Settling motion of interacting solid particles in the vicinity of a plane solid boundary

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Received 5 November 2004; accepted after revision 14 February 2005
Available online 8 April 2005
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Abstract

The sedimentation of \( N \geq 1 \) small arbitrarily-shaped solid bodies near a solid plane is addressed by discarding inertial effects and using \( 6N \) boundary-integral equations. Numerical results for 2 or 3 identical spheres reveal that combined wall-particle and particle-particle interactions deeply depend on the cluster’s geometry and distance to the wall and may even cancel for a sphere which then moves as it were isolated. To cite this article: A. Sellier, C. R. Mecanique 333 (2005). © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Sédimentation d’un ensemble de particules solides en présence d’une paroi solide plane. La sédimentation en régime de Stokes de \( N \geq 1 \) corps solides quelconques situés près d’une paroi plane est étudiée à l’aide de \( 6N \) équations de frontière. Les résultats pour 2 ou 3 sphères identiques montrent que la résultante des interactions particule-particule et paroi-particule est très sensible à la disposition des sphères et peut même s’annuler pour l’une d’elles qui dans ce cas migre comme si elle était seule. Pour citer cet article : A. Sellier, C. R. Mecanique 333 (2005). © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Keywords: Fluid mechanics; Sedimentation; Particle-particle interactions; Wall-particle interactions; Boundary elements

Mots-clés : Mécánique des fluides ; Sédimentation ; Interactions particule-particule ; Interactions particule-paroi ; Éléments de frontière

1. Introduction

The new approach advocated in [1] to compute the low-Reynolds-number falling motions of \( N \geq 1 \) arbitrarily-shaped solid bodies investigates pure particle-particle interactions in sedimentation. However, boundaries are also
encountered in practice and the case of a solid wall has been only handled in [2,3] for \( N = 1 \) and for several spheres in axisymmetric motion in [4]. The present work thus investigates combined particle–particle and wall–particle interactions for arbitrary clusters lying near a plane solid wall \( \Sigma \) by extending [1]. This is achieved by using this time a Green’s tensor [5] that vanishes on \( \Sigma \) and therefore again solving \( 6N \) boundary-integral equations on the entire cluster’s surface.

2. Governing linear system

We look at \( N \geq 1 \) solid arbitrarily-shaped particle(s) \( \mathcal{P}_n \) (\( n = 1, \ldots, N \)) immersed in a Newtonian fluid of uniform viscosity \( \mu \) and density \( \rho \) above the solid and motionless \( x_3 = 0 \) plane \( \Sigma \). For example, the case of a few spheres is sketched in Fig. 1.

Under the uniform gravity \( g \) each \( \mathcal{P}_n \) with center of mass \( O_n \) settles with respect to the Cartesian frame \((O, x_1, x_2, x_3)\) at the unknown angular velocity \( \Omega^{(n)} \) and translational velocity \( \mathbf{U}^{(n)} \) (the velocity of \( O_n \)). The fluid and each \( \mathcal{P}_n \) with volume \( V_n \), center of volume \( O_n' \), mass \( M_n \) and surface \( S_n \) have negligible inertia. Hence, the liquid has at a current point \( M \) quasi-steady [1] velocity \( \mathbf{u} \), pressure \( p + \rho \mathbf{g} \cdot \mathbf{OM} \) and stress tensor \( \sigma \) that obey

\[
\mu \nabla^2 \mathbf{u} = \mathbf{V} p \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \ \Omega, \quad (\mathbf{u}, p) \to (0, 0) \quad \text{as} \ |\mathbf{OM}| \to \infty
\]

\[
\mathbf{u} = \mathbf{0} \quad \text{on} \ \Sigma \quad \text{and} \quad \mathbf{u} = \mathbf{U}^{(n)} + \mathbf{\Omega}^{(n)} \times \mathbf{O}_n \mathbf{M} \quad \text{on} \ S_n \quad (n \geq 1)
\]

\[
\int_{S_n} \sigma \cdot \mathbf{n} dS_n + (M_n - \rho \mathbf{V}_n) g = 0, \quad \int_{S_n} \mathbf{O}_n \mathbf{M} \cdot \sigma \cdot \mathbf{n} dS_n + \rho \mathbf{V}_n \mathbf{g} \cdot \mathbf{O}_n \mathbf{O}_n' = 0 \quad (n \geq 1)
\]

with \( \Omega \) the fluid domain and \( \mathbf{n} \) the unit outward normal on the cluster’s surface \( S = \bigcup_{n=1}^{N} S_n \). In order to rewrite (3), that requires zero net force and torque (with respect to \( O_n \)) on each \( \mathcal{P}_n \) of ignored inertia, let us introduce \( 6N \) flows \( (\mathbf{u}_L^{(n)}, \mathbf{p}_L^{(n)}) \) with stress tensor \( \sigma_{L}^{(n)} \) for \( L \in \{T, R\}, i \in \{1, 3\} \) and \( n = 1, \ldots, N \). Those flows fulfill (1) and the conditions

\[
\mathbf{u}_L^{(n), i} = \mathbf{0} \quad \text{on} \ \Sigma, \quad \mathbf{u}_L^{(n), i} = \mathbf{0} \quad \text{on} \ S_m \text{ if } m \neq n, \quad \mathbf{u}_T^{(n), i} = \mathbf{e}_i \quad \text{and} \quad \mathbf{u}_R^{(n), i} = \mathbf{e}_i \times \mathbf{O}_n \mathbf{M} \quad \text{on} \ S_n \quad (4)
\]

\[\begin{array}{c}
(a) \\
(b) \\
(c)
\end{array}\]

Fig. 1. Identical spheres near the \( x_3 = 0 \) solid plane \( \Sigma \): (a) 2-sphere cluster with \( O_1 O_2 \cdot \mathbf{e}_3 = 0 \); (b) 2-sphere cluster with \( O_1 O_2 \) normal to \( \Sigma \); (c) 3-sphere cluster with \( O_1 O_2 = O_1 O_3 = O_2 O_3, O_1 O_3 \cdot \mathbf{e}_3 = 0 \) and \( g = -g \mathbf{e}_3 \).

Fig. 1. Sphères identiques au voisinage du plan solide \( \Sigma \) (\( x_3 = 0 \)): (a) \( N = 2 \) et \( O_1 O_2 \cdot \mathbf{e}_3 = 0 \); (b) \( N = 2 \) et \( O_1 O_2 \) perpendiculaire à \( \Sigma \); (c) \( N = 3 \) et \( O_1 O_2 = O_1 O_3 = O_2 O_3, O_1 O_3 \cdot \mathbf{e}_3 = 0, g = -g \mathbf{e}_3 \).
Because $\mathbf{u} = \mathbf{0}$ on $\Sigma$ where $\mathbf{n} = e_3$ the usual reciprocal identity [6] provides, for any flow ($\mathbf{u}', p'$) with stress tensor $\sigma'$ satisfying the equations and far-field behavior (1), the relation

$$\int_{\Sigma} \mathbf{u}' \cdot \sigma' \cdot n \, dS = \int_{\Sigma} \mathbf{u} \cdot \sigma'' \cdot n \, dS = \sum_{m=1}^{N} \int_{\Sigma} \mathbf{u} \cdot \sigma' \cdot n \, dS_m$$

(5)

Upon introducing the vector $\mathbf{f}^{(m)\,i}_L = \sigma^{(m)\,i}_L \cdot n$ on $S$, the quantities $A^{(n)\,i,j}_L$ and $B^{(n)\,i,j}_L$ with

$$A^{(n)\,i,j}_L = - \int_{S_m} \mathbf{e}_j \cdot \mathbf{f}^{(n)\,i}_L \, dS_m, \quad B^{(n)\,i,j}_L = - \int_{S_m} (\mathbf{e}_j \wedge \mathbf{O}_m \mathbf{e}_l) \cdot \mathbf{f}^{(n)\,i}_L \, dS_m$$

(6)

and adopting henceforth the tensor summation convention with $\mathbf{U}^{(n)} = U_j^{(n)} \mathbf{e}_j$ and $\mathbf{\Omega}^{(n)} = \Omega_j^{(n)} \mathbf{e}_j$, the choice ($\mathbf{u}', p' = (\mathbf{u}^{(n)}_L, i), p^{(n)}_L$) in (5) easily shows that (3) becomes

$$\begin{cases} 
A^{(n)\,i,j}_L + B^{(n)\,i,j}_L \Omega_j^{(m)} e_j = (\mathbf{M}_n - \rho \mathbf{V}_n) e_j = T^{(n)}_j \\
A^{(n)\,i,j}_L \mathbf{U}_j^{(m)} + B^{(n)\,i,j}_L \mathbf{\Omega}_j^{(m)} e_j = \rho \mathbf{V}_n (\mathbf{g} \wedge \mathbf{O}_n \mathbf{O}_m) \cdot \mathbf{e} = \mathbf{C}^{(n)}
\end{cases}$$

(7)

(8)

Setting $\mathbf{Y} = (T^{(1)}, \ldots, T^{(N)}, C^{(1)}, \ldots, C^{(N)})$, the linear system (7), (8) with $6N \times 6N$ matrix $\mathbf{A}$ also reads $\mathbf{A} \cdot \mathbf{X} = \mathbf{V}$ with $\mathbf{X} = (U^{(1)}, \ldots, U^{(N)}, \mathbf{O}^{(1)}, \ldots, \mathbf{O}^{(N)})$ the unknown generalized velocity and $\mathbf{V}$ the transposed of $\mathbf{V}$. As seen by putting $(\mathbf{u}, p) = (\mathbf{u}^{(n)}_L, i), p^{(n)}_L$ and $(\mathbf{u}', p') = (\mathbf{u}^{(n)}_L, i), p^{(n)}_L$ in the first equality (5) the matrix $\mathbf{A}$ is symmetric. Moreover, if $V[\mathbf{u} \cdot \mathbf{e}_l] = V[u_i \mathbf{e}_j] = u_i \mathbf{e}_j$ and $e_{ij} = (u_{i,j} + u_{j,i})/2$, the divergence theorem and (1) yield

$$E := \int_{\Sigma} \mathbf{u} \cdot \sigma \cdot n \, dS = -2\mu \int_{\Omega} e_{ij} e_{ij} \, d\Omega < 0$$

(9)

Since (2) and (4) show that $\mathbf{e}_{n} = U_i^{(n)} e_i + \Omega_i^{(n)} e_j$ on $S$ and $\mathbf{u} = U_j^{(m)} e_j + \Omega_j^{(m)} (e_j \wedge \mathbf{O}_m \mathbf{e}_l)$ on $S_m$ it follows from (9), (2) and (6) that $E = -\mathbf{X} \cdot \mathbf{A} \cdot \mathbf{X} < 0$ whatever $\mathbf{X}$. Hence, $\mathbf{A}$ is not only real-valued and symmetric but also positive-definite and (7), (8) thus admit a unique solution $\mathbf{X}$, here obtained (see (6)) by solely solving the surface tractions $\mathbf{f}^{(n)\,i}_L$ on the multiply-connected (if $N \geq 2$) cluster's boundary $S$.

3. Relevant integral representations and boundary-integral equations

We denote by $M'(x_1, x_2, -x_3)$ the symmetric with respect to the plane $\Sigma$ of any point $M(x_1, x_2, x_3)$ located in $\Sigma \cup \Sigma' \cup \Sigma$ and introduce for $P$ on $S$ the pseudo-functions [5]

$$G_0^0 (P, M) = \delta_{jk} / PM + (PM' \cdot \mathbf{e}_j) / PM^3$$

(10)

$$G_0^1 (P, M) = - G_0^0 (P, M') - 2c_j [(OM \cdot \mathbf{e}_j) / PM^3] [\delta_{jk} PM' \cdot \mathbf{e}_j$$

$$- \delta_{kj} PM' \cdot \mathbf{e}_k + \mathbf{O}_P \cdot \mathbf{e}_3] [\delta_{jk} - 3(PM' \cdot \mathbf{e}_j) / PM^2]$$

(11)

with $c_1 = c_2 = 1, c_3 = -1$ and $\delta_{jk}$ the Kronecker delta. Extending in our case $N \geq 1$ the result obtained in [7,8] for a single particle it is found that $\mathbf{u}^{(n)\,i}_L$, subject to (1) and (4), then admits the key single-layer integral representation

$$-8\pi \mu (\mathbf{u}^{(n)\,i}_L \cdot \mathbf{e}_j) (M) = \int_{S} G_0^0 (P, M') [f^{(n)\,i}_L (P) \cdot \mathbf{e}_k] \, dS_m \quad \text{for} \quad M \in \Omega \cup \Omega' \cup \Sigma$$

(12)

The above key result (12) appeals to the following remarks and basic consequences:
(i) Of course $\mathbf{u}_{L}^{(n),i}$ vanishes on $\Sigma$ because $[G_{ij}^{(n)} + G_{ij}^{(P,M)}](P, M) = 0$ if $M$ lies on $\Sigma$ [5]. However, (12) in general also involves for $(u^p, p^p)$ subject to (1) and the property $\mathbf{u}^p = \mathbf{0}$ on $\Sigma$ an additional double-layer integral which only vanishes if $\mathbf{u}^p$ is a rigid-body motion on each $S_n$ (as is each $\mathbf{u}_{L}^{(n),i}$).

(ii) Each unknown traction $\mathbf{t}_{L}^{(n),i}$ obeys on $S$ a Fredholm boundary-integral equation of the first kind obtained by combining (4) and (12). One thus determines $\mathbf{X} = (\mathbf{U}^{(1)}, \ldots, \mathbf{U}^{N}, \mathbf{Q}^{(1)}, \ldots, \mathbf{Q}^{(N)})$ by solving $6N$ integral equations on the cluster's surface.

(iii) Once all vectors $\mathbf{t}_{L}^{(n),i}$ and $\mathbf{X}$ have been evaluated, (12) finally provides if necessary the velocity fields $\mathbf{u}_{L}^{(n),i}$ and therefore $\mathbf{u} = \mathbf{U}_{L}^{(n),i} + \mathbf{Q}_{R}^{(n),i}$ in the liquid domain $\Omega$.

4. Numerical method and preliminary results

As in [1], the integral equation (12) for $\mathbf{t}_{L}^{(n),i}$ is inverted by a boundary element technique [9] with 6-node isoparametric curved triangular elements and $N_{m}$ nodes on each $S_{n}$ and a $LU$ factorization algorithm to solve the discretized counterpart of (12). The procedure which readily recovers [1] far from the wall (see (10), (11)) is tested for a single spheroid with uniform density $\rho_{b}$, inequality $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} + \frac{\mu}{\lambda} \leq 2\left(1 + \frac{c}{a}\right)$ and separation ratio $h = H/(a_0) > 1$. Isolated ($h = \infty$) this body only translates for $g = g_{e_3}$ at the velocity $\mathbf{U}^{(1)} = g a^2(\rho_b - \rho) v(\epsilon)/\mu e_3$ with $v(1) = 2/9$ for a sphere and for oblate spheroids $[6]$

$$v(\epsilon) = \frac{p(p^2 + 3) \arctan(1/p) - p^3}{12}$$

with $p = \epsilon/(1 - \epsilon^2)^{1/2}$ and $0 < \epsilon < 1$  (13)

Symmetries and linearity confine the analysis to the settings $g = g_{e_1}$ and $g = -g_{e_3}$ with $g > 0$. For $\rho_b \neq \rho$ the non-zero Cartesian velocities, normalized by the velocity of the isolated spheroid and solely depending upon $(\epsilon, h)$, are found to be

$$u_1 = \frac{\mu a^2 \mathbf{U}^{(1)} \cdot e_1}{g(\rho_b - \rho)v(\epsilon)}, \quad u_2 = \frac{\mu a^2 \mathbf{Q}^{(1)} \cdot e_2}{g(\rho_b - \rho)v(\epsilon)} \quad \text{if } \mathbf{g} = g_{e_1}; \quad u_3 = \frac{\mu a^2 \mathbf{U}^{(1)} \cdot e_3}{g(\rho_b - \rho)v(\epsilon)} \quad \text{if } \mathbf{g} = -g_{e_3}$$

(14)

The computed values are compared in Table 1, for different $N_1$-node meshes on $S_1$, both with the analytical bipolar coordinates method [10] for a sphere ($\epsilon = 1$) and the numerical results of [3] for the $\epsilon = 1/2$ oblate spheroid.

Clearly, the agreement is excellent for the sphere and very good for the oblate spheroid. Actually, [2,3] kept in (12) the extra weakly-singular double-layer integral although (remind our remark (i) below (12)) it vanishes and this might explain the small observed discrepancies for $\epsilon = 1/2$.

Although the advocated procedure holds for $N \geq 1$ arbitrary bodies, we henceforth present results for clusters (see Fig. 1) of 2 or 3 identical spheres $P_n$ with center $O_n$, radius $a$ and uniform density $\rho_b \neq \rho$. We put 242 nodes

<table>
<thead>
<tr>
<th>$N_1$</th>
<th>$k$</th>
<th>$u_1; \epsilon = 1$</th>
<th>$u_2; \epsilon = 1$</th>
<th>$u_3; \epsilon = 1$</th>
<th>$u_4; \epsilon = 0.5$</th>
<th>$u_5; \epsilon = 0.5$</th>
<th>$u_6; \epsilon = 0.5$</th>
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<tbody>
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<td>74</td>
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<td>0.4463</td>
<td>0.0245</td>
<td>0.1087</td>
<td>0.6433</td>
<td>-0.0534</td>
<td>0.246</td>
</tr>
<tr>
<td>242</td>
<td>1.1</td>
<td>0.4424</td>
<td>0.0259</td>
<td>0.0886</td>
<td>0.6413</td>
<td>-0.0538</td>
<td>0.244</td>
</tr>
<tr>
<td>1058</td>
<td>1.1</td>
<td>0.4430</td>
<td>0.0270</td>
<td>0.0871</td>
<td>0.6411</td>
<td>-0.0538</td>
<td>0.244</td>
</tr>
<tr>
<td>[10, 3]</td>
<td>1.1</td>
<td>0.4430</td>
<td>0.0270</td>
<td>0.0873</td>
<td>0.6464</td>
<td>-0.0522</td>
<td>0.241</td>
</tr>
<tr>
<td>74</td>
<td>2.0</td>
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<td>0.0034</td>
<td>0.4726</td>
<td>0.7910</td>
<td>-0.0250</td>
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</tr>
<tr>
<td>242</td>
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<td>0.0035</td>
<td>0.4707</td>
<td>0.7890</td>
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<td>0.472</td>
</tr>
<tr>
<td>1058</td>
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<td>0.7232</td>
<td>0.0035</td>
<td>0.4705</td>
<td>0.7888</td>
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<tr>
<td>[10, 3]</td>
<td>2.0</td>
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<td>0.0035</td>
<td>0.4705</td>
<td>0.7892</td>
<td>-0.0252</td>
<td>0.472</td>
</tr>
</tbody>
</table>

Tableau 1

Vitesse adimensionées $u_1, u_2$ et $u_3$ (voir (14)) pour une sphère ($\epsilon = 1$) et un ellipsoïde de révolution aplati ($\epsilon = 1/2$) en fonction du nombre $N_1$ de points de collocation
Fig. 2. Normalized velocities $u$ and $w$ in Cases $k$ if $N = 2$ and $O_1O_2 \cdot e_3 = 0$ for $\delta = 0.1$ ($k = 1(\circ), k = 2(\ast)$, $k = 3(\ast\ast)$), $\delta = 0.5$ ($k = 1(\Theta)$, $k = 2(\Omega), k = 3(\bigcirc)$) and $\delta = 0.9$ ($k = 1(\Lambda), k = 2(\alpha), k = 3(\nu)$): (a) $u$ with dashed ($k = 1, 2$) and solid ($k = 3$) curves for $\delta = 0$; (b) $w$ with dashed ($k = 2, 3$) and solid ($k = 1$) curves for $\delta = 0$.

Fig. 2. Vitesses adimensionnées $u$ et $w$ dans les Cas $k$ si $N = 2$ et $O_1O_2 \cdot e_3 = 0$ pour $\delta = 0.1(k = 1(\circ), k = 2(\ast), k = 3(\ast\ast))$, $\delta = 0.5(k = 1(\Theta), k = 2(\Omega), k = 3(\bigcirc))$ et $\delta = 0.9(k = 1(\Lambda), k = 2(\alpha), k = 3(\nu))$. (a) $u$ avec des courbes en trait pointillé ($k = 1, 2$) et plein ($k = 3$) pour $\delta = 0$; (b) $w$ avec des courbes en trait pointillé ($k = 2, 3$) et plein ($k = 1$) pour $\delta = 0$.

on $S_n$ and introduce the positive wall–sphere and sphere–sphere separation parameters as $\delta = a/O_1 \cdot e_3 < 1$ and $\lambda = 2a/O_1O_2 < 1$, respectively. By linearity we consider the settings $g = g e_1$ (Case 1), $g = g e_2$ (Case 2), $g = -g e_3$ (Case 3) with $g > 0$ and use in Case $k$ the normalized velocities

$$u_i^{(n), (k)} = \frac{9\mu a^{-2}u_i^{(n)} \cdot e_i}{2g(\rho_1 - \rho) c_k}, \quad w_i^{(n), (k)} = \frac{9\mu a^{-2} \Omega_i^{(n)} \cdot e_i}{2g(\rho_2 - \rho) c_k},$$

with $c_1 = c_2 = 1$, $c_3 = -1$ (15)

For 2 spheres and $O_1O_2 \cdot e_3 = 0$ (see Fig. 1(a)) only $u = u_1^{(1), (k)} = u_2^{(2), (k)}$ in each Case $k$, $w = w_2^{(1), (1)} = w_2^{(2), (1)}$ in Case 1, $w = w_2^{(1), (2)} = -w_3^{(2), (2)}$ in Case 2 and $w = w_2^{(1), (3)} = -w_2^{(2), (3)}$ in Case 3 are non-zero. These quantities are plotted in Fig. 2 versus $\lambda$.

As seen in Fig. 2(a), pure wall–sphere ($\lambda = 0$) interactions slow down the spheres ($u < 1$) and increase with $\delta$ and pure sphere–sphere ($\delta = 0$) interactions speed up the spheres ($u > 1$) and increase with $\lambda$. For $\delta \lambda ω 0$ both interactions interact and $u - 1$ deeply depends on ($\delta, \lambda$). If $\delta = 0.1$ (all Cases $k$) and $\delta = 0.5$ (Case 1) we may have $u = 1$ (a sphere ignores the other one and $\Sigma$) or also $u > 1$ if $\lambda$ and $\delta$ are large and small enough, respectively. In other cases wall–particle interactions are dominant and spheres move slower than if isolated ($u < 1$). This actually occurs near the wall whatever $\lambda$ since $u$ then weakly depends on $\lambda$, as observed for $\lambda = 0.9$. Finally, note that $u$ strongly depends on Case $k$ and $u_1^{(1), (1)} > u_1^{(1), (2)} > u_1^{(1), (3)}$ for any pair ($\delta, \lambda$) with $\delta \lambda ω 0$. In Fig. 2(b) similar trends are obtained for $w$ with $w_1^{(2), (1)} > w_1^{(1), (3)} > w_1^{(1), (2)}$ and $w \rightarrow 0$ as $\lambda \rightarrow 1$ in Case 1 (not in Cases 2 or 3).

If $O_1O_2$ is normal to $\Sigma$ (see Fig. 1(b)) non-zero velocities read $u(n) = u_1^{(n), (1)} = u_2^{(n), (2)}$ in Case 1 (or 2) and $u(n) = u_3^{(n), (3)}$ in Case 3. As depicted in Fig. 3(a), $u(1) < u(2)$ in each Case $k$ for $\delta > 0$ since $P_2$ experiences stronger wall–sphere interactions than $P_2$. As in Fig. 2(a), $u(n)$ decreases as $\delta$ increases for any $\lambda$ and $P_n$ might ignore the other sphere ($u(n) = 1$) for $(n, \delta) = (1, 0.3)$ in Case 1 and $(n, \delta) = (2, 0.3)$ in Cases 1, 3. In addition, $u(n)$ is smaller in Case 3 than in Case 1 and $u(2)$ strongly decreases as $\lambda$ increases for $\delta = 0.9$.

Finally, we consider in Case 3 the 3-sphere cluster sketched in Fig. 1(c) by plotting in Fig. 3(b) the velocities $u(n) = u(n)^{(n), (3)}$ for $10\lambda = 1, 5, 9$. Clearly, $u(1)$ and $u(2) = u(3)$ decrease with $1/\delta$ or $\lambda$ and for a given sphere–sphere separation $\lambda$ there exist wall positions $\delta_1$ such that $u(1) = \delta_1$ such that $u(2) = 1$ and $\delta_2$ at which all spheres adopt the same velocity ($u(1) = u(2) > 1$) whereas $u(2) - u(1)$ has sign of $\delta - \delta_c$ for $\delta \neq \delta_c$. 
Fig. 3. (a) Normalized velocities $u(n)$ for $N = 2$ and $O_2O_3$ normal to $\Sigma$ in Case 1 for $\delta = 0.3$ (n = 1(3), n = 2(7)) or $\delta = 0.9$ (n = 1(5), n = 2(4)) and in Case 3 for $\delta = 0.3$ (n = 1(5), n = 2(7)) or $\delta = 0.9$ (n = 1(5), n = 2(7)); (b) velocities $u(n)$ versus $\delta$ for $N = 3$ in Case 3 if $\lambda = 0.1$ (n = 1(3), n = 2(7)), $\lambda = 0.5$ (n = 1(3), n = 2(7)) and $\lambda = 0.9$ (n = 1(5), n = 2(7)).

5. Conclusions

The proposed procedure has a reasonable cpu-time cost and may therefore be embedded in a Runge--Kutta march-in-time algorithm to track a time-dependent cluster's geometry as time evolves. This task is under investigation both for spheres and non-spherical bodies. As obtained in [3] for one spheroid, we expect to find equilibrium orientations of non-spherical particles for a few specific initial clusters.

References