Far East J. Appl. Math. 19(3) (2005), 249-263

ON THE SLOW GRAVITY-DRIVEN MOTION OF ARBITRARY CLUSTERS OF SOLID PARTICLES AND BUBBLES: GENERAL THEORY

A. SELLIER

LadHyX, Ecole Polytechnique
91128 Palaiseau Cédex, France

Abstract

A new theory is proposed to determine the migration of a collection of arbitrary-shaped solid bodies and spherical bubbles under the action of gravity when inertial effects are negligible. The advocated procedure appeals only to surface quantities on the entire cluster's boundary and is therefore quite suitable for a future numerical implementation. The well posedness of the approach is established and the relevant boundary-integral equations governing all the needed surface quantities are also derived.

1. Introduction

Determining the motion of a collection of particles (solid bodies or bubbles) remains a tremendous challenge for many applications encountered in multiphase flows. As explained in [3] and [5] one usually adopts the simplified Stokes flow approximation for small particles with weak inertia. However, even within this framework it remains very difficult to investigate the particle-particle interactions for fully three-dimensional clusters made of more than two particles and the available results therefore confine the analysis special cases. Among others, one

2000 Mathematics Subject Classification: 76D07, 30E25, 31A10.

Key words and phrases: Stokes flow, reciprocal identity, boundary-integral equations.

Communicated by Kumar K. Tamma

Received September 24, 2004; Revised January 17, 2005

can cite the case of spherical (not arbitrary-shaped) solid bodies [4], the axisymmetric configurations of solid spheres and spheroids [2, 6] and the thermocapillary migration of chains of bubbles [1, 7, 8, 10, 13]. New boundary methods have been recently proposed to obtain the thermocapillary motion of arbitrary clusters of spherical bubbles in Sellier [11] and the gravity-driven motion of a collection of arbitrary-shaped solid bodies in Sellier [12]. However, neither [11] nor [12] is able to cope with the gravity-driven motion of clusters consisting of both spherical bubbles and arbitrary-shaped solid bodies. The present study thus introduces a generalized boundary method which adequately addresses such clusters when subject to the gravity and recovers [11, 12] as special cases.

2. Governing Equations

We consider, as sketched in Figure 1, N solid arbitrary-shaped bodies $\mathcal{P}_n (n = 1, \ldots, N)$ and M spherical bubbles $\mathcal{B}_m (m = 1, \ldots, M)$ immersed in a quiescent and unbounded Newtonian fluid of uniform viscosity μ and density ρ and subject to the uniform gravity field $\mathbf{g}$. The cluster migration is studied in a given Cartesian framework (O, x_1, x_2, x_3) and we henceforth adopt the usual tensor summation convention with $\mathbf{OM} = \mathbf{x} = x_i \mathbf{e}_i$ and $r = |\mathbf{x}| = (x_i x_i)^{1/2}$. Each not necessarily homogeneous solid $\mathcal{P}_n$ with length scale a_n, boundary S_n and center of mass O_n experiences a rigid-body motion of unknown translational velocity $\mathbf{U}^{(n)}$ (the velocity of O_n) and angular velocity $\mathbf{\Omega}^{(n)}$ whereas each bubble $\mathcal{B}_m$ with boundary B_m, center C_m and radius a'_m only translates at the unknown velocity $\mathbf{U}'^{(m)}$. For negligible inertial effects, i.e., if $\mathrm{Re} = \rho U a/\mu << 1$ with $a = Max(a_n, a'_m)$ and $U = Max(|\mathbf{U}^{(n)}|, a_n|\mathbf{\Omega}^{(n)}|, |\mathbf{U}'^{(m)}|)$, the fluid flow is quasistatic and has velocity $\mathbf{u}$, pressure $p + \rho \mathbf{g} \cdot \mathbf{x}$ and stress tensor $\boldsymbol{\sigma}$ such that

$$\mu \nabla^2 \mathbf{u} = \nabla p \text{ and } \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, (\mathbf{u}, p) \to (\mathbf{0}, 0) \text{ as } r = (x_i x_i)^{1/2} \to \infty, \quad (1)$$

$$\mathbf{u} = \mathbf{U}^{(n)} + \mathbf{\Omega}^{(n)} \wedge \mathbf{O_n M} \text{ on } S_n \text{ for } n = 1, \ldots, N, \quad (2)$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U}'^{(m)} \cdot \mathbf{n} \text{ and } \boldsymbol{\sigma} \cdot \mathbf{n} = (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n})\mathbf{n} \text{ on } B_m \text{ for } m = 1, ..., M \tag{3}$$

with Ω the unbounded fluid domain and $\mathbf{n}$ the unit outward normal on each surface S_n or B_m. The boundary conditions (2)-(3) are explained in the Appendix. The flow $(\mathbf{u}, p + \rho\mathbf{g} \cdot \mathbf{x})$ applies on S_n a net force $\mathbf{R}_n$ and a net torque $\mathbf{\Gamma}_n$ (about O_n) whereas it exerts on B_m a net force $\mathbf{R}'_m$ and, by virtue of (3), a zero net torque about C_m. If $\mathcal{P}_n$ has center of volume O'_n and volume $\mathcal{V}_n$, then one easily obtains

$$\mathbf{R}_n = \int_{S_n} \boldsymbol{\sigma} \cdot \mathbf{n} dS + (M_n - \rho\mathcal{V}_n)\mathbf{g},$$

$$\mathbf{\Gamma}_n = \int_{S_n} \mathbf{O_n M} \wedge \boldsymbol{\sigma} \cdot \mathbf{n} dS - \rho\mathcal{V}_n \mathbf{O_n O'_n} \wedge \mathbf{g}. \tag{4}$$

In addition, for a spherical bubble $\mathcal{B}_m$ with radius a'_m the quantity $\mathbf{R}'_m$ reads

$$\mathbf{R}'_m = \int_{B_m} \boldsymbol{\sigma} \cdot \mathbf{n} dS_n - 4\pi(a'_m)^3 \rho\mathbf{g}/3. \tag{5}$$

Assuming solid bodies $\mathcal{P}_n$ and bubbles $\mathcal{B}_m$ of negligible inertia, we require that $\mathbf{R}_n = \mathbf{\Gamma}_n = \mathbf{0}$ and that $\mathbf{R}'_m = \mathbf{0}$. Exploiting (4)-(5) those conditions immediately become, for $n = 1, ..., N$ and $m = 1, ..., M$,

$$\int_{S_n} \boldsymbol{\sigma} \cdot \mathbf{n} dS = (\rho\mathcal{V}_n - M_n)\mathbf{g}, \int_{S_n} \mathbf{O_n M} \wedge \boldsymbol{\sigma} \cdot \mathbf{n} dS = \rho\mathcal{V}_n \mathbf{O_n O'_n} \wedge \mathbf{g}, \tag{6}$$

$$\int_{B_m} \boldsymbol{\sigma} \cdot \mathbf{n} dS = 4\pi(a'_m)^3 \rho\mathbf{g}/3. \tag{7}$$

Hence, we look at the unknown generalized velocity $\mathbf{X} = (\mathbf{U}^{(1)}, ..., \mathbf{U}^{(N)}; \mathbf{\Omega}^{(1)}, ..., \mathbf{\Omega}^{(N)}; \mathbf{U}'^{(1)}, ..., \mathbf{U}'^{(M)})$ such that (1)-(3) and (6)-(7) hold. A possible iterative treatment might consist in appealing, as many times as necessary, to a Finite Element Code to obtain $(\mathbf{u}, p)$ solution to (1)-(3) for a given entry $\mathbf{X}$, then subsequently evaluate all the integrals occurring in (6)-(7) by computing the surface traction $\boldsymbol{\sigma} \cdot \mathbf{n}$ on S_n and B_m and finally stop the iterative procedure as soon as both (6) and (7) are satisfied by the tested value of $\mathbf{X}$. Unfortunately, this method suffers from the following drawbacks:

(i) One needs to initiate the iterative scheme with a good enough guess value $\mathbf{X}_g$ but there is no efficient rule to choose it.

(ii) The liquid domain is not bounded and must therefore be truncated for a Finite Element implementation. However, it is well-known [3, 5] that a body $\mathcal{B}$ (here a solid particle $\mathcal{P}_n$ or a bubble $\mathcal{B}_m$) induces a fluid velocity that exhibits a weak $1/\lambda$ decay at any point M located far from $\mathcal{B}$ with $\lambda = |O_n M|$ if $\mathcal{B} = \mathcal{P}_n$ and $\lambda = |C_m M|$ if $\mathcal{B} = \mathcal{B}_m$, respectively. This behavior requires to use a large truncated fluid domain and this would yield tremendously cpu time consuming computations when iteratively appealing to the Finite Element Code.

(iii) For any tested entry $\mathbf{X}$ the Finite Element Code would produce $\mathbf{u}$ and p in the truncated liquid domain at a given accuracy. However, one further needs to evaluate the associated surface traction $\boldsymbol{\sigma} \cdot \mathbf{n}$ on the entire cluster's surface (i.e., the Cartesian derivatives of each velocity component $\mathbf{u} \cdot \mathbf{e}_i$) when dealing with (6)-(7) and this key step would actually yield a dramatic loss of accuracy.

Because of the previously alluded to significant drawbacks (i)-(iii), a quite different and boundary approach free from such troubles is then advocated in Sections 3 and 4.

3. A Key and Well-posed Linear System

This section derives a well-posed linear system for the $6N + 3M$ unknown Cartesian velocity components $U_j^{(n)} = \mathbf{U}^{(n)} \cdot \mathbf{e}_j$, $\Omega_j^{(n)} = \boldsymbol{\Omega}^{(n)} \cdot \mathbf{e}_j$ and $U_j'^{(m)} = \mathbf{U}'^{(m)} \cdot \mathbf{e}_j$. The trick consists in using for $i = 1, 2, 3$; $n = 1, \ldots, N$ and $m = 1, \ldots, M$ this time $6N + 3M$ auxiliary Stokes flows $(\mathbf{u}_T^{(n),i}, p_T^{(n),i})$, $(\mathbf{u}_R^{(n),i}, p_R^{(n),i})$ and $(\mathbf{u}^{(m),i}, p^{(m),i})$ with stress tensors $\boldsymbol{\sigma}_T^{(n),i}$, $\boldsymbol{\sigma}_R^{(n),i}$ and $\boldsymbol{\sigma}^{(m),i}$ that obey (1), i.e., are free from body forces and quiescent far from the cluster, and fulfill the following specific boundary conditions:

$$\mathbf{u}_T^{(n),i} = \delta_{nn'}\mathbf{e}_i \text{ on } S_{n'}; \; \mathbf{u}_T^{(n),i} \cdot \mathbf{n} = 0 \text{ and } \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0} \text{ on } B'_m, \tag{8}$$

$$\mathbf{u}_R^{(n),i} = \delta_{nn'}\mathbf{e}_i \wedge \mathbf{O_n M} \text{ on } S_{n'};\ \mathbf{u}_R^{(n),i} \cdot \mathbf{n} = 0 \text{ and } \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0} \text{ on } B'_m, \quad (9)$$

$$\mathbf{u}^{(m),i} = \mathbf{0} \text{ on } S_{n'};\ \mathbf{u}^{(m),i} \cdot \mathbf{n} = \delta_{mm'}\mathbf{e}_i \cdot \mathbf{n} \text{ and } \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0} \text{ on } B'_m, \quad (10)$$

with δ_{kl} the Kronecker delta. Clearly, $\mathbf{u}_T^{(n),i}$ and $\mathbf{u}_R^{(n),i}$ are the flows occurring when all the bubbles and solid bodies are motionless except $\mathcal{P}_n$ that translates or rotates respectively at the velocity $\mathbf{e}_i$. Subscripts T and R are thus employed for a translation and a rotation, respectively, of only one solid body. In a similar fashion, only $\mathcal{B}_m$ is not at rest and translates at the velocity $\mathbf{e}_i$ for the flow $\mathbf{u}^{(m),i}$. Using the above flows $\mathbf{u}_T^{(n),i}$, $\mathbf{u}_R^{(n),i}$ and $\mathbf{u}^{(m),i}$ makes it possible to express the conditions (6)-(7). This is achieved by exploiting the key Lorentz reciprocal identity [3, 5]

$$\int_S \mathbf{u} \cdot \boldsymbol{\sigma}' \cdot \mathbf{n} dS = \int_S \mathbf{u}' \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS \quad (11)$$

that holds for two arbitrary Stokes flows $(\mathbf{u}, p)$ and $(\mathbf{u}', p')$ with stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ obeying (1) with S the entire surface of the cluster, i.e., such that $S = S_s \cup S_b$ if $S_s = \cup_{n=1}^{N} S_n$ and $S_b = \cup_{m=1}^{M} B_m$ henceforth denote the entire solid and non-solid surfaces, respectively.

Exploiting the conditions $\mathbf{u}_T^{(n),i} = \delta_{nn'}\mathbf{e}_i$ on $S_{n'}$, $\mathbf{u}_T^{(n),i} \cdot \mathbf{n} = 0$ and $\boldsymbol{\sigma} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0}$ on S_b and finally the identity (11) for $(\mathbf{u}', p') = (\mathbf{u}_T^{(n),i}, p_T^{(n),i})$ yields

$$\int_{S_n} \mathbf{e}_i \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_{S_s} \mathbf{u}_T^{(n),i} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_S \mathbf{u}_T^{(n),i} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS$$
$$= \int_S \mathbf{u} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} dS. \quad (12)$$

In a similar fashion, using the boundary conditions (9), one also obtains

$$\mathbf{e}_i \cdot \left\{ \int_{S_n} \mathbf{O_n M} \wedge \cdot\, \boldsymbol{\sigma} \cdot \mathbf{n} dS \right\} = \int_{S_n} (\mathbf{e}_i \wedge \mathbf{O_n M}) \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_S \mathbf{u} \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} dS. \quad (13)$$

For $L \in \{T, R\}$ recall that $\boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0}$ on each surface $B_{m'}$. Therefore, $\mathbf{u} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} = (\mathbf{u} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n})$ on the surface S_b. Taking into account the conditions (2)-(3) for $\mathbf{u}$ and adopting the following definitions:

$$A_{(n'),T}^{(n),i,j} = \int_{S_{n'}} \mathbf{e}_j \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} dS,$$

$$B_{(n'),T}^{(n),i,j} = \int_{S_{n'}} (\mathbf{e}_j \wedge \mathbf{O}_{\mathbf{n}'}\mathbf{M}) \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} dS, \tag{14}$$

$$A_{(n'),R}^{(n),i,j} = \int_{S_{n'}} \mathbf{e}_j \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} dS,$$

$$B_{(n'),R}^{(n),i,j} = \int_{S_{n'}} (\mathbf{e}_j \wedge \mathbf{O}_{\mathbf{n}'}\mathbf{M}) \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} dS, \tag{15}$$

$$C_{(m'),T}^{(n),i,j} = \int_{B_{m'}} (\mathbf{e}_j \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n}) dS,$$

$$C_{(m'),R}^{(n),i,j} = \int_{B_{m'}} (\mathbf{e}_j \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n}) dS, \tag{16}$$

in conjunction with the equalities (12)-(13) thus easily yields (under the standard tensor summation convention) the relations

$$\int_{S_n} \mathbf{e}_i \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = A_{(n'),T}^{(n),i,j} U_j^{(n')} + B_{(n'),T}^{(n),i,j} \Omega_j^{(n')} + C_{(m'),T}^{(n),i,j} U_j'^{(m')}, \tag{17}$$

$$\mathbf{e}_i \cdot \left\{ \int_{S_n} \mathbf{O}_{\mathbf{n}}\mathbf{M} \wedge \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS \right\} = A_{(n'),R}^{(n),i,j} U_j^{(n')} + B_{(n'),R}^{(n),i,j} \Omega_j^{(n')} + C_{(m'),R}^{(n),i,j} U_j'^{(m')}. \tag{18}$$

Let us now deal with the conditions (7). Since $\boldsymbol{\sigma} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0}$ and $\mathbf{u}^{(m),i} \cdot \mathbf{n} = \delta_{mm'} \mathbf{e}_i \cdot \mathbf{n}$ on each $B_{m'}$ we readily obtain

$$\int_{B_m} \mathbf{e}_i \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_{B_m} (\mathbf{u}^{(m),i} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}) dS$$

$$= \int_{S_b} (\mathbf{u}^{(m),i} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}) dS. \tag{19}$$

Using successively the equality $\boldsymbol{\sigma} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0}$ on S_b, the property $\mathbf{u}^{(m),i} = \mathbf{0}$ on S_s, the identity (11) and the condition $\boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} \wedge \mathbf{n} = \mathbf{0}$ on S_b it follows that

$$\int_{B_m} \mathbf{e}_i \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_{S_b} \mathbf{u}^{(m),i} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS$$

$$= \int_{S} \mathbf{u}^{(m),i} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_{S} \mathbf{u} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} dS$$

$$= \int_{S_s} \mathbf{u} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} dS + \int_{S_b} (\mathbf{u} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}) dS. \quad (20)$$

Upon introducing the new coefficients

$$D_{(n'),T}^{(m),i,j} = \int_{S_{n'}} \mathbf{e}_j \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} dS, \; D_{(n'),R}^{(m),i,j} = \int_{S_{n'}} (\mathbf{e}_j \wedge \mathbf{O_n M}) \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} dS, \quad (21)$$

$$F_{(m')}^{(m),i,j} = \int_{B_{m'}} (\mathbf{e}_j \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}) dS, \quad (22)$$

and using the boundary conditions (2)-(3) for $\mathbf{u}$ one thus arrives at

$$\int_{B_m} \mathbf{e}_i \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = D_{(n),T}^{(m),i,j} U_j^{(n')} + D_{(n),R}^{(m),i,j} \Omega_j^{(n')} + F_{(m')}^{(m),i,j} U_j'^{(m')}. \quad (23)$$

In summary, the previous relations (17)-(18) and (23) yield, when combined with the conditions (6)-(7), the following $(6N + 3M)$-equation linear system ($i = 1, 2, 3$; $n = 1, \ldots, N$ and $m = 1, \ldots, M$)

$$A_{(n'),T}^{(n),i,j} U_j^{(n')} + B_{(n'),T}^{(n),i,j} \Omega_j^{(n')} + C_{(m'),T}^{(n),i,j} U_j'^{(m')} = (\rho \mathcal{V}_n - M_n) \mathbf{g} \cdot \mathbf{e}_i, \quad (24)$$

$$A_{(n'),R}^{(n),i,j} U_j^{(n')} + B_{(n'),R}^{(n),i,j} \Omega_j^{(n')} + C_{(m'),R}^{(n),i,j} U_j'^{(m')} = \rho \mathcal{V}_n [\mathbf{O_n O'_n} \wedge \mathbf{g}] \cdot \mathbf{e}_i, \quad (25)$$

$$D_{(n'),T}^{(m),i,j} U_j^{(n')} + D_{(n'),R}^{(m),i,j} \Omega_j^{(n')} + F_{(m')}^{(m),i,j} U_j'^{(m')} = 4\pi (a'_m)^3 \rho \mathbf{g} \cdot \mathbf{e}_i / 3, \quad (26)$$

for the unknown Cartesian velocity components $U_j^{(n')}$, $\Omega_j^{(n')}$ and $U_j'^{(m')}$. If one denotes by $\mathbf{A}$ the matrix associated to the second-rank tensor $A^{i,j} \mathbf{e}_i \otimes \mathbf{e}_j$ and by ${}^t\mathbf{X}$ the transposed value of $\mathbf{X}$ the equations (24)-(26) also admit the condensed notation ${}^t\mathbf{Y} = \mathcal{M} \cdot {}^t\mathbf{X}$ with

$$\mathbf{Y} = ((\rho \mathcal{V}_1 - M_1)\mathbf{g}, \ldots, (\rho \mathcal{V}_N - M_M)\mathbf{g}; \rho \mathcal{V}_1 \mathbf{O_1 O'_1} \wedge \mathbf{g}, \ldots, \rho \mathcal{V}_N \mathbf{O_N O'_N} \wedge \mathbf{g};$$
$$4\pi (a'_1)^3 \rho \mathbf{g}, \ldots, 4\pi (a'_M)^3 \rho \mathbf{g})$$

and the following real-valued matrix:

$$\mathcal{M} = \begin{pmatrix} \mathbf{A}_{(1),T}^{(1)} & \cdots & \mathbf{A}_{(N),T}^{(1)} & \mathbf{B}_{(1),T}^{(1)} & \cdots & \mathbf{B}_{(N),T}^{(1)} & \mathbf{C}_{(1),T}^{(1)} & \cdots & \mathbf{C}_{(M),T}^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{(1),T}^{(N)} & \cdots & \mathbf{A}_{(N),T}^{(N)} & \mathbf{B}_{(1),T}^{(N)} & \cdots & \mathbf{B}_{(N),T}^{(N)} & \mathbf{C}_{(1),T}^{(N)} & \cdots & \mathbf{C}_{(M),T}^{(N)} \\ \mathbf{A}_{(1),R}^{(1)} & \cdots & \mathbf{A}_{(N),R}^{(1)} & \mathbf{B}_{(1),R}^{(1)} & \cdots & \mathbf{B}_{(N),R}^{(1)} & \mathbf{C}_{(1),R}^{(1)} & \cdots & \mathbf{C}_{(M),R}^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{(1),R}^{(N)} & \cdots & \mathbf{A}_{(N),R}^{(N)} & \mathbf{B}_{(1),R}^{(N)} & \cdots & \mathbf{B}_{(N),R}^{(N)} & \mathbf{C}_{(1),R}^{(N)} & \cdots & \mathbf{C}_{(M),R}^{(N)} \\ \mathbf{D}_{(1),T}^{(1)} & \cdots & \mathbf{D}_{(N),T}^{(1)} & \mathbf{D}_{(1),R}^{(1)} & \cdots & \mathbf{D}_{(N),R}^{(1)} & \mathbf{F}_{(1)}^{(1)} & \cdots & \mathbf{F}_{(M)}^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{D}_{(1),T}^{(M)} & \cdots & \mathbf{D}_{(N),T}^{(M)} & \mathbf{D}_{(1),R}^{(M)} & \cdots & \mathbf{D}_{(N),R}^{(M)} & \mathbf{F}_{(1)}^{(M)} & \cdots & \mathbf{F}_{(M)}^{(M)} \end{pmatrix}. \quad (27)$$

Let us prove that $\mathcal{M}$ is symmetric and negative-definite. First, we note that (12) also holds for the flows $\mathbf{u}_T^{(n'),j}$ and $\mathbf{u}_R^{(n'),j}$, i.e., not only the flow $\mathbf{u}$. By virtue of (14)-(15), one thus arrives at

$$A_{(n),T}^{(n'),j,i} = \int_S \mathbf{u}_T^{(n'),j} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} dS, \; A_{(n),R}^{(n'),j,i} = \int_S \mathbf{u}_R^{(n'),j} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} dS. \quad (28)$$

On this side (13) actually also holds for the flows $\mathbf{u}_T^{(n'),j}$ and $\mathbf{u}_R^{(n'),j}$ and a similar treatment permits us to obtain

$$B_{(n),T}^{(n'),j,i} = \int_S \mathbf{u}_T^{(n'),j} \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} dS, \; B_{(n),R}^{(n'),j,i} = \int_S \mathbf{u}_R^{(n'),j} \cdot \boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} dS. \quad (29)$$

Now, applying the key identity (11) to each integral arising in (28)-(29) immediately establishes the following properties:

$$A_{(n),T}^{(n'),j,i} = A_{(n'),T}^{(n),i,j}, \quad A_{(n),R}^{(n'),j,i} = B_{(n'),T}^{(n),i,j},$$

$$B_{(n),T}^{(n'),j,i} = A_{(n'),T}^{(n),i,j}, \quad B_{(n),R}^{(n'),j,i} = B_{(n'),R}^{(n),i,j}. \quad (30)$$

Furthermore, using the definitions (16), (21) and the conditions (8)-(10) makes it possible to show for $L = T, R$ that

$$C_{(m'),L}^{(n),i,j} = \int_S (\mathbf{u}^{(m'),j} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n}) dS = \int_S \mathbf{u}^{(m'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS, \quad (31)$$

$$D_{(n),L}^{(m'),j,i} = \int_S (\mathbf{u}_L^{(n),i} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}^{(m'),j} \cdot \mathbf{n}) dS = \int_S \mathbf{u}^{(n),i} \cdot \boldsymbol{\sigma}_L^{(m'),j} \cdot \mathbf{n} dS. \quad (32)$$

By virtue of (11), it follows that

$$D_{(n),T}^{(m'),j,i} = C_{(m'),T}^{(n),i,j}, \quad D_{(n),R}^{(m'),j,i} = C_{(m'),R}^{(n),i,j}. \tag{33}$$

Finally, exploiting the conditions (10) for the flows $u^{(m),i}$ and $u^{(m'),j}$ in conjunction with the definition (22) and the identity (11) gives

$$F_{(m')}^{(m),i,j} = \int_S (\mathbf{u}^{(m'),j} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n})dS$$

$$= \int_S (\mathbf{u}^{(m),i} \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}^{(m'),j} \cdot \mathbf{n})dS = F_{(m)}^{(m'),j,i}. \tag{34}$$

In summary, the relations (30), (33) and (34) readily show that $\mathcal{M}$ is symmetric. Let us now consider, for an arbitrary generalized velocity $\mathbf{X} = (\mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(N)}; \boldsymbol{\Omega}^{(1)}, \ldots, \boldsymbol{\Omega}^{(N)}; \mathbf{U}'^{(1)}, \ldots, \mathbf{U}'^{(M)})$, the flow $(\mathbf{u}, p)$ subject to (1)-(3). Since (1) holds the rate E of dissipation of mechanical energy for this flow satisfies [3]

$$E = \int_S \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}dS < 0. \tag{35}$$

In order to express E we note that, by superposition (under the tensor summation convention) we have

$$\mathbf{u} = U_j^{(n')}\mathbf{u}_T^{(n'),j} + \Omega_j^{(n')}\mathbf{u}_R^{(n'),j} + U_j'^{(m')}\mathbf{u}^{(m'),j}, \tag{36}$$

$$\boldsymbol{\sigma} = U_i^{(n)}\boldsymbol{\sigma}_T^{(n),i} + \Omega_i^{(n)}\boldsymbol{\sigma}_R^{(n),i} + U_i'^{(m)}\boldsymbol{\sigma}^{(m),i}. \tag{37}$$

Accordingly, $E = I_T^{(n),i}U_i^{(n)} + I_R^{(n),i}\Omega_i^{(n)} + I^{(m),i}U_i'^{(m)}$ with for $L = T, R$,

$$I_L^{(n),i} = \int_S \mathbf{u} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n}dS, \quad I^{(m),i} = \int_S \mathbf{u} \cdot \boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}dS. \tag{38}$$

Owing to the decomposition (36) note that

$$I_L^{(n),i} = U_j^{(n')} \int_S \mathbf{u}_T^{(n'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n}dS + \Omega_j^{(n')} \int_S \mathbf{u}_R^{(n'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n}dS$$

$$+ U_j'^{(m')} \int_S \mathbf{u}^{(m'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n}dS \tag{39}$$

with, by virtue of the boundary conditions (8)-(10), the relations

$$\int_S \mathbf{u}_T^{(n'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS = \int_{S'_n} \mathbf{e}_j \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS, \tag{40}$$

$$\int_S \mathbf{u}_R^{(n'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS = \int_{S'_n} (\mathbf{e}_j \wedge \mathbf{O}'_\mathbf{n}\mathbf{M}) \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS, \tag{41}$$

$$\int_S \mathbf{u}^{(m'),j} \cdot \boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n} dS = \int_{B'_m} (\mathbf{e}_j \cdot \mathbf{n})(\mathbf{n} \cdot \boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n}) dS. \tag{42}$$

Combining (39) with (40)-(42) and the definitions (14)-(16) thus yields

$$I_L^{(n),i} = A_{(n'),L}^{(n),i,j} U_j^{(n')} + B_{(n'),L}^{(n),i,j} \Omega_j^{(n')} + C_{(m'),L}^{(n),i,j} U_j'^{(m')} \text{ for } L = T,\ R. \tag{43}$$

Adopting the same method, the reader may easily establish that

$$I^{(m),i} = D_{(n),T}^{(m),i,j} U_j^{(n')} + D_{(n),R}^{(m),i,j} \Omega_j^{(n')} + F_{(m')}^{(m),i,j} U_j'^{(m')}. \tag{44}$$

Hence, the inequality (35) shows that for any generalized velocity $\mathbf{X}$,

$$E = I_T^{(n),i} U_i^{(n)} + I_R^{(n),i} \Omega_i^{(n)} + I^{(m),i} U_i'^{(m)} = \mathbf{X} \cdot \mathcal{M} \cdot {}^t\mathbf{X} < 0. \tag{45}$$

By virtue of (45), the real-valued and symmetric matrix $\mathcal{M}$ is negative-definite and the system (24)-(26) therefore admits a unique solution $\mathbf{X}$. This property establishes the well posedness of the advocated general theory which consists, when solving (24)-(26) under the definitions (14)-(16) and (21)-(22), in only evaluating the surface tractions $\boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n}$, $\boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n}$ and $\boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}$ on the entire surface S of the cluster and thereafter circumvents the calculation of the fluid flow about the cluster when determining the motion of the solid bodies and spherical bubbles. Note that our approach is free from all the drawbacks (i)-(iii) alluded to at the end of Section 2. The next section gives the boundary-integral equations that govern the required surface quantities $\boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n}$, $\boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n}$ and $\boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}$.

4. Relevant Boundary-integral Equations

First, let us recall [9] that the velocity $\mathbf{u} = u_j \mathbf{e}_j$ of a Stokes flow

$(\mathbf{u}, p)$ subject to (1) and exerting on S the surface traction $\boldsymbol{\sigma} \cdot \mathbf{n}$, obeys in the whole fluid domain Ω the following integral representation:

$$u_j(M) = \frac{1}{8\pi} \int_S \left\{ u_k(P) T_{kjl}(P, M) n_l(P) - G_{jk}(P, M) \left[\frac{\mathbf{e}_k \cdot \boldsymbol{\sigma} \cdot \mathbf{n}}{\mu} \right](P) \right\} dS \quad (46)$$

if one sets $\mathbf{n} = n_l \mathbf{e}_l$ on S and denotes by $G_{jk}(P, M)$ and $T_{kjl}(P, M)$ the Oseen-Burgers second-rank free-space Green's tensor and the associated third-rank stress tensor Cartesian components, respectively, such that

$$G_{jk}(P, M) = \frac{\delta_{jk}}{PM} + \frac{[\mathbf{PM} \cdot \mathbf{e}_k][\mathbf{PM} \cdot \mathbf{e}_j]}{PM^3}, \quad (47)$$

$$T_{kjl}(P, M) = \frac{6[\mathbf{PM} \cdot \mathbf{e}_k][\mathbf{PM} \cdot \mathbf{e}_j][\mathbf{PM} \cdot \mathbf{e}_l]}{PM^5}. \quad (48)$$

By virtue of (46), (36)-(37) and (24)-(26) one is therefore able to compute the fluid velocity $\mathbf{u}$ about the migrating cluster by solely evaluating (for $L = T, R$) both the surface tractions $\boldsymbol{\sigma}_L^{(n),i} \cdot \mathbf{n}$, $\boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n}$ and the velocities $u_L^{(n),i}$, $u^{(m),i}$ on the entire surface S. In order to obtain all those surface quantities it is fruitful to let M tend to S in the relation (46) associated to an arbitrary Stokes flow $\mathbf{v} = v_j \mathbf{e}_j$ with pressure q, subject to (1) and exerting on S the surface traction $\mu \mathbf{f} = \mu f_k \mathbf{e}_k$. Curtailing elementary manipulations, one arrives for M located on S at the key identity

$$8\pi v_j(M) = \int_{S'} [v_k(P) - v_k(\mathbf{x})] T_{kjl}(P, M) n_l(P) dS$$
$$+ \int_{S \setminus S'} v_k(P) T_{kjl}(P, M) n_l(P) dS - \int_S G_{jk}(P, M) f_k(P) dS \quad (49)$$

with $S' = S_n$ (respectively $S' = B_m$) if M lies on S_n (respectively B_m). Clearly, the relation (49) provides a link between the velocity $\mathbf{v}$ and the vector $\mathbf{f}$ on the boundary S. Henceforth, we assume that $\mathbf{v}$ is one auxiliary flow $u_L^{(n),i}$ or $u^{(m),i}$ and introduce on the non-solid surface S_b the functions d, a and the vector $\mathbf{a} = a_k \mathbf{e}_k$ such that

$$\mathbf{v} = d\mathbf{n} + \mathbf{a} \text{ and } \mathbf{a} \cdot \mathbf{n} = 0, \ \mathbf{f} = a\mathbf{n}. \quad (50)$$

Of course (recall (8)-(10)) both $\mathbf{v}$ and d are prescribed on surfaces S_s and S_b, respectively. The unknown quantities $(a, \mathbf{a})$ on S_b and $\mathbf{f} = f_k\mathbf{e}_k$ on S_s are found, owing to (49), to satisfy the coupled boundary-integral equations

$$\int_{S_b} a_k(P)T_{kjl}(P, M)n_l(P)dS - \int_{S_b} G_{jk}(P, M)[an_k](P)dS$$

$$- \int_{S_s} G_{jk}(P, M)f_k(P)dS = s_j^n[d, \mathbf{v}](M) \text{ for } M \text{ on } S_n, \tag{51}$$

$$8\pi a_j(M) - \int_{B_m} [a_k(P) - a_k(M)]T_{kjl}(P, M)n_l(P)dS$$

$$- \int_{S_b \setminus B_m} a_k(P)T_{kjl}(P, M)n_l(P)dS - \int_{S_b} G_{jk}(P, M)[an_k](P)dS$$

$$- \int_{S_s} G_{jk}(P, M)f_k(P)dS = t_j^m[d, \mathbf{v}](M) \text{ for } M \text{ on } B_m, \tag{52}$$

with the definitions

$$s_j^n[d, \mathbf{v}](M) = 8\pi v_j(M) - \int_{S_n} [v_k(P) - v_k(M)]T_{kjl}(P, M)n_l(P)dS$$

$$- \int_{S_s \setminus S_n} v_k(P)T_{kjl}(P, M)n_l(P)dS$$

$$- \int_{S_b} [dn_k](P)T_{kjl}(P, M)n_l(P)dS, \tag{53}$$

$$t_j^m[d, \mathbf{v}](M) = \int_{B_m} [(dn_k)(P) - (dn_k)(M)]T_{kjl}(P, M)n_l(P)dS$$

$$+ \int_{S_b \setminus B_m} [dn_k](P)T_{kjl}(P, M)n_l(P)dS$$

$$+ \int_{S_s} v_k(P)T_{kjl}(P, M)n_l(P)dS. \tag{54}$$

The general theory advocated in this paper hence finally reduces to the treatment of $6N + 3M$ boundary-integral equations (51)-(52) associated to the following cases:

(i) $\mathbf{v} = \delta_{nn'}\mathbf{e}_i$ on $S_{n'}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on S_b. In this case one obtains $\boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} = \mu\mathbf{f}$ on S_s; $\mathbf{u}_T^{(n),i} = \mathbf{a}$ and $\boldsymbol{\sigma}_T^{(n),i} \cdot \mathbf{n} = \mu a\mathbf{n}$ on S_b.

(ii) $\mathbf{v} = \delta_{nn'}(\mathbf{e}_i \wedge \mathbf{O_n M})$ on $S_{n'}$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on S_b. In this case one obtains $\boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} = \mu\mathbf{f}$ on S_s; $\mathbf{u}_R^{(n),i} = \mathbf{a}$ and $\boldsymbol{\sigma}_R^{(n),i} \cdot \mathbf{n} = \mu a\mathbf{n}$ on S_b.

(iii) $\mathbf{v} = \mathbf{0}$ on S_s and $\mathbf{v} \cdot \mathbf{n} = \delta_{mm'}\mathbf{e}_i \cdot \mathbf{n}$ on $B_{m'}$. In this case one obtains $\boldsymbol{\sigma}^{(m),i} \cdot \mathbf{n} = \mu\mathbf{f}$ on S_s; $\mathbf{u}^{(m),i} = \mathbf{a} + \delta_{mm'}(\mathbf{e}_i \cdot \mathbf{n})\mathbf{n}$ and $\boldsymbol{\sigma}^{(n),i} \cdot \mathbf{n} = \mu a\mathbf{n}$ on each $B_{m'}$.

Before closing this section, let us mention that for any closed surface $\mathcal{S}$ (for example, $\mathcal{S} = S_n$ or $\mathcal{S} = B_m$) and any point M located on $\mathcal{S}$, inside or outside the domain bounded by $\mathcal{S}$ the following identities hold [9]:

$$\int_{\mathcal{S}} G_{jk}(P, M)n_k(P)dS = 0, \quad j = 1, 2, 3. \tag{55}$$

Accordingly, the boundary-integral equations (51)-(52) do not admit a unique solution $(a, \mathbf{a})$ on S_b and $\mathbf{f}$ on S_s: one can readily add to a an arbitrary constant on each surface B_m and to $\mathbf{f}$ an arbitrary multiple of the unit vector $\mathbf{n}$ on each surface S_n. Note however that the coefficients introduced by (14)-(16) are not sensitive to those arbitrary quantities, i.e., that our system (24)-(26) and therefore its unique solution $\mathbf{X}$ are unchanged.

5. Concluding Remarks

A general procedure has been established to determine the migration of a collection of $M \geq 1$ spherical bubbles and $N \geq 1$ arbitrary-shaped solid bodies subject to the gravity and the resulting velocity field about the cluster. The proposed theory is proved to be well-posed and solely appeals to the knowledge of the velocity and the surface traction arising on the entire cluster's surface S for $6N + 3M$ auxiliary Stokes flows. These key surface quantities are governed by coupled boundary-integral equations on S and one thus only needs to mesh the cluster's surface when achieving a numerical implementation of the theory. Such a suitable implementation is under current progress and resorts, as achieved in [12] for only N solid bodies and different boundary-integral equations, to standard boundary elements.

Appendix

This Appendix introduces the adopted boundary conditions (2) on the surface S_n of a solid particle $\mathcal{P}_n$ and (3) on the surface B_m of the bubble $\mathcal{B}_m$ as follows:

(i) The solid $\mathcal{P}_n$ rotates at the unknown angular velocity $\mathbf{\Omega}^{(n)}$ and its center of mass O_n translates at the unknown velocity $\mathbf{U}^{(n)}$ with respect to the selected Cartesian framework (O, x_1, x_2, x_3). Since the liquid is viscous one thus immediately prescribes at S_n the no-slip velocity boundary condition (2).

(ii) The bubble $\mathcal{B}_m$ only translates at the velocity $\mathbf{U}'_m$ and the condition (3) on the normal fluid velocity $\mathbf{u} \cdot \mathbf{n}$ arises from the impenetrability requirement. The remaining condition of zero tangential stress on B_m is the balance of tangential stress at the bubble-liquid interface while the normal stress balance is ignored since we assume a surface tension high enough to keep the bubble spherical [10].

References

[1] F. Feuillebois, Thermocapillary migration of two equal bubbles parallel to their line of centers, J. Colloid Interface Sci. 131 (1989), 267-274.

[2] M. J. Gluckman, R. Pfeffer and R. Weinbaum, A new technique for treating multiparticle slow viscous flow, axisymmetric flow past spheres and spheroids, J. Fluid Mech. 50 (1971), 705-740.

[3] J. Happel and H. Brenner, Low Reynolds Number Hydrodynamics, Martinus Nijhoff, 1973.

[4] Q. Hassonjee, P. Ganatos and R. Pfeffer, A strong-interaction theory for the motion of arbitrary three-dimensional clusters of spherical particles at low Reynolds number, J. Fluid Mech. 197 (1988), 1-37.

[5] S. Kim and S. J. Karrila, Microhydrodynamics: Principles and Selected Applications, Butterworth, 1991.

[6] W. H. Liao and D. A. Krueger, Multipole expansion calculation of slow viscous flow about spheroids of different sizes, J. Fluid Mech. 96 (1980), 223-241.

[7] M. Meyyappan, W. Wilcox and R. S. Subramanian, The slow axisymmetric motion of two bubbles in a thermal gradient, J. Colloid Interface Sci. 94 (1983), 243-257.

[8] M. Meyyappan, W. Wilcox and R. S. Subramanian, The thermocapillary motion of two bubbles oriented arbitrarily relative to a thermal gradient, J. Colloid Interface Sci. 97 (1984), 291-294.

[9] C. Pozrikidis, Boundary Integral and Singularity Methods for Linearized Viscous Flow, Cambridge University Press, 1992.

[10] J. V. Satrape, Interactions and collisions of bubbles in thermocapillary motion, Phys. Fluids A 4 (1992), 1883-1900.

[11] A. Sellier, On the capillary motion of arbitrary clusters of spherical bubbles, Part 1, General theory, J. Fluid Mech. 197 (2004), 391-401.

[12] A. Sellier, On the slow gravity-driven migration of arbitrary clusters of small solid particles, Comptes Rendus Mécanique 332(12) (2004), 987-992.

[13] H. Wei and R. S. Subramanian, Thermocapillary migration of a small chain of bubbles, Phys. Fluids A 5 (1993), 1583-1595.

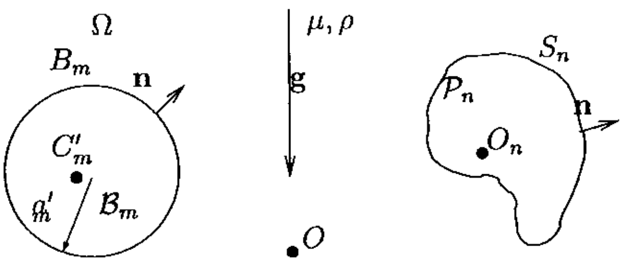

Figure 1. Employed notations for the addressed cluster

■