On the slow gravity-driven migration of arbitrary clusters of small solid particles

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Abstract

A new approach is advocated to compute at a low cpu time cost the rigid-body motions of settling solid particles when inertial effects are negligible. In addition to the relevant boundary-integral equations, the numerical implementation and a few convincing benchmark tests we address two configurations of equivalent spheres and spheroids, i.e. that exhibit when isolated the same settling velocity. To cite this article: A. Sellier, C. R. Mecanique 332 (2004).

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Résumé

Sur la sédimentation d’une assemblée quelconque de particules solides. On propose une approche originale pour déterminer le mouvement d’une assemblée de particules solides et de formes arbitraires soumise à l’action de la pesanteur dans l’approximation de Stokes. Outre les intégrales de frontière et la méthode numérique associée on présente quelques comparaisons et examine le cas de deux configurations de sphères et ellipsoides de résolution équivalents, c’est-à-dire dotés lorsqu’ils sont seuls de la même vitesse de sédimentation. Pour citer cet article : A. Sellier, C. R. Mecanique 332 (2004).

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Mots-clés : Mécanique des fluides ; Sédimentation ; Interactions ; Équations de frontière

1. Introduction

Evaluating the rigid-body motion(s) of \( N \geq 1 \) arbitrarily-shaped solid particles subject to the gravity and adopting a general (not necessary periodic) configuration remains a tremendous task even within the usual Stokes flow

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approximation. Consequently, the available works only deal with spheres [1–4] or axisymmetric chains of spheroids aligned with the gravity [5,6]. This study thus introduces a new method valid whatever the shapes of the particles. The advocated approach appeals to $6N$ boundary-integral equations and makes it possible to calculate the rigid-body of each particle without determining the unbounded fluid flow about the cluster.

2. General theory

We consider, as depicted in Fig. 1, a collection of $N \geq 1$ arbitrarily-shaped solid, small and not necessarily homogeneous particles $P_n$ with surface $S_n$, center of mass $O_n$ and mass $M_n$. Using Cartesian coordinates $(O, x_1, x_2, x_3)$ and the usual tensor summation convention with $\mathbf{OM} = x_i e_i$, we assume that the particles are subject to the uniform gravity field $\mathbf{g} = -g \mathbf{e}_3$ (with $g > 0$) and immersed in a quiescent and unbounded Newtonian fluid of uniform viscosity $\mu$ and density $\rho$.

The solid $P_n$ with small length scale $a_n$ experiences a quasi-static rigid-body motion of unknown angular velocity $\mathbf{\Omega}^{(n)}$ and translational velocity $\mathbf{U}^{(n)}$ (the velocity of $O_n$). Neglecting inertial effects, i.e., assuming that $Re = \rho U a_n / \mu \ll 1$ with $a = \text{Max}(a_n)$ and $U = \text{Max}(|U^{(i)}|, a_n|\Omega^{(i)})$, the quasisteady fluid velocity field $\mathbf{u}$ and pressure field $\mathbf{p} + \rho g \mathbf{e}_3$ obey

$$\mu \nabla^2 \mathbf{u} = \nabla \mathbf{p} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega$$

$$\mathbf{u} \rightarrow (0, 0) \quad \text{as} \quad r = (x_i x_i)^{1/2} \rightarrow \infty, \quad \mathbf{u} = \mathbf{U}^{(n)} + \mathbf{\Omega}^{(n)} \wedge \mathbf{O_n M} \quad \text{on} \quad S_n$$

with $\Omega$ the fluid domain. The generalized velocity $\mathbf{X} := (\mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(N)}; \mathbf{\Omega}^{(1)}, \ldots, \mathbf{\Omega}^{(N)})$ is unknown and one thus needs to supplement (1)–(2) with additional relations. Denoting by $n$ the unit outward normal on $S_n$, the flow $(\mathbf{u}, \mathbf{p})$ with stress tensor $\sigma$, the static pressure $\rho g \mathbf{e}_3$ and the gravity $\mathbf{g}$ apply on $P_n$ with volume $V_n$ and center of volume $O'_n$ a net force $\mathbf{R}^{(n)}$ and a net torque $\mathbf{C}^{(n)}$ (about the center of mass $O_n$) such that

$$\mathbf{R}^{(n)} = \int_{S_n} \sigma \cdot n \, dS_n + (M_n - \rho V_n) \mathbf{g}, \quad \mathbf{C}^{(n)} = \int_{S_n} \mathbf{O_n M} \wedge \sigma \cdot n \, dS_n - \rho V_n \mathbf{O_n O'_n} \wedge \mathbf{g}$$

Neglecting particle inertia, the required conditions read $\mathbf{R}^{(n)} = \mathbf{C}^{(n)} = 0$, i.e.

$$\mathbf{F}^{(n)} := \int_{S_n} \sigma \cdot n \, dS_n = (\rho V_n - M_n) \mathbf{g}, \quad \mathbf{\Gamma}^{(n)} := \int_{S_n} \mathbf{O_n M} \wedge \sigma \cdot n \, dS_n = \rho V_n \mathbf{O_n O'_n} \wedge \mathbf{g}$$

Fig. 1. A 2-sphere cluster in (a) axisymmetric or (b) asymmetric cases and (c) a cluster of 4 equivalent spheres and spheroids.

Fig. 1. Deux sphères disposées de façon (a) axissymétrique ou (b) asymétrique et (c) un ensemble de 4 sphères et ellipsoïdes de révolution équivalents.
When determining \( X \) fulfilling (1), (2) and (4) one may think about using a Finite Element Method to calculate the above net forces \( \mathbf{F}^{(i)} \) and torques \( \mathbf{T}^{(n)} \) induced by \((u, p)\) solution to (1), (2) for a given guess value \( X_0 \) and then iteratively modify \( X_0 \) until (4) is satisfied. Unfortunately, for fully three-dimensional \( N \)-particle clusters this strategy is tremendously cpu time consuming and yields a poor accuracy for the velocity \( X \) because one has to numerically evaluate the traction \( \mathbf{f} \cdot \mathbf{n} \) on each boundary \( S_m \) for the computed flow \((u, p)\). The approach advocated in this work is free from such drawbacks and appeals to \( 6N \) steady Stokes flows \((u^{(i)}_L, p^{(i)}_L)\) with \( L \in \{T, R\} \), \( i \in \{1, 3\} \) and \( e \in \{1, \ldots, N\} \). These flows, free from body force and quiescent far from the cluster, obey (1) and the boundary conditions

\[
\begin{align*}
\mathbf{u}^{(i)}_L &= \delta_{en} \mathbf{e}_i, \\
\mathbf{u}^{(i)}_R &= \delta_{en} (\mathbf{e}_i \otimes \mathbf{O}_n \mathbf{M}) \quad \text{on } S_m
\end{align*}
\]  

with \( \delta_{en} \) the Kronecker symbol. The flow \((u^{(i)}_L, p^{(i)}_L)\) has stress tensor \( \sigma^{(i)}_L \) and thus applies the surface traction \( \mathbf{f}^{(i)}_L \cdot \mathbf{n} \) on the entire cluster’s boundary \( S = \bigcup_{m=1}^{N} S_m \). Noting that, since \((u, p)\) obeys (1), the usual reciprocal theorem \([7]\) yields

\[
\int_S \mathbf{u}^{(i)}_L \cdot \mathbf{r} \cdot \mathbf{n} \, dS = \int_S \mathbf{u} \cdot \mathbf{r}^{(i)}_L \cdot \mathbf{n} \, dS = \int_S \mathbf{u} \cdot \mathbf{f}^{(i)}_L \, dS
\]

it is then straightforward, by exploiting the boundary conditions (2) and (5), to cast the conditions (4) under the following \( 6N \)-equation linear system

\[
\begin{align*}
\mathbf{A}^{(i),j}_{(m),L} \mathbf{U}^{(m)}_{(j)} + \mathbf{B}^{(i),j}_{(m),R} \mathbf{Q}^{(m)}_{(j)} &= (\rho \mathbf{V}_n - \mathbf{M}_n) \mathbf{g}_j / \mu \quad (7) \\
\mathbf{A}^{(i),j}_{(m),R} \mathbf{U}^{(m)}_{(j)} + \mathbf{B}^{(i),j}_{(m),L} \mathbf{Q}^{(m)}_{(j)} &= \rho g \mathbf{V}_n (\mathbf{O}_n \mathbf{O}_n' \otimes \mathbf{e}_3) \mathbf{e}_j / \mu \quad (8)
\end{align*}
\]

if one sets \( \mathbf{U}^{(m)}_{(j)} = \mathbf{U}^{(m)} \mathbf{e}_j \), \( \mathbf{Q}^{(m)}_{(j)} = \mathbf{Q}^{(m)} \mathbf{e}_j \) and makes use of the definitions

\[
-\mu \mathbf{A}^{(i),j}_{(m),L} = \int_{S_m} \mathbf{e}_j \cdot \mathbf{f}^{(i)}_L \, dS_m, \\
-\mu \mathbf{B}^{(i),j}_{(m),L} = \int_{S_m} (\mathbf{e}_j \otimes \mathbf{O}_m \mathbf{M}) \cdot \mathbf{f}^{(i)}_L \, dS_m
\]

The key system (7), (8) admits a \( 6N \times 6N \) square, real-valued, symmetric and positive-definite matrix \([8]\) and therefore a unique solution \( X \) for any \( N \)-particle cluster and settings \( \mathbf{P}_n, \rho_n, \mathbf{M}_n, \mathbf{V}_n, \mathbf{O}_n, \mathbf{O}_n' \). It also shows that is sufficient to compute the very few surface tractions \( \mathbf{f}^{(i)}_L \) on the entire cluster’s boundary \( S \) to obtain the required rigid-body motions of the particles. As nicely established in \([9]\), the velocity field \( \mathbf{u}^{(i)}_L \) admits both in the unbounded fluid domain \( \Omega \) and on the surface \( S \) the integral representation

\[
\mathbf{u}^{(i)}_L (\mathbf{e}_k) [\mathbf{M}] = -\int_S \left( \frac{\partial \mathbf{f}^{(i)}_L}{\partial \mathbf{f}^{(i)}_L} + \frac{\mathbf{P} \mathbf{M} \mathbf{e}_k}{\mathbf{M}^3} \right) \left( \frac{\mathbf{f}^{(i)}_L}{\mathbf{f}^{(i)}_L} \cdot \mathbf{e}_j \right) \mathbf{d}S
\]

for \( k = 1, 2, 3 \) (10)

The proposed strategy then consists of the following steps:

(i) First, obtain each traction \( \mathbf{f}^{(i)}_L \) by exploiting the representation (10) on \( S \). One thus ends up with a Fredholm boundary-integral equation of the first kind that admits a solution defined up to an arbitrary constant multiple of \( n \) on each subdomain \( S_m \) \([9]\).

(ii) Solve the governing system (7)–(8) by computing the coefficients \( \mathbf{A}^{(i),j}_{(m),L} \) and \( \mathbf{B}^{(i),j}_{(m),L} \) (which are readily uniquely determined for \( \mathbf{f}^{(i)}_L \) defined up to a multiple of \( n \) on \( S_m \)).

(iii) If needed, evaluate the velocity \( \mathbf{u} \) in the unbounded domain \( \Omega \) by using (10) where \( \mathbf{u}^{(i)}_L \) and \( \mathbf{f}^{(i)}_L \) are replaced with \( \mathbf{u} \) and \( \mathbf{f} = \sum_{m=1}^{N} \sum_{i=1}^{3} (\mathbf{f}^{(i)}_L + \mathbf{Q}^{(m)}_{(j)} \mathbf{f}^{(i)}_L) \), respectively.

Clearly, the advocated approach applies to \( N \)-particle clusters made of arbitrarily-shaped and not necessarily homogeneous particles. Moreover, the derived boundary formulation permits us in practice to accurately compute the rigid-body motions of the particles without determining the fluid flow (by only using the previous steps (i)–(iii)).
3. Numerical implementation and benchmarks

The 6N boundary-integral equations encountered in step (i) are discretized by using on Sₜ a Nₜ(λ) node mesh of 6-node isoparametric, curvilinear triangular and boundary elements [10,9]. This results in a Nₜ-node mesh on the cluster’s surface S and the obtained linear systems with dense and non-symmetric 3Nₜ × 3Nₜ matrix are solved by Gaussian elimination. Henceforth, we confine the analysis to the case of homogeneous (Ωₚ = Ωₙₚ) spheroids and spheroids. Appealing to [8], a single spheroid P₁ of inequation x₁² + x₂² + x₃²/λ² ≤ a₁² is found to settle without rotating at the velocity \( \mathbf{U}^{(1)} = g \alpha_{2}^2 (\rho - \rho_{1}) \nu \omega_{2} / \mu \) with \( \nu = 2/9 \) for a sphere and for \( \lambda \neq 1 \), under the notation \( p = \lambda |\mathbf{A}|^{2} - 1 \). 

\[
v = \frac{P}{12} \left[ (p^2 + 1) \log \left( \frac{p + 1}{p - 1} \right) - 2p \right] \quad \text{if } \lambda > 1,
\]

\[
v = \frac{p}{6} \left[ p + (1 - p^2) \arctan \left( \frac{1}{p} \right) \right] \quad \text{if } \lambda < 1
\]

Our numerical implementation is benchmarked against the above exact solution for a sphere and oblate (\( \lambda = 1/2 \)) or prolate (\( \lambda = 2 \)) spheroids. As shown in Table 1, the computed velocity \( v \) nicely converges towards its analytical value as the number \( N_{d}^{(1)} \) of collocation points on \( \Sigma_{1} \) increases with an excellent relative precision of order of 0.1% for \( N_{d}^{(1)} = 242 \). Even for \( N_{d}^{(1)} = 74 \), each computed angular velocity component is of order of 10⁻³.

The case of two identical spheres (\( \lambda_1 = \lambda_2 = 1, \alpha_1 = \alpha_2 = \alpha, \rho_1 = \rho_2 \)) in axisymmetric and asymmetric configurations (see Fig. 1(a), (b)) is also compared with [1] and [2], respectively. Following those works, we list in

### Table 1
Computed normalized velocity \( v \) for a sphere and oblate or prolate spheroids using different numbers \( N_{d}^{(1)} \) of collocation points

<table>
<thead>
<tr>
<th>( N_{d}^{(1)} )</th>
<th>( v (\lambda = 1/2) )</th>
<th>( v (\lambda = 1) )</th>
<th>( v (\lambda = 2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>74</td>
<td>0.1237682</td>
<td>0.222682</td>
<td>0.370022</td>
</tr>
<tr>
<td>242</td>
<td>0.122767</td>
<td>0.222279</td>
<td>0.359245</td>
</tr>
<tr>
<td>1058</td>
<td>0.122736</td>
<td>0.222227</td>
<td>0.359164</td>
</tr>
<tr>
<td>exact</td>
<td>0.122733</td>
<td>0.222222</td>
<td>0.359185</td>
</tr>
</tbody>
</table>

### Table 2
Computed non-zero normalized velocity components \( u_{3}^{(1)} \) and \( u_{2}^{(2)} \) of two identical spheres (\( \alpha_1 = \alpha_2 = \alpha \)) settling in the (a) axisymmetric and (b) asymmetric configurations sketched in Fig. 1 for two separation parameters \( h = 0.2/(\alpha_1 + \alpha_2) \)

<table>
<thead>
<tr>
<th>( N_{d}^{(1)} )</th>
<th>( N_{d}^{(2)} )</th>
<th>( h )</th>
<th>( (a) u_{3}^{(1)} = u_{3}^{(2)} )</th>
<th>( (b) u_{1}^{(1)} = u_{2}^{(2)} )</th>
<th>( (b) u_{2}^{(1)} = -u_{2}^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>74</td>
<td>1.12763</td>
<td>1.15101</td>
<td>1.36481</td>
<td>0.13141</td>
<td></td>
</tr>
<tr>
<td>242</td>
<td>1.12763</td>
<td>1.15101</td>
<td>1.36481</td>
<td>0.13141</td>
<td></td>
</tr>
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<td>1058</td>
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<td>0.13141</td>
<td></td>
</tr>
<tr>
<td>exact [1,2]</td>
<td>1.12763</td>
<td>1.15101</td>
<td>1.36481</td>
<td>0.13141</td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>2.33241</td>
<td>1.30485</td>
<td>1.16410</td>
<td>0.03383</td>
<td></td>
</tr>
<tr>
<td>242</td>
<td>2.33241</td>
<td>1.30485</td>
<td>1.16410</td>
<td>0.03383</td>
<td></td>
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<td>1058</td>
<td>2.33241</td>
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<td>1.30485</td>
<td>1.16410</td>
<td>0.03383</td>
<td></td>
</tr>
</tbody>
</table>
Table 2 the non-zero Cartesian velocity components normalized by the settling velocity of an isolated sphere, i.e. the quantities

\[
\begin{align*}
\mathbf{u}_1^{(n)}(h) &= \frac{9\mu U_{x}^{(n)}}{2g(\rho - \rho_0)a_{3}^{2}}, \\
\mathbf{w}_2^{(n)}(h) &= \frac{9\mu \Omega^{(n)}}{2g(\rho - \rho_0)a_{3}^{2}}, \\
\mathbf{h} &= \frac{O_{1}O_{2}}{a_{1} + a_{2}}.
\end{align*}
\]

(12)

Again, using 242-node meshes yields a nice 4-digit accuracy even for \( h = 1.12763 \).

4. Results for configurations of homogeneous equivalent spheres and spheroids

We present preliminary results for two \( N \)-particle clusters (\( N = 4,5 \)) consisting (see Fig. 1(c)) of \( N - 2 \) equal spheres \( \mathcal{P}_s \) arranged at the corners of a regular polygon in the horizontal \( x_3 = 0 \) plane with \( 3 \leq n \leq N \), \( a_0 = a_3 \), \( O_{i}O_{j} = d \) and two identical spheroids \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) of inequalities \( x_1^2 + x_2^2 + [x_3 + (-1)^n h]^2 \leq \lambda^2 a_3^2 \) for \( n = 1,2 \) with \( (\lambda, d, h) \) so that the particles do not touch. All the particles admit the same uniform density \( \rho_s \) and settling velocity when isolated (equivalent particles), i.e. \( a_1/a_3 = (9\mu/2a_3^2)^{-1/2} \) with \( \mu \) given by (11) and we resort to a 242-node mesh on each \( S_n \). Symmetries show that the only non-zero normalized Cartesian velocity components are \( u_1^{(1)} = u_1^{(2)} \) and \( u_1^{(3)} = \cdots = u_1^{(N)} \) with \( u_1^{(n)} = 9U_{x}^{(n)} / 2a_3^2 (\rho - \rho_0) \) for \( n = 1, \ldots, N \).

These quantities are plotted in Fig. 2 versus the ratio \( h/a_3 \geq 2 \) for \( d/a_3 = 2, \lambda = 0.5, 2 \) and \( N = 4,5 \). Note that each particle moves faster than if isolated (\( u_1^{(n)} > 1 \)) and the interactions become strong for \( h - d = 2a_3 \) (with \( u^{(n)} \sim 2 \)). The velocities are bigger for five particles \( (N = 5) \) than for four particles \( (N = 4) \) and for a given value of \( N \). It is found that \( u_1^{(1)} \) and \( u_1^{(3)} \) weakly and strongly decrease with \( \lambda \), respectively. Finally, curves for \( u_1^{(1)} \) and \( u_1^{(3)} \) cross for a critical value \( h = h_c \) at which the cluster keeps a steady configuration when falling, all particles adopting the same velocity.

The critical setting \( h_c/d \) has been found by an iterative (bisection) scheme stopping as soon as \( |u_1^{(1)} - u_1^{(3)}| \leq 5 \times 10^{-6} \) for \( 1.5 \leq d \leq 10 \) and both the computed ratio \( h_c/d \) and the associated cluster's settling velocity \( u_1^{(1)} = u_1^{(3)} \) are displayed in Fig. 3. The curves \( h_c/d \) previously given in [4] for spheres \( (\lambda = 1) \) are perfectly recovered and as \( d \) increases \( h_c/d \) increases and asymptotes to a constant value because particles behave like point forces for large distances \( d \) and \( h \). Moreover, \( h_c/d \) increases both with \( \lambda \) and \( N \) for a given value of \( d \). As revealed by Fig. 3(b), \( u_c \) not only increases with \( d \) because interactions become strong but also with \( 1/\lambda \) and \( N \) for a

![Fig. 2. Normalized velocities \( u_1^{(1)} (N = 4(\circ) \text{ or } N = 5(\bullet)) \) and \( u_1^{(3)} (N = 4(\odot) \text{ or } N = 5(\bullet)) \) for \( d/a_3 = 2, (a) \lambda = 0.5 \text{ and } (b) \lambda = 2 \).](image)

![Fig. 2. Vitesse adimensionnées \( u_1^{(1)} (N = 4(\circ) \text{ ou } N = 5(\bullet)) \) et \( u_1^{(3)} (N = 4(\odot) \text{ ou } N = 5(\bullet)) \) pour \( d/a_3 = 2, (a) \lambda = 0, 5 \text{ et } (b) \lambda = 2 \).](image)
Fig. 3. (a) Critical ratio $h_c/d$, and (b) settling velocity $u_c$, of the steady configurations for $\lambda = 0.5$ ($N = 4(*)$ or $N = 5(||)$), $\lambda = 1$ ($N = 4(o)$ or $N = 5(*)$), and $\lambda = 2$ ($N = 4(\bigstar)$ or $N = 5(|)$).

Fig. 3. (a) Rapport critique $h_c/d$, et (b) vitesse de sedimentation $u_c$, des configurations rigides pour $\lambda = 0.5$ ($N = 4(*)$ ou $N = 5(||)$), $\lambda = 1$ ($N = 4(o)$ ou $N = 5(*)$) et $\lambda = 2$ ($N = 4(\bigstar)$ ou $N = 5(|)$).

given spacing $d$. For example, the critical $N$-particle cluster with ratio $d/a_3$ settles faster when involving prolate spheroids than when consisting of $N$ spheres.

5. Concluding remarks

The present method nicely recovers previous results for spheres and permits us to deal with non-spherical particles. The exhibited critical steady configurations of $N = 4$, 5 spheres and spheroids are likely to be unstable and the challenging stability analysis of such clusters, assuming a fluid flow of small but non-zero Reynolds number as achieved in [11] for identical spheres lying in the same horizontal plane, is under current investigation.

References