x}) = 0 \text{ (conducting) or } \frac{\partial \Phi}{\partial z}(\mathbf{x}) = 0 \text{ (insulating) on } \Sigma \text{ in Case 1}, \quad (28)$$

$$U_x(\mathbf{x}) = U_y(\mathbf{x}) = 0, U_z(\mathbf{x}) = -2 v_z(\mathbf{x}_0, \mathbf{x}) \text{ on } \Sigma \text{ in Case 2}, \quad (29)$$

$$\Phi(\mathbf{x}) = 0 \text{ (conducting) or } \frac{\partial \Phi}{\partial z}(\mathbf{x}) = -2 \frac{\partial \psi}{\partial z}(\mathbf{x}_0, \mathbf{x}) \text{ (insulating) on } \Sigma \text{ in Case 2}. \quad (30)$$

In summary, the problem has been reduced to the determination of the *auxiliary* MHD Stokes flow $(\mathbf{U}, P)$ and electric potential Φ governed by (24)–(30).

3. Solution for a point force oriented normal to the wall

This section treats the previous Case 1 of a point force oriented normal to the bounding plane wall, i.e. the choice $\mathbf{e} = \mathbf{e}_z$.

3.1. Governing problem for a relevant 'generating' function

In view of (26)–(28), the quantities $(\mathbf{U}, P, \Phi)$ are axisymmetric about the axis $(M_0, \mathbf{e}_z)$ and $\mathbf{U}$ is *without* swirl. The last equation (25) then yields $\Delta \Phi = 0$ in $\mathcal{D}$. Invoking the far-field behaviour and boundary conditions for Φ then gives $\Phi = 0$ whatever the wall nature. As in (6), the axisymmetric flow $(\mathbf{U}, P)$ is then sought under the following form

$$\mu w_x(\mathbf{x}) = \left[\frac{\partial^2 F}{\partial x \partial z}\right](\mathbf{x}_0, \mathbf{x}), \mu w_y(\mathbf{x}) = \left[\frac{\partial^2 F}{\partial y \partial z}\right](\mathbf{x}_0, \mathbf{x}), \quad (31)$$

$$\mu w_z(\mathbf{x}) = -\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right](\mathbf{x}_0, \mathbf{x}), P(\mathbf{x}) = \left[\frac{\partial}{\partial z}\left\{\Delta F - \frac{F}{d^2}\right\}\right](\mathbf{x}_0, \mathbf{x}) \quad (32)$$

with unknown auxiliary 'generating' function $F(\mathbf{x}_0, \mathbf{x})$ obeying

$$\Delta(\Delta F) - \frac{1}{d^2}\frac{\partial^2 F}{\partial z^2} = 0 \text{ for } z = \mathbf{x}.\mathbf{e}_z > 0. \quad (33)$$

As seen in Section 2.2, the Cartesian components of the free-space velocity $\mathbf{v}(\mathbf{x}_0, \mathbf{x})$ are related to the free-space function $H(\mathbf{x}_0, \mathbf{x})$ by similar relations. From (26)–(28), the differential Equation (33) is thus supplemented with

$$\left(\frac{\partial^2 F}{\partial x \partial z}, \frac{\partial^2 F}{\partial y \partial z}, \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)(\mathbf{x}_0, \mathbf{x}) \to (0, 0, 0) \text{ as } |\mathbf{x} - \mathbf{x}_0| \to \infty, \quad (34)$$

$$\left[\frac{\partial^2 F}{\partial x \partial z} + 2\frac{\partial^2 H}{\partial x \partial z}\right](\mathbf{x}_0, \mathbf{x}) = \left[\frac{\partial^2 F}{\partial y \partial z} + 2\frac{\partial^2 H}{\partial y \partial z}\right](\mathbf{x}_0, \mathbf{x}) = 0 \text{ on } \Sigma(z = 0), \quad (35)$$

$$\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right](\mathbf{x}_0, \mathbf{x}) = 0 \text{ on } \Sigma(z = 0). \quad (36)$$

In summary, the task reduces to the determination of F governed by (33)–(36).

3.2. Solution in two-dimensional Fourier space

Here, $z_0 = \mathbf{x}_0.\mathbf{e}_z > 0$ and $F(\mathbf{x}_0, \mathbf{x}) = F(t_1, t_2, z; z_0)$ where $t_1 = x - x_0$ and $t_2 = y - y_0$. Similarly, $H(\mathbf{x}_0, \mathbf{x}) = H(t_1, t_2, z; z_0)$. For convenience, we resort to the two-dimensional Fourier transform $\hat{f}$ of a function $f(t_1, t_2)$ such that

$$\hat{f}(\boldsymbol{q}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1, t_2)\mathrm{e}^{i\boldsymbol{q}.\mathbf{t}}\mathrm{d}t_1\mathrm{d}t_2, \widehat{\frac{\partial f}{\partial x}} = -iq_1\hat{f}, \widehat{\frac{\partial f}{\partial y}} = -iq_2\hat{f} \quad (37)$$

where i denotes the complex number such that $i^2 = -1, \mathbf{t} = t_1\mathbf{e}_x + t_2\mathbf{e}_y$ and $\boldsymbol{q} = q_1\mathbf{e}_x + q_2\mathbf{e}_y$ is the vector in the Fourier space with $q = |\boldsymbol{q}| = \{q_1^2 + q_2^2\}^{1/2}$. Omitting henceforth the dependence upon z_0 gives $\hat{H} = \hat{H}(\boldsymbol{q}, z)$ and $\hat{F} = \hat{F}(\boldsymbol{q}, z)$. In the two-dimensional Fourier space, the problem (33)–(36) becomes

$$q^4\hat{F} - \left(2q^2 + \frac{1}{d^2}\right)\frac{\partial^2 \hat{F}}{\partial^2 z} + \frac{\partial^4 \hat{F}}{\partial z^4} = 0, \hat{F}(\boldsymbol{q}, z) \to 0 \text{ as } z \to +\infty, \quad (38)$$

$$\frac{\partial \hat{F}}{\partial z}(\boldsymbol{q}, 0) = -2\frac{\partial \hat{H}}{\partial z}(\boldsymbol{q}, 0), \hat{F}(\boldsymbol{q}, 0) = 0. \quad (39)$$

As shown in Tsinober (1973a), the general solution to (38) reads

$$\hat{F}(\boldsymbol{q}, z) = A_1(\boldsymbol{q})\mathrm{e}^{\alpha_1 z} + A_2(\boldsymbol{q})\mathrm{e}^{\alpha_2 z} \quad (40)$$

with arbitrary functions A_1, A_2 and the useful definitions and properties

$$\alpha_1 = -\frac{1}{2d} - \left(q^2 + \frac{1}{4d^2}\right)^{1/2} < \alpha_2 = \frac{1}{2d} - \left(q^2 + \frac{1}{4d^2}\right)^{1/2} < 0, \tag{41}$$

$$\alpha_1^2 + \frac{\alpha_1}{d} - q^2 = 0, \alpha_2^2 - \frac{\alpha_2}{d} - q^2 = 0, \alpha_1\alpha_2 = q^2. \tag{42}$$

Enforcing the conditions (39) then easily provides the solution

$$\hat{F}(\boldsymbol{q}, z) = -4d \sinh\left(\frac{z}{2d}\right) e^{-\sqrt{q^2 + \frac{1}{4d^2}}z} \left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0). \tag{43}$$

Appealing to the Appendix 2 (see (B7)), it immediately follows that

$$\hat{F}(\boldsymbol{q}, z) = -\frac{d^2}{\pi} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \left[\left(q^2 + \frac{1}{4d^2}\right)^{-1/2}\right] e^{-\sqrt{q^2 + \frac{1}{4d^2}}(z+z_0)}. \tag{44}$$

3.3. Resulting generating function and auxiliary MHD Stokes flow

The function $F(t_1, t_2, z)$ is determined by applying to $\hat{F}$ the inverse two-dimensional Fourier transform defined as

$$f(t_1, t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\boldsymbol{q}) e^{-i\boldsymbol{q}.\mathbf{t}} dq_1 dq_2. \tag{45}$$

From (44) note that $\hat{F}(\boldsymbol{q}, z) = \hat{F}(q, z)$ with $q = |\boldsymbol{q}|$. Setting $\rho = |\mathbf{t}| = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}$ gives $\boldsymbol{q}.\mathbf{t} = \rho q \theta$ with θ the angle made by vectors $\boldsymbol{q}$ and $\mathbf{t}$. We shall also use the relations

$$\int_0^{2\pi} e^{-i\rho q \cos\theta} d\theta = 2\pi J_0(\rho q), \int_0^{2\pi} \cos\theta e^{-i\rho q \cos\theta} d\theta = -2\pi i J_1(\rho q) \tag{46}$$

where J_n designates the usual Bessel function (of the first kind) of integer order n. Consequently, one gets

$$F(\mathbf{x}_0, \mathbf{x}) = F(\rho, z; z_0) = \int_0^{\infty} \hat{F}(q, z) J_0(\rho q) q dq. \tag{47}$$

Moreover, as shown in Appendix 2, note that

$$\int_0^{\infty} \left[\frac{e^{-\sqrt{q^2 + \frac{1}{4d^2}}(z+z_0)}}{\sqrt{q^2 + \frac{1}{4d^2}}}\right] J_0(\rho q) q dq = g(R'), g(u) = \frac{e^{-u/(2d)}}{u}, R' = |\mathbf{x} - \mathbf{x}_0'|. \tag{48}$$

Combining (44) with (47)–(48) finally provides the desired 'generating' function

$$F(\mathbf{x}_0,\mathbf{x}) = -\frac{d^2}{\pi}\sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right], R' = |\mathbf{x}-\mathbf{x}_0'|. \quad (49)$$

The auxiliary flow $(\mathbf{U}, P)$ is then analytically obtained from (31)–(32) and (49). This task appeals to many elementary manipulations not detailed here and uses the identities (with primes denoting derivatives)

$$g''(u) = \frac{g(u)}{4d^2}\left[1+\frac{4d}{u}+\frac{8d^2}{u^2}\right], \left(\frac{g'(u)}{u}\right)' = \frac{g(u)}{4d^2u}\left[1+\frac{6d}{u}+\frac{12d^2}{u^2}\right]. \quad (50)$$

The auxiliary velocity components and pressure, obtained in closed-form, read

$$U_x(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\left[\frac{x-x_0}{R'}\right]\frac{A}{\mu}, U_y(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\left[\frac{y-y_0}{R'}\right]\frac{A}{\mu}, \quad (51)$$

$$A = \left\{\cosh\left(\frac{z}{2d}\right)\left[1+\frac{2d}{R'}\right] - \sinh\left(\frac{z}{2d}\right)\left[\frac{z+z_0}{R'}\right]\left(1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right)\right\} \times \left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right], \quad (52)$$

$$U_z(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\frac{B}{\mu}, P(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\cosh\left(\frac{z}{2d}\right)\frac{B}{d}, \quad (53)$$

$$B = \left\{1+\frac{2d}{R'}+\frac{4d^2}{R'^2} - \left[\frac{z+z_0}{R'}\right]^2\left(1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right)\right\}\left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right]. \quad (54)$$

Because $R = R'$ for $z = 0$, the above results (use (11)–(12) for $z = 0$) satisfy the boundary conditions (27). In addition, as $d \to \infty$, one gets

$$U_x \sim \frac{z_0(x-x_0)}{4\pi\mu R'^3}\left[1-\frac{3z(z+z_0)}{R'^2}\right], U_y \sim \frac{z_0(y-y_0)}{4\pi\mu R'^3}\left[1-\frac{3z(z+z_0)}{R'^2}\right], \quad (55)$$

$$U_z \sim \frac{z_0 z}{4\pi\mu R'^3}\left[1-\frac{3(z+z_0)^2}{R'^2}\right], P \sim \frac{z_0}{2\pi R'^3}\left[1-\frac{3(z+z_0)^2}{R'^2}\right] \quad (56)$$

therefore retrieving the results for the usual Stokes flow (see, for instance, Pozrikidis, 1992).

4. Solution for a point force oriented parallel to the wall

This section considers a point force oriented parallel to the bounding plane wall, i.e. the Case 2 with $\mathbf{e} = \mathbf{e}_x$. As will be seen below, the treatment is more involved than for the axisymmetric problem solved in Section 3.

4.1. Solution in two-dimensional Fourier space

This time the coupled MHD flow $(\mathbf{U}, P)$ and electric potential Φ obey (24)–(26) and (29)–(30). Exploiting first (24)–(26) easily shows (see details in Priede, 2013) that

$$\mathcal{L}(U_x) = \mathcal{L}(U_y) = \mathcal{L}(U_z) = \mathcal{L}(P) = \mathcal{L}(\Phi) = 0, \mathcal{L}(F) := \Delta(\Delta F) - \frac{1}{d^2}\frac{\partial^2 F}{\partial z^2}. \tag{57}$$

Since P, Φ and each velocity component vanish as z becomes large, so do the associated (again omitting the dependance upon $\mathbf{x}_0$) two-dimensional Fourier transforms $\hat{P}(\boldsymbol{q}, z)$, $\hat{\Phi}(\boldsymbol{q}, z)$, $\hat{U}_x(\boldsymbol{q}, z)$, $\hat{U}_y(\boldsymbol{q}, z)$ and $\hat{U}_z(\boldsymbol{q}, z)$. By virtue of (57) and recalling (40), it turns out that

$$\hat{U}_t(\boldsymbol{q}, z) = \hat{U}_t^{(1)}(\boldsymbol{q})\mathrm{e}^{\alpha_1 z} + \hat{U}_t^{(2)}(\boldsymbol{q})\mathrm{e}^{\alpha_2 z} \text{ for } t = x, y, z; \tag{58}$$

$$\hat{P}(\boldsymbol{q}, z) = \hat{P}^{(1)}(\boldsymbol{q})\mathrm{e}^{\alpha_1 z} + \hat{P}^{(2)}(\boldsymbol{q})\mathrm{e}^{\alpha_2 z}, \hat{\Phi}(\boldsymbol{q}, z) = \hat{\Phi}^{(1)}(\boldsymbol{q})\mathrm{e}^{\alpha_1 z} + \hat{\Phi}^{(2)}(\boldsymbol{q})\mathrm{e}^{\alpha_2 z}. \tag{59}$$

One thus ends up with 10 unknown functions: $\hat{U}_t^{(k)}, \hat{P}^{(k)}$ and $\hat{\Phi}^{(k)}$ for $k = 1, 2$ and $t = x, y, z$. Omitting henceforth for those functions the dependence vs. $\boldsymbol{q}$, the velocity boundary conditions (29) yield

$$\hat{U}_x^{(2)} = -\hat{U}_x^{(1)}, \hat{U}_y^{(2)} = -\hat{U}_y^{(1)}, \hat{U}_z^{(1)} + \hat{U}_z^{(2)} = \hat{T}, \hat{T} = \frac{2iq_1}{\mu}\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0). \tag{60}$$

Moreover, the second Equation (25) coupling the electric potential Φ with the velocity $\mathbf{U}$ becomes, when combined with (60),

$$\left[\frac{\partial^2}{\partial z^2} - q^2\right]\hat{\Phi} = iB[q_2\hat{U}_x - q_1\hat{U}_y] = iB[q_2\hat{U}_x^{(1)} - q_1\hat{U}_y^{(1)}][\mathrm{e}^{\alpha_1 z} - \mathrm{e}^{\alpha_2 z}]. \tag{61}$$

The solution to (61), *of the form* (59), is readily (recall the relations (42))

$$\hat{\Phi} = -idB\hat{G}\left[\frac{\mathrm{e}^{\alpha_1 z}}{\alpha_1} + \frac{\mathrm{e}^{\alpha_2 z}}{\alpha_2}\right], \hat{G} = q_2\hat{U}_x^{(1)} - q_1\hat{U}_y^{(1)}. \tag{62}$$

Accordingly, the boundary condition (30) for Φ on the $z = 0$ plane wall gives

$$\hat{G} = 0 \text{ for conducting wall}, \hat{G} = -\frac{q_2}{\mu d}\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0) \text{ for insulating wall}. \tag{63}$$

The two-dimensional Fourier transform of product of (24) with $\mathbf{e}_z$ reads

$$\mu\left[\frac{\partial^2}{\partial z^2} - q^2\right]\hat{U}_z = \frac{\partial \hat{P}}{\partial z} = \alpha_1 \hat{P}^{(1)} e^{\alpha_1 z} + \alpha_2 \hat{P}^{(2)} e^{\alpha_2 z}. \tag{64}$$

Consequently, the pressure function $\hat{P}$ is (use again (42))

$$\hat{P} = \mu\left[\frac{\alpha_1^2 - q^2}{\alpha_1}\right]\hat{U}_z^{(1)} e^{\alpha_1 z} + \mu\left[\frac{\alpha_2^2 - q^2}{\alpha_2}\right]\hat{U}_z^{(2)} e^{\alpha_2 z} = -\frac{\mu}{d}\left[\hat{U}_z^{(1)} e^{\alpha_1 z} - \hat{U}_z^{(2)} e^{\alpha_2 z}\right]. \tag{65}$$

The unknown functions $\hat{U}_x^{(1)}, \hat{U}_y^{(1)}, \hat{U}_z^{(1)}$ and $\hat{U}_z^{(2)}$ are obtained from (60), (62) and the Fourier transform of the first Equation (25). The resulting linear system is

$$\hat{U}_z^{(1)} + \hat{U}_z^{(2)} = \hat{T}, q_2 \hat{U}_x^{(1)} - q_1 \hat{U}_y^{(1)} = \hat{G}, \tag{66}$$

$$\alpha_1 \hat{U}_z^{(1)} = i[q_1 \hat{U}_x^{(1)} + q_2 \hat{U}_y^{(1)}], \alpha_2 \hat{U}_z^{(2)} = -i[q_1 \hat{U}_x^{(1)} + q_2 \hat{U}_y^{(1)}]. \tag{67}$$

Using the property $\alpha_1\alpha_2 = q^2$ and recalling (62) and (65), one finally gets the following solution in the two-dimensional Fourier space

$$\hat{U}_x^{(1)} = \frac{q_2}{q^2}\hat{G} - idq_1\hat{T}, \hat{U}_x^{(2)} = -\hat{U}_x^{(1)}, \hat{U}_z^{(1)} = d\alpha_2\hat{T}, \tag{68}$$

$$\hat{U}_y^{(1)} = -\frac{q_1}{q^2}\hat{G} - idq_2\hat{T}, \hat{U}_y^{(2)} = -\hat{U}_y^{(1)}, \hat{U}_z^{(2)} = -d\alpha_1\hat{T}, \tag{69}$$

$$\hat{P}^{(1)} = -\mu\alpha_2\hat{T}, \hat{P}^{(2)} = -\mu\alpha_1\hat{T}, \hat{\Phi}^{(1)} = -i\frac{dB}{\alpha_1}\hat{G}, \hat{\Phi}^{(2)} = -i\frac{dB}{\alpha_2}\hat{G}. \tag{70}$$

Now, one must check that Fourier transforms of the products of (24) with $\mathbf{e}_x$ and $\mathbf{e}_y$ are satisfied by the above solution. Since functions $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ are linearly independent, the Fourier transform of (24). $\mathbf{e}_x$ yields the following relations (recall the definition $d^2 = \mu/(\sigma B^2)$)

$$(\alpha_1^2 - q^2)\hat{U}_x^{(1)} = -i\frac{q_1}{\mu}\hat{P}^{(1)} - i\frac{q_2\sigma B}{\mu}\hat{\Phi}^{(1)} + \frac{1}{d^2}\hat{U}_x^{(1)}, \tag{71}$$

$$(\alpha_2^2 - q^2)\hat{U}_x^{(2)} = -i\frac{q_1}{\mu}\hat{P}^{(2)} - i\frac{q_2\sigma B}{\mu}\hat{\Phi}^{(2)} + \frac{1}{d^2}\hat{U}_x^{(2)}. \tag{72}$$

As the reader may easily check using the relations (42) and the solution (68)–(70), the two above relations are satisfied whatever $\hat{T}$ and $\hat{G}$. In a similar fashion, the Fourier transform of (24). $\mathbf{e}_y$ is also satisfied by the solution (68)–(70).

4.2. Auxiliary MHD flow and electric potential for a conducting wall

From (63) it appears that $\hat{G} = 0$ for a plane conducting wall. In such circumstances, $\Phi = 0$ since (62) gives $\hat{\Phi} = 0$. In addition, from the definition (60) of $\hat{T}$

$$\hat{U}_x = \frac{2dq_1q_1}{\mu}[\mathrm{e}^{\alpha_1 z} - \mathrm{e}^{\alpha_2 z}]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0),$$

$$\hat{P} = -2iq_1[\alpha_2 \mathrm{e}^{\alpha_1 z} + \alpha_1 \mathrm{e}^{\alpha_2 z}]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0), \tag{73}$$

$$\hat{U}_y = \frac{2dq_1q_2}{\mu}[\mathrm{e}^{\alpha_1 z} - \mathrm{e}^{\alpha_2 z}]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0),$$

$$\hat{U}_z = \frac{2idq_1}{\mu}[\alpha_2 \mathrm{e}^{\alpha_1 z} - \alpha_1 \mathrm{e}^{\alpha_2 z}]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0). \tag{74}$$

Now, inspecting (47)–(48) shows that

$$[\widehat{g(R')}](\boldsymbol{q}) = [\widehat{g(R')}](q) = \frac{\mathrm{e}^{-\sqrt{q^2 + \frac{1}{4d^2}}(z+z_0)}}{\sqrt{q^2 + \frac{1}{4d^2}}} \quad \text{for } g(R') = \frac{\mathrm{e}^{-R'/(2d)}}{R'} \tag{75}$$

while some manipulations, using the value of $[\frac{\partial \hat{H}}{\partial z}](\boldsymbol{q}, 0)$ obtained in Appendix 2 and the definitions of α_1 and α_2 permit one to recast (73)–(74) as follows

$$\hat{U}_x = -\frac{q_1^2 d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \hat{g}(q),$$

$$\hat{U}_y = -\frac{q_1 q_2 d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \hat{g}(q), \tag{76}$$

$$\hat{U}_z = \frac{iq_1 d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \left\{\frac{1}{2d} \cosh\left(\frac{z}{2d}\right) \hat{g}(q) - \sinh\left(\frac{z}{2d}\right) \left[\frac{\partial \hat{g}}{\partial z}\right](q)\right\}, \tag{77}$$

$$\hat{P} = \frac{iq_1 d}{\pi} \sinh\left(\frac{z_0}{2d}\right) \left\{\frac{1}{2d} \sinh\left(\frac{z}{2d}\right) \hat{g}(q) - \cosh\left(\frac{z}{2d}\right) \left[\frac{\partial \hat{g}}{\partial z}\right](q)\right\}. \tag{78}$$

From (75)–(78) it is then clear that the auxiliary MHD flow velocity components and pressure for the conducting wall are given by

$$U_x^{cw} = \frac{d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \frac{\partial^2}{\partial x^2}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right], \tag{79}$$

$$U_y^{cw} = \frac{d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \frac{\partial^2}{\partial y \partial x}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right], \tag{80}$$

$$U_z^{cw} = \frac{d^2}{\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \left\{\sinh\left(\frac{z}{2d}\right) \frac{\partial^2}{\partial z \partial x}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right] - \frac{1}{2d} \cosh\left(\frac{z}{2d}\right) \frac{\partial}{\partial x}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right]\right\}, \tag{81}$$

$$P^{cw} = \frac{d}{\pi} \sinh\left(\frac{z_0}{2d}\right) \left\{\cosh\left(\frac{z}{2d}\right) \frac{\partial^2}{\partial z \partial x}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right] - \frac{1}{2d} \sinh\left(\frac{z}{2d}\right) \frac{\partial}{\partial x}\left[\frac{\mathrm{e}^{-R'/(2d)}}{R'}\right]\right\}. \tag{82}$$

where the upper script *cw* refers to the conducting wall case. Additional simple calculations using (50) finally yield the following analytical results

$$U_x^{cw}(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\frac{C}{\mu}, \tag{83}$$

$$U_y^{cw}(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\left[\frac{(x-x_0)(y-y_0)}{R'^2}\right]\frac{D}{\mu}, \tag{84}$$

$$U_z^{cw}(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\left[\frac{x-x_0}{R'}\right]\frac{E_1}{\mu},\ P^{cw}(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\left[\frac{x-x_0}{R'}\right]\frac{E_2}{d}, \tag{85}$$

$$C = \left\{\left[\frac{x-x_0}{R'}\right]^2\left(1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right) - \frac{2d}{R'}\left[1+\frac{2d}{R'}\right]\right\}\left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right], \tag{86}$$

$$D = \left[1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right]\left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right], \tag{87}$$

$$E_1 = \left\{\cosh\left(\frac{z}{2d}\right)\left[1+\frac{2d}{R'}\right] + \sinh\left(\frac{z}{2d}\right)\left[\frac{z+z_0}{R'}\right]\left(1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right)\right\}\left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right], \tag{88}$$

$$E_2 = \left\{\sinh\left(\frac{z}{2d}\right)\left[1+\frac{2d}{R'}\right] + \cosh\left(\frac{z}{2d}\right)\left[\frac{z+z_0}{R'}\right]\left(1+\frac{6d}{R'}+\frac{12d^2}{R'^2}\right)\right\}\left[\frac{\mathrm{e}^{-R'/(2d)}}{4\pi R'}\right]. \tag{89}$$

Taking $z = 0$ and thus $R = R'$ in the above results immediately shows, using (17), that (29) indeed holds for the obtained velocity field $\mathbf{U}^{cw}$. In addition, inspecting (83)–(89) gives in the Stokes limit $d \to \infty$ the asymptotic behaviours

$$U_x^{cw} \sim \frac{z_0 z}{4\pi\mu R'^3}\left[\frac{3(x-x_0)^2}{R'^2} - 1\right],\ U_y^{cw} \sim \frac{3z_0 z(x-x_0)(y-y_0)}{4\pi\mu R'^5}, \tag{90}$$

$$U_z^{cw} \sim \frac{z_0(x-x_0)}{4\pi\mu R'^3}\left[1+\frac{3z(z+z_0)}{R'^2}\right],\ P^{cw} \sim \frac{3z_0 z(z+z_0)(x-x_0)}{2\pi\mu R'^5} \tag{91}$$

which perfectly agree with the pure Stokes flow results given in Pozrikidis (1992).

4.3. Auxiliary MHD flow and electric potential for an insulating wall

This time $\hat{G}$ is non-zero and given by (63). Using now the upper script *iw* for this conducting wall case, one gets

$$U_x^{iw} = U_x^{cw} + V_x,\ U_y^{iw} = U_y^{cw} + V_y,\ U_z^{iw} = U_z^{cw},\ P^{cw} = P^{iw} \tag{92}$$

with velocity components $\hat{V}_x$ and $\hat{V}_y$ gained by setting $\hat{T} = 0$ in (68)–(69). Thus, only the electric potential Φ and the velocity components parallel with the wall are sensitive to the wall nature. From Section 4.1, one gets $\hat{V}_x = q_2 \hat{V}$, $\hat{V}_y = -q_1 \hat{V}$ and

$$\hat{\Phi}^{iw} = \frac{iBq_2}{\mu}\left[\frac{e^{\alpha_1 z}}{\alpha_1} + \frac{e^{\alpha_2 z}}{\alpha_2}\right]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0),\ \hat{V} = -\frac{q_2}{\mu d q^2}\left[e^{\alpha_1 z} - e^{\alpha_2 z}\right]\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0). \tag{93}$$

Looking at $\partial \hat{\Phi}^{iw}/\partial z$ from (93) easily provides the result

$$\left[\frac{\partial \Phi^{iw}}{\partial z}\right](\mathbf{x}_0, \mathbf{x}) = B \sinh\left(\frac{z_0}{2d}\right) \cosh\left(\frac{z}{2d}\right) \left[\frac{y - y_0}{R'}\right] \left(1 + \frac{2d}{R'}\right) \left[\frac{e^{-R'/(2d)}}{4\pi \mu R'}\right] \tag{94}$$

which, as expected, agrees with the second boundary condition (30) since (20) holds and $R' = R$ for $z = 0$. The determination of $\hat{\Phi}^{iw}$ itself and of V_x and V_y requires more efforts. Introducing the function W such that (see Appendix 2)

$$W(\mathbf{x}_0, \mathbf{x}) = \frac{2d(y - y_0)}{\rho^2}\left[e^{-R'/(2d)} - e^{-(z+z_0)/(2d)}\right],$$
$$\hat{W} = -\frac{iq_2}{q^2}\left[\frac{e^{-\sqrt{q^2 + \frac{1}{4d^2}}(z+z_0)}}{\sqrt{q^2 + \frac{1}{4d^2}}}\right] \tag{95}$$

yields from (93) the basic identities

$$\hat{V}_x = \frac{iq_2}{2\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \hat{W},\ \hat{V}_y = \frac{-iq_1}{2\pi \mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \hat{W}, \tag{96}$$

$$\hat{\Phi}^{iw} = \frac{Bd}{4\pi\mu} \sinh\left(\frac{z_0}{2d}\right)\left\{\frac{1}{d}\sinh\left(\frac{z}{2d}\right)\hat{W} - 2\cosh\left(\frac{z}{2d}\right)\frac{\partial \hat{W}}{\partial z}\right\}. \tag{97}$$

Accordingly, the velocity components V_x, V_y and the electric potential Φ^{iw} are

$$V_x = -\frac{1}{2\pi\mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \frac{\partial W}{\partial y},\ V_y = \frac{1}{2\pi \mu} \sinh\left(\frac{z_0}{2d}\right) \sinh\left(\frac{z}{2d}\right) \frac{\partial W}{\partial x}, \tag{98}$$

$$\Phi^{iw} = \frac{Bd}{4\pi\mu} \sinh\left(\frac{z_0}{2d}\right)\left\{\frac{1}{d}\sinh\left(\frac{z}{2d}\right) W - 2\cosh\left(\frac{z}{2d}\right)\frac{\partial W}{\partial z}\right\}. \tag{99}$$

Injecting the first equality (95) in (98)–(99) then finally gives the formulae

$$V_x(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\frac{E_x}{\mu},\ V_y(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\sinh\left(\frac{z}{2d}\right)\frac{E_y}{\mu}, \tag{100}$$

$$\Phi^{iw}(\mathbf{x}_0,\mathbf{x}) = \sinh\left(\frac{z_0}{2d}\right)\frac{E}{\mu}, \tag{101}$$

$$E = \frac{Bd(y-y_0)}{2\pi\rho^2}\left\{\left[\sinh\left(\frac{z}{2d}\right) + \left(\frac{z+z_0}{R'}\right)\cosh\left(\frac{z}{2d}\right)\right]e^{-R'/(2d)} - e^{-z_0/(2d)}\right\}, \tag{102}$$

$$E_x = \frac{d}{\pi\rho^2}\left\{\frac{4(y-y_0)^2}{dR'}e^{-R'/(2d)} - \left[1 - \frac{2(y-y_0)^2}{\rho^2}\right]\left[e^{-R'/(2d)} - e^{-(z+z_0)/(2d)}\right]\right\}, \tag{103}$$

$$E_y = -\frac{2d(x-x_0)(y-y_0)}{\pi\rho^2}\left\{\left[\frac{e^{-R'/(2d)} - e^{-(z+z_0)/(2d)}}{\rho^2}\right] + \frac{2e^{-R'/(2d)}}{dR'}\right\}. \tag{104}$$

When B vanishes and $d \to \infty$, all quantities V_x, V_y and Φ^{iw} vanish.

5. Concluding remarks

The coupled fundamental MHD flow $(\mathbf{u}, p)$ and electric potential ϕ produced by a point force with arbitrary unit strength $\mathbf{e}$ located in a conducting Newtonian liquid at $\mathbf{x}_0$ have been obtained when the liquid is subject to a uniform magnetic field $\mathbf{B}$ and bounded by a plane wall normal to $\mathbf{B}$ and either insulating or perfectly conducting. Following the procedure employed by previous authors Blake (1971), Pozrikidis (1992) in absence of magnetic field (case of a pure Stokes flow), the treatment uses a decomposition (see (22)–(23)) of $\mathbf{u}, p$ and ϕ in several terms: the free-space solution produced by a source with strength $\mathbf{e}$ located at $\mathbf{x}_0$, the free-space solution produced by a source with strength $-\mathbf{e}$ placed at the symmetric $\mathbf{x}'_0$ of $\mathbf{x}_0$, with respect to the wall and a third auxiliary term. The associated auxiliary quantities $(\mathbf{U}, P, \Phi)$ have been *analytically* obtained, whatever the wall nature (insulating or conducting), for $\mathbf{e} = e_x$ or $\mathbf{e} = e_z$ using a two-dimensional Fourier transform. Since easily deduced from the $\mathbf{e} = e_x$ case, the case $\mathbf{e} = e_y$ is let to the reader.

As a result, it is possible to introduce the so-called second-rank velocity Green tensor $\mathbf{G} = G_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, with α and β in $\{x, y, z\}$, such that

$$u_\alpha(\mathbf{x}_0,\mathbf{x}) = \mathbf{u}(\mathbf{x}_0,\mathbf{x}).\mathbf{e}_\alpha = G_{\alpha\beta}(\mathbf{x}_0,\mathbf{x})/(8\pi\mu) \text{ for } \mathbf{e} = \mathbf{e}_\beta \text{and } \alpha = x, y, z. \tag{105}$$

As the reader can check after elementary manipulations and using the results of the present work, it turns out here that $G_{\alpha\beta}(\mathbf{x}_0,\mathbf{x}) = G_{\beta\alpha}(\mathbf{x},\mathbf{x}_0)$ for both the insulating and the conducting wall. Such a nice symmetry property, well known Kim and Karrila (1983), Pozrikidis (1992) for a pure Stokes flow ($\mathbf{B} = \mathbf{0}$), is not trivial for the present problem. One should also note that it is in general not necessarily true depending upon the handled problem (for instance, this property does not hold for a Stokes flow above a motionless porous slab where the velocity is not required to vanish (Khabthani, Sellier, Elasmi, & Feuillebois, 2012).

For most applications, the magnetic field $\mathbf{B}$ is, as considered in the present study, normal to the plane boundary. The case of $\mathbf{B}$ parallel with the wall is very likely to be tackled by a similar procedure. It should however require many additional efforts.

Finally, the material derived in this paper is of utmost importance when building two key axisymmetric fundamental MHD flows produced by radial and axial distributions of forces spread on a circular ring located in a $z = cste > 0$ plane in a liquid domain bounded by an insulating or conducting $z = 0$ plane wall. Such a task has been recently done in Sellier and Aydin (2016b) for the *unbounded* liquid and it would be nice to investigate to which extent the wall affects the flows obtained and discussed in Sellier and Aydin (2016b). This challenging issue is postponed to another work.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1. Functions T_1 and T_2

Recalling that $R = |\mathbf{x} - \mathbf{x}_0|$ the functions $T_1(\mathbf{x}_0, \mathbf{x})$ and $T_2(\mathbf{x}_0, \mathbf{x})$ read

$$T_1(\mathbf{x}_0,\mathbf{x}) = \frac{\mathrm{e}^{-R/(2d)}}{R}\left\{\frac{\mathrm{e}^{(z-z_0)/(2d)}}{R-(z-z_0)} + \frac{\mathrm{e}^{-(z-z_0)/(2d)}}{R+z-z_0}\right\} - \frac{2}{R^2-(z-z_0)^2}, \tag{A1}$$

$$T_2(\mathbf{x}_0,\mathbf{x}) = \frac{\mathrm{e}^{-R/(2d)}}{R^2}\left\{\left[\frac{R+2d}{2dR}\right]\left[\frac{\mathrm{e}^{(z-z_0)/(2d)}}{R-(z-z_0)} + \frac{\mathrm{e}^{-(z-z_0)/(2d)}}{R+z-z_0}\right] + \frac{\mathrm{e}^{(z-z_0)/(2d)}}{[R-(z-z_0)]^2} + \frac{\mathrm{e}^{-(z-z_0)/(2d)}}{[R+z-z_0]^2}\right\} - \frac{4}{[R^2-(z-z_0)^2]^2}. \tag{A2}$$

From (A1)–(A2), it is clear that $T_i(\mathbf{x}_0, \mathbf{x}) = (\mathbf{x}_0', \mathbf{x})$ for $i = 1, 2$ when $\mathbf{x}$ is located on the $z = 0$ plane wall Σ.

Appendix 2. Two auxiliary identities

This Appendix derives the relations (48) and (95). One can think about two ways to calculate the required quantity $\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0)$. The first one is it to apply the two-dimensional Fourier transform (37)–(9) taking also $z = 0$. Using the first relation (46), it immediately follows that, since $\rho = |\mathbf{t}|$,

$$\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q},0) = \left[\frac{\partial \hat{H}}{\partial z}\right](q,0) = \frac{d}{4\pi}\sinh\left(\frac{z_0}{2d}\right)\int_0^\infty \left[\frac{\mathrm{e}^{-\sqrt{\rho^2+z_0^2}/(2d)}}{\sqrt{\rho^2+z_0^2}}\right] J_0(q\rho)\rho\,\mathrm{d}\rho. \tag{B1}$$

A second way is to set $t_3 = z - z_0$ and apply to (8) the Fourier transform (37) to get first $\hat{H}(\boldsymbol{q}, t_3)$. This latter function thus obeys

$$\left(\frac{\partial^2}{\partial t_3^2} - q^2\right)^2 \hat{H} - \frac{1}{d^2}\frac{\partial^2 \hat{H}}{\partial t_3^2} = \frac{\delta(t_3)}{2\pi}. \tag{B2}$$

One has then to determine the function $\hat{h}$ such that (since $t_3 = -z_0$ for $z = 0$)

$$\hat{h}(s) = \int_{-\infty}^{\infty} \frac{\hat{H}(\boldsymbol{q}, t_3)\mathrm{e}^{ist_3}}{\sqrt{2\pi}}\mathrm{d}t_3, \left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0) = -i\int_{-\infty}^{\infty} \frac{s\hat{h}(s)\mathrm{e}^{isz_0}}{\sqrt{2\pi}}\mathrm{d}s. \tag{B3}$$

From (B2) and after some algebra, it is found that

$$\hat{h}(s) = \frac{1}{(2\pi)^{3/2}[s^2/d^2 + (s^2 + q^2)^2]} = \frac{1}{(2\pi)^{3/2}(s^2 + s_1^2)(s^2 + s_2^2)}, \tag{B4}$$

$$s_1 = \sqrt{q^2 + \frac{1}{4d^2}} + \frac{1}{2d}, s_2 = \sqrt{q^2 + \frac{1}{4d^2}} - \frac{1}{2d}, s_1^2 - s_2^2 = \frac{2}{d}\sqrt{q^2 + \frac{1}{4d^2}}. \tag{B5}$$

Since $z_0 > 0$, note that (use for instance Gradshteyn & Ryzhik, 1965, p. 410)

$$\int_0^{\infty} \frac{s\sin(z_0 s)\mathrm{d}s}{(s^2 + s_1^2)(s^2 + s_2^2)} = -\frac{\pi}{2}\left[\frac{\mathrm{e}^{-s_1 z_0} - \mathrm{e}^{-s_2 z_0}}{s_1^2 - s_2^2}\right]. \tag{B6}$$

Therefore, the second relation (B2) gives

$$\left[\frac{\partial \hat{H}}{\partial z}\right](\boldsymbol{q}, 0) = \frac{d}{4\pi}\sinh\left(\frac{z_0}{2d}\right)\left[\frac{\mathrm{e}^{-\sqrt{q^2 + \frac{1}{4d^2}}z_0}}{\sqrt{q^2 + \frac{1}{4d^2}}}\right]. \tag{B7}$$

Comparing (B1) with (B7) then provides the announced result (48).

The function W is obtained by applying to $\hat{W}$ the inverse two-dimensional Fourier transform (45). Noting that $|\mathbf{t}| = \rho$ and setting in polar coordinates $\mathbf{t} = (\rho, \alpha)$ and $\boldsymbol{q} = (q, \alpha + \theta)$ easily yield (recall the second relation (46))

$$W = -\sin\alpha\mathcal{I}, \mathcal{I} = \int_0^{\infty}\left[\frac{\mathrm{e}^{-\sqrt{q^2 + \frac{1}{4d^2}}(z + z_0)}}{\sqrt{q^2 + \frac{1}{4d^2}}}\right]J_1(\rho q)\mathrm{d}q. \tag{B8}$$

Fortunately, it is possible to analytically calculate the integral $\mathcal{I}$. Indeed, $\mathcal{I} = I_{\frac{1}{2}}(u_1)K_{\frac{1}{2}}(u_2)$ (see Gradshteyn & Ryzhik, 1965, p. 719) with $I_{\frac{1}{2}}$ and $K_{\frac{1}{2}}$ the usual Bessel functions of fractional order 1/2 and

$$u_1 = \frac{1}{4d}\left[\sqrt{\rho^2 + (z + z_0)^2} - (z + z_0)\right], u_2 = \frac{1}{4d}\left[\sqrt{\rho^2 + (z + z_0)^2} + z + z_0\right]. \tag{B9}$$

Because $\rho\sin\alpha = y - y_0$ and

$$I_{\frac{1}{2}}(u_1) = \frac{1}{\sqrt{2\pi u_1}}[\mathrm{e}^{u_1} - \mathrm{e}^{-u_1}], K_{\frac{1}{2}}(u_2) = \sqrt{\frac{\pi}{2u_2}}\mathrm{e}^{-u_2} \tag{B10}$$

one therefore ends up with (95).