Asymptotic Analysis of Low Reynolds Number Flow with a Linear Shear Past a Circular Cylinder*

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Two-dimensional steady flow of an incompressible viscous fluid around a circular cylinder in the case where the velocity field at large distances is the combination of a simple shear and a uniform stream is described in terms of matched asymptotic expansions valid at a low Reynolds number. The main purpose of the present paper is (1) to examine the validity of the assumptions used by Bretherton (1961) and (2) to construct an alternative approach without using such assumptions. In the present paper is constructed a system of governing integral equations for vorticity and stream function based on an Oseen-type equation. Local solutions, inner and outer solutions, are obtained from these equations by using the method proposed by Kida (1991), which is so systematic that we do not need the detailed physical consideration. Finally aerodynamic forces are compared with those obtained by Bretherton. The present paper shows that Bretherton's assumptions are correct within the first approximation. One cycle higher order solutions are obtained in this paper.

***Key Words*:** Fluid Dynamics, Viscous Flow, Shear Flow, Low Reynolds Number Flow, Two-Dimensional Flow, Asymptotic Analysis, Singular Perturbation Method

1. Introduction

This paper treats a steady, two-dimensional low Reynolds number flow of an incompressible viscous fluid around a circular cylinder in the case where the velocity at large distances is given by the combination of a uniform flow and a simple shear. Low Reynolds number flow problems are classic and have been studied by many investigators (e.g., Pozrikidis[1]), however, the present shear flow problems have not been studied in detail except by Bretherton[2], who analyzed this problem using a method of matched asymptotic expansions with respect to Reynolds number Re, which was based on the incoming flow velocity to the cylinder and the radius of the cylinder. In his analysis, the inner solutions of the stream function were assumed to approach the shear flow as $|\vec{x}|\to\infty$ and they were obtained by using the assumption : (a) the inner solutions which satisfy the boundary conditions on the surface of the cylinder and the far-field condition, $o(|\vec{x}|\log|\vec{x}|)$ as $|\vec{x}|\to\infty$, are identically zero. The first approximation of the outer solutions was assumed to be the shear flow and the second approximation was assumed to be a uniform flow. Furthermore, he assumed : (b) the outer solutions which satisfy the conditions, that they are $o(|\vec{X}|\log|\vec{X}|)$ as $|\vec{X}|\to 0$ and their derivative becomes zero as $|\vec{X}|\to\infty$, must be constant, where $\vec{X}$ is the outer variable ($\vec{X}=(Re/2)^{1/2}\vec{x}$) ($Re=Ga^2/\nu$, where G is the shear rate). In order to obtain the additional outer solutions, the following assumption was used : (c) the outer solutions are given by superposing solutions for the instantaneous Oseen solution of an unsteady point source in a shear flow and they are expressible as a power series of $1/\log Re$. The first and the second approximations mentioned above are of the order of unity and $Re^{1/2}$, respectively, and the higher approximations are expressed as $Re^{1/2}/(\log Re)^n$ ($n=1, 2, \cdots$). However,

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these assumptions have not been proved yet from the point of view of the asymptotic analysis.

Michaelides[3] reviewed earlier works which investigated forces on an object immersed in a fluid flow. In these works, the Reynolds number of flows based on a typical slip velocity on the surface of a body and a typical length of the body is very low. Therefore, the interaction between the body and the ambient flow is mainly based on the theoretical and experimental results of the Stokes or Oseen flows. Earlier theoretical works for finite low Reynolds number flows were mainly based on the matched asymptotic approach completed by Proudman and Pearson[4], Kaplun and Lagerstrom[5] and Kaplun[6] for two- and three-dimensional bodies in a uniform steady flow. As pointed out by Kaplun[6], there is an essential difference between two-dimensional and three-dimensional cases in the method of matched asymptotic analysis: in a two-dimensional flow, inner solutions are obtained by matching with outer solutions (the first solution of the outer flow is the uniform flow), however, in a three-dimensional flow, outer solutions are obtained by matching with inner solutions (the first solution of the inner flow is the Stokes flow solution). This difference results from the Stokes and Whitehead paradoxes in the iterative method with respect to the Reynolds number (see Van Dyke[7]). The matched asymptotic approach has been applied to unsteady flow problems by many investigators; an impulsively started motion, a sudden change of motion and an oscillatory motion (e.g., Bentwich and Miloh[8], Sano[9], Lovalenti and Brady[10]-[12], Nakanishi and coworkers[13],[14]).

With regard to a shear flow, three-dimensional flows past a sphere have been studied by many investigators (e.g., Bretherton[15], Saffman[16], Drew[17], McLaughlin[18], and Feng and Joseph[19]). However, two-dimensional flows have not been studied except by Bretherton[2]. Bretherton[15] pointed out that the lift force was not generated on the basis of the creeping flow equations regardless of the velocity profile and relative size of particle. Saffman[16], therefore, analysed the lift force on a sphere in a shear flow at a low Reynolds number using the matched asymptotic method, in order to take into account the inertia term. Drew[17] extended Saffman's method to pure rotation and pure shear in a far-field flow and derived the hyrodynamic force $\vec{F}$ for the shear flow as

$$\vec{F}=6\pi\mu a\vec{U}\cdot\left\{\left[1+0.10\left(\frac{1}{2}\kappa Re\right)^{1/2}\right](\vec{e}_1\vec{e}_1+\vec{e}_3\vec{e}_3)+0.502\left(\frac{1}{2}\kappa Re\right)^{1/2}(\vec{e}_1\vec{e}_3+\vec{e}_3\vec{e}_1)\right\}.$$

Here, $\vec{U}$ is the fluid velocity far from the sphere, a the radius of the sphere, μ the viscosity, Re the Reynolds number $(=\rho Ua/\mu)$, and κ a dimensionless measure of the shearing. On the other hand, Bretherton[2] derived the force $\vec{F}$ on a circular cylinder under the assumptions (a) - (c): $\vec{F}=4\pi\mu\left[\text{Real}\left(\frac{HU_o+KV_o}{\tau-\frac{1}{2}\log Re}\right), \text{Real}\left(\frac{EU_o+FV_o}{\tau-\frac{1}{2}\log Re}\right)\right]$, where $Re=Ga^2/\nu$, $\vec{U}=(U_o, V_o)$, $E=-2.11i$, $F=-1+0.289i$, $H=1+0.289i$, $K=-0.513i$, and $\tau=0.679+0.798i$ and "Real" denotes the real part. Thus, the lift force is of the order of $1/(\log Re)^2$, although the drag force is of the order of $1/\log Re$. We note in his analysis that the outer solutions to any finite order approximation are governed by the Oseen equation for the shear flow from his assumption (c).

The main purpose of the present paper is to confirm Bretherton's assumptions, that is, whether or not the assumptions from (a) to (c) are reasonable. In two-dimensional flows, the key point of the asymptotic analysis is to obtain outer solutions. Bretherton[2] gave the outer solutions using the unsteady problem of diffusing substance which is instantaneously released at the origin at time $t=0$, that is, the assumption (c), however, it is hard to extend systematically his method to higher order approximations without a detailed physical consideration. In the present paper is constructed an alternative asymptotic approach based on integral expressions proposed in a series of papers by Kida and coworkers[20]-[30]. This approach is so systematic that a detailed physical consideration is not necessary and it does not lead to incorrect solutions (e.g., Kida and Miyai[23], Nakanishi et al.[13]).

In the present paper, governing integral expressions will be first constructed in section 3 for an Oseen-type approximation of the combination flow of a uniform flow and a simple shear flow. Second, inner and outer solutions will be derived from these integral expressions in section 4. In particular, it will be shown that Bretherton's outer solutions will be obtained without assumption (c). Furthermore, the second-order approximation of the aerodynamic forces will be obtained in section 5.

2. Governing Equations

We consider a two-dimensional incompressible steady fluid flow past a circular cylinder as a combination of a uniform flow and a simple shear flow, as shown in Fig. 1. Cartesian coordinates are taken as (x_1, x_2) and the origin is taken as the center of the circular cylinder. The ratio of the simple shear is defined as G and the uniform velocity is denoted as (U, V).

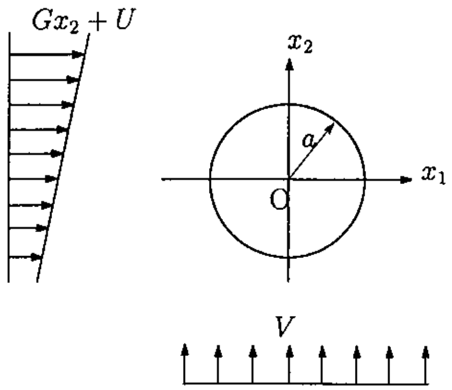

Fig. 1 Physical state and coordinate systems

The governing equation for vorticity, $(0, 0, \omega)$, is derived from the two-dimensional Navier-Stokes equations:

$$u_i \partial\omega/\partial x_i = \nu \nabla^2 \omega, \tag{1}$$

where $\vec{u}(\vec{x}, t)$ $(=(u_1, u_2))$ is the velocity vector at point $\vec{x}(=(x_1, x_2))$, ∇ is the nabla operator, and ν is the kinematic viscosity. We introduce the stream function, Ψ, then the vorticity is related to Ψ:

$$\omega = -\nabla^2 \Psi. \tag{2}$$

No-slip condition and far-field condition are imposed in this problem:

$$\vec{u} \to (Gx_2 + U, V) \quad \text{as } |\vec{x}| \to \infty, \tag{3}$$

$$\vec{u} = 0 \quad \text{on } S, \tag{4}$$

where S is the surface of the circular cylinder.

We normalize lengths and velocities with respect to the radius of the circular cylinder a and the uniform speed $U_c = (U^2 + V^2)^{1/2}$. Here, we introduce the following dimensionless perturbation stream function ψ.

$$u_1 = \beta_o x_2 + U_o + \frac{\partial \psi}{\partial x_2}, \; u_2 = V_o - \frac{\partial \psi}{\partial x_1}, \tag{5}$$

where $\beta_o = Ga/U_c$ and (U_o, V_o) is the normalized uniform flow with U_c. Then, the dimensionless forms of Eqs.(1), (2) are

$$(\beta_o x_2 + U_o)\frac{\partial \omega}{\partial x_1} + V_o \frac{\partial \omega}{\partial x_2} = \frac{1}{Re}\nabla^2 \omega + f, \tag{6}$$

$$\omega = -\nabla^2 \psi, \tag{7}$$

where f and the Reynolds number Re are defined by

$$f = -\frac{\partial \psi}{\partial x_2}\frac{\partial \omega}{\partial x_1} + \frac{\partial \psi}{\partial x_1}\frac{\partial \omega}{\partial x_2}, \tag{8}$$

$$Re = U_c a/\nu. \tag{9}$$

The boundary conditions for the perturbation stream function ψ are obtained from Eqs.(3), (4) as

$$\frac{\partial \psi}{\partial x_1} \to 0, \; \frac{\partial \psi}{\partial x_2} \to 0, \text{ as } |\vec{x}| \to \infty, \tag{10}$$

$$\frac{\partial \psi}{\partial x_1} = V_o, \; \frac{\partial \psi}{\partial x_2} = -\sin\theta - U_o, \text{ on } S, \tag{11}$$

where θ is defined by $\vec{x} = (\cos\theta, \sin\theta)$ on S.

In this problem, Bretherton[2] defined the alternative Reynolds number Ga^2/ν based on the incoming shear flow and assumed it to be of the same order of Re, that is, $\beta_o = O(1)$. We note that we cannot treat the special flow of $\beta_o = 0$ under this assumption, that is, the flow past a circular cylinder in the uniform flow.

Here we define the small parameter ε for convenience of the description as

$$\varepsilon \equiv Re/2. \tag{12}$$

The basic governing equation of vorticity, Eq.(6), is rewritten as

$$\nabla^2 \omega - 2\varepsilon\left((\beta_o x_2 + U_o)\frac{\partial \omega}{\partial x_1} + V_o \frac{\partial \omega}{\partial x_2}\right) = -2\varepsilon f. \tag{13}$$

Here, we introduce $\tilde{y}$ and Ω as

$$\tilde{y} \equiv x_2 + U_o/\beta_o, \; \Omega \equiv \exp(-\varepsilon V_o \tilde{y})\omega. \tag{14}$$

Then, Ω and ψ are governed from Eqs.(7), (13) by

$$\nabla^2 \Omega - 2\varepsilon\beta_o \tilde{y}\frac{\partial \Omega}{\partial x_1} - (\varepsilon V_o)^2 \Omega = -2\varepsilon \exp(-\varepsilon V_o \tilde{y}) f, \tag{15}$$

$$\nabla^2 \psi = -\exp(\varepsilon V_o \tilde{y})\Omega. \tag{16}$$

3. Integral Expressions

Let us define a fundamental function as $G_f(\vec{x} - \vec{x}_o; \tilde{y}_o)$ satisfying the following governing equation:

$$\nabla^2 G_f + 2\varepsilon\beta_o \tilde{y}\frac{\partial G_f}{\partial x_1} - (\varepsilon V_o)^2 G_f = \delta(x_1 - x_{o1})\delta(\tilde{y} - \tilde{y}_o), \tag{17}$$

$$G_f \to 0 \quad \text{as } |\vec{x} - \vec{x}_o| \to \infty, \tag{18}$$

where $\delta(x)$ is the Dirac delta function. From the Green formula and the boundary conditions at far field, $|\Omega(\vec{x})| \to 0$ as $|\vec{x}| \to \infty$, $\Omega(\vec{x})$ is given by

$$\begin{aligned}\Omega(\vec{x}_o) = &\int_0^{2\pi}\left(G_f(\vec{x}_s - \vec{x}_o; \tilde{y}_o)\frac{\partial \Omega(\vec{x}_s)}{\partial r} - \Omega(\vec{x}_s)\frac{\partial G_f(\vec{x}_s - \vec{x}_o; \tilde{y}_o)}{\partial r}\right)d\theta \\ &- 2\varepsilon\int_0^{2\pi}(\beta_o \sin\theta + U_o)G_f(\vec{x}_s - \vec{x}_o; \tilde{y}_o) \times \Omega(\vec{x}_s)\cos\theta\, d\theta - 2\varepsilon\int_D \exp(-\varepsilon V_o \tilde{y}) \\ &\times f(\vec{x})G_f(\vec{x} - \vec{x}_o; \tilde{y}_o)dv,\end{aligned} \tag{19}$$

where $\vec{x}_s = (\cos\theta, \sin\theta)$ is on S, $\rho = |\vec{x} - \vec{x}_o|$ for $x_o \in D$, D is the entire flow region outside the circular cylinder and $dv = dx_1 dx_2$. From Eqs.(10), (16), $\psi(\vec{x}_o)$ is also given by

$$\psi(\vec{x}_o) = \frac{1}{2\pi}\int_D \exp(\varepsilon V_o \tilde{y})\Omega(\vec{x})\log\frac{1}{\rho}dv + \frac{1}{2\pi}\int_0^{2\pi}\left(\psi(\vec{x}_s)\frac{\partial}{\partial r}\log\frac{1}{\rho} - \frac{\partial \psi(\vec{x}_s)}{\partial r}\log\frac{1}{\rho}\right)d\theta. \tag{20}$$

The boundary conditions given by Eq.(11) yield the following:

$$\frac{\partial \psi(\vec{x}_s)}{\partial r} = V_o \cos\theta - U_o \sin\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta, \tag{21}$$

$$\frac{\partial \psi(\vec{x}_s)}{\partial \theta} = -V_o \sin\theta - U_o \cos\theta - \frac{1}{2}\sin 2\theta. \tag{22}$$

The fundamental function G_f is obtained from Eqs.(17), (18). The detailed derivation is shown in Appendix 1:

$$G_f(X, Y; \tilde{y}_o) = -\frac{1}{4\pi}\int_0^\infty \frac{1}{(1+\beta^2)^{1/2}}$$
$$\times\exp\Big[-\frac{\varepsilon\beta_o}{4\cdot 3^{1/2}}\beta(X^2+Y^2)$$
$$-\varepsilon\frac{3^{1/2}}{\beta\beta_o}V_o^2 - \frac{\varepsilon}{2}\frac{\beta^2\beta_o}{\beta^2+1}(XY+2X\tilde{y}_o)$$
$$-\varepsilon\frac{3^{1/2}\beta\beta_o}{\beta^2+1}\Big(Y\tilde{y}_o+\tilde{y}_o^2+\frac{Y^2}{4}-\frac{X^2}{12}\Big)\Big]d\beta, \quad (23)$$

where $X=x_1-x_{o1}$ and $Y=x_2-x_{o2}$.

4. Asymptotic Solutions

We easily see from Eq. (23) that there are two significant local regions: $(X, Y)=O(1)$ and $(X, Y)=O(1/\varepsilon^{1/2})$ (see Kida[27]). We call the former and latter regions "inner region" and "outer region", respectively. The inner and outer variables are defined as (X, Y) and $(\tilde{X}, \tilde{Y})=(\varepsilon^{1/2}X, \varepsilon^{1/2}Y)$.

The inner solution of Ω, say Ω^i, is obtained by substituting Eq. (23) into Eq. (19) and using the concept proposed by Kida[27]. Let us consider the integral operator:

$$G_f\hat{f} = \int_D G_f\hat{f}dv. \quad (24)$$

The above relation is rewritten as

$$G_f\hat{f} = \int_0^{2\pi}\int_1^{1/\delta} G_f^i\hat{f}rdrd\theta + \int_0^{2\pi}\int_{1/\delta}^\infty G_f\hat{f}rdrd\theta, \quad (25)$$

where δ is a small parameter with $\delta \geq \varepsilon_o^{1/2}$, where ε_o is a small parameter with $\varepsilon \leq \varepsilon_o$. G_f^i is the asymptotic expansion of G_f with respect to the inner variable (X, Y), which is given in Appendix 3. Since G_f^i has only a logarithmic singularity, the asymptotic expansion of the first term on the right-hand side of Eq. (25) is given by substituting G_f^i. The concept proposed by Kida[27] states that in the second term on the right-hand side, $\tilde{f}$ is indeterminate. In this term, $|\vec{x}_o|<1/\delta$ and $|\vec{x}|>1/\delta$, therefore, we can expand G_f asymptotically with respect to ε. Thus, we can obtain the functional form of the second term on the right-hand side of Eq. (25) with unknown coefficients. We easily see that the asymptotic form of $\int_0^{2\pi}\mathrm{Pf}\int_{1/\delta}^\infty G_f^i\hat{f}rdrd\theta$ is the same as that of the second term on the right-hand side of Eq. (25). Thus, the above relation is rewritten as

$$G_f\hat{f} = \int_0^{2\pi}\mathrm{Pf}\int_1^\infty G_f^i\hat{f}rdrd\theta + \int_0^{2\pi}\mathrm{Pf}\int_{1/\delta}^\infty G_f\tilde{f}rdrd\theta, \quad (26)$$

where $\tilde{f}$ is an indeterminate function. Using this concept, the first inner expansion of Ω^i is given by

$$\Omega^i \approx \frac{A_o}{2}\Big(\gamma+\log\Big(\frac{\varepsilon\beta_o r_o^2}{8\cdot 3^{1/2}}\Big)\Big)$$
$$+\sum_{m=1}\frac{1}{r_o^m}(A_m^c\cos m\varphi + A_m^s\sin m\varphi). \quad (27)$$

Let us consider the outer solutions. Here, we define the integral operator G_f^o:

$$G_f^o\hat{f} \equiv \int_0^{2\pi}\int_{\varepsilon^{1/2}}^\infty G_f\hat{f}RdRd\theta. \quad (28)$$

Let us define G_f^o as the outer expansion of G_f, then G_f^o is given in Appendix 3. The above relation (28) is rewritten as

$$G_f^o\hat{f} = \int_0^{2\pi}\int_\delta^\infty G_f^o\hat{f}RdRd\theta + \int_0^{2\pi}\int_{\varepsilon^{1/2}}^\delta G_f\hat{f}RdRd\theta,$$
$$=\int_0^{2\pi}\mathrm{Pf}\int_0^\infty G_f^o\hat{f}RdRd\theta$$
$$+\int_0^{2\pi}\mathrm{Pf}\int_{\varepsilon^{1/2}}^\delta (G_f-G_f^o)\hat{f}RdRd\theta, \quad (29)$$

where δ is a small parameter $\delta \geq \varepsilon_o^{1/2}$. Using the concept of Kida[27], $\hat{f}$ for $R<\delta$ is indeterminate, where $R=|\tilde{\vec{x}}|$, $(\tilde{\vec{x}}=\varepsilon^{1/2}\vec{x})$. In the second term on the right-hand side of the first line of Eq. (29), the asymptotic expansion of G_f for $R_o>\delta$ and $R<\delta$ is obtained using Theorem A given in Appendix 2, where $R_o=|\tilde{\vec{x}}_o|$, $(\tilde{\vec{x}}_o=\varepsilon^{1/2}\vec{x}_o)$. We see that the asymptotic functional form of $\int_0^{2\pi}\mathrm{Pf}\int_0^\delta G_f^o\hat{f}RdRd\theta$ is the same as that of $\int_0^{2\pi}\int_{\varepsilon^{1/2}}^\delta G_f\hat{f}RdRd\theta$.

Here we use the following assumption for obtaining the outer solution.

Assumption A: The first approximation of the nonlinear term, that is, the third term on the right-hand side of Eq. (19), is higher order than that of the first and second terms on the right-hand side of Eq. (19) with respect to ε.

□

Then, we have from Eq. (19)

$$\Omega^o \approx \frac{1}{2\pi}\mathrm{Pf}\int_0^\infty \frac{1}{(1+t^2)^{1/2}}\exp(\beta_o CR_o^2)[a_o$$
$$+\varepsilon^{1/2}\beta_o R_o(b_1^c A + b_1^s B)]dt, \quad (30)$$

where $\tilde{\vec{x}}=R_o(\cos\varphi, \sin\varphi)$, and A, B and C are defined as

$$A=\frac{t}{2\cdot 3^{1/2}}\cos\varphi - \frac{1}{2}\frac{t^2}{1+t^2}\sin\varphi - \frac{3^{1/2}}{6}\frac{t}{1+t^2}\cos\varphi,$$
$$B=\frac{t}{2\cdot 3^{1/2}}\sin\varphi + \frac{1}{2}\frac{t^2}{1+t^2}\cos\varphi - \frac{3^{1/2}}{2}\frac{t}{1+t^2}\sin\varphi,$$
$$C=-\frac{t}{4\cdot 3^{1/2}} - \frac{3^{1/2}}{12}\frac{t}{1+t^2} + \frac{1}{4}\frac{t^2}{1+t^2}\sin 2\varphi$$
$$+\frac{3^{1/2}}{6}\frac{t}{1+t^2}\cos 2\varphi.$$

Bretherton[2] says that the outer solution of Ω is given as $2\tilde{A}\partial\zeta/\partial\xi + 2\tilde{B}(\partial\zeta/\partial\xi - \partial\zeta/\partial\tilde{C})$, as shown in Eqs. (19), (20) in his paper, where ζ is given by Eq. (15) in his paper and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ correspond to A, B, C in his paper, respectively. The present result, Eq. (30), reveals that Ω is expressed as $a_o\zeta + \varepsilon^{1/2}(b\partial\zeta/\partial\xi + c\partial\zeta/\partial\eta) + O(\varepsilon\cdot\varepsilon^{1/2}a_o)$. As will be shown by Eqs. (32), (36), (37), a_o is of a much higher order than $\varepsilon^{1/2}b_1^{c,s}$. Thus, we see that the first approximation of the present outer solution is essentially identical with Bretherton's[2], that is, the first part of the assumption (c) mentioned in section 1 is correct within the first approximation.

The inner expansion of the outer solution, Ω^{oi}, is given from Eq. (30) by using Eqs. (91) - (99) in Appendix 3:

$$\Omega^{oi} \approx -\frac{a_o}{2\pi}\left(\gamma+\log\left(\frac{\beta_o R_o^2}{8\cdot 3^{1/2}}\right)\right) + \frac{\varepsilon^{1/2}}{\pi}(b_1^c\cos\varphi + b_1^s\sin\varphi)\frac{1}{R_o}. \quad (31)$$

Thus, we have from the matching requirement, $\Omega^{io}=\Omega^{oi}$:

$$a_o \approx -\pi A_o,\ b_1^{c,s} \approx \pi A_1^{c,s}, \quad (32)$$

$$A_m^{c,s} \approx 0 \quad \text{for } m>2. \quad (33)$$

The inner and outer solutions of ψ are also obtained and we can match these solutions. Using further the boundary conditions given by Eqs. (10), (11), we finally obtain the following relations (the detailed derivation is omitted due to the limitation of pages):

$$-\frac{\varepsilon}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_1^\infty \Omega^i r dr d\theta \approx \frac{\varepsilon}{2}+\frac{1}{\pi\log\varepsilon}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega^o \log R R dR d\theta, \quad (34)$$

$$\frac{1}{2\pi\varepsilon^{1/2}}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega^o(\cos\theta, \sin\theta)dRd\theta - \frac{1}{4}A_1^{c,s}\log\varepsilon \approx (V_o, -U_o) - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_1^\infty \Omega^i(\cos\theta,\sin\theta)drd\theta. \quad (35)$$

Substituting Eqs. (27), (30) into Eq. (34), we can obtain A_o as

$$A_o \approx \frac{\varepsilon\log\varepsilon}{2}\Big/\Big[\frac{\varepsilon}{4}\left(\gamma+\log\left(\frac{\varepsilon\beta_o}{8\cdot 3^{1/2}}\right)-1\right)\log\varepsilon + \mathrm{Pf}\int_0^\infty R\log R dR\,\mathrm{Pf}\int_0^\infty \frac{1}{(1+t^2)^{1/2}} \times\exp(-\beta_o\hat{C}R^2)I_o(\beta_o(a^2+b^2)^{1/2}R^2)dt\Big], \quad (36)$$

where

$$\hat{C}=\frac{1}{4\cdot 3^{1/2}}\left(t+\frac{t}{1+t^2}\right),\ a=\frac{1}{4}\frac{t^2}{1+t^2},$$

$$b=\frac{1}{2\cdot 3^{1/2}}\frac{t}{1+t^2}.$$

Thus, we see that A_o is $O(\varepsilon)$. Therefore, the first order of $A_1^{c,s}$ is obtained from Eq. (35)

$$A_1^{c,s} \approx -\frac{4}{\log\varepsilon}(V_o, -U_o). \quad (37)$$

From these results, we see that $A_1^{c,s}$ is of $O(1/\log\varepsilon)$ and Ω^o is of $O(1/\log\varepsilon)$. Therefore, $f=O(\varepsilon/(\log\varepsilon)^2)$, that is, we see that *Assumption A* is reasonable.

Let us extend this to one cycle higher approximation. The first approximation of Ω^i becomes,

$$\Omega^i \approx \frac{1}{r_o}(A_1^c\cos\varphi + A_1^s\sin\varphi), \quad (38)$$

since $A_o=O(\varepsilon)$. Substituting Eq. (38) into Eq. (20) and taking into account Eqs. (32), (33), we have

$$\psi^i \approx -U_o r_o\sin\varphi + V_o r_o\cos\varphi - \frac{1}{2}\log r_o + \frac{1}{4}r_o^2\cos 2\varphi + \frac{1}{2}(A_1^c\cos\varphi + A_1^s\sin\varphi) \times\left(\frac{r_o}{2}-\frac{1}{2r_o}-r_o\log r_o\right). \quad (39)$$

Therefore, the first approximation of f in the inner region, say f^i, is given by

$$f^i \approx \frac{1}{r_o^2}((A_1^s V_o - A_1^c U_o)\cos 2\varphi - (A_1^c V_o + A_1^s U_o)\sin 2\varphi) - \frac{1}{2r_o}(A_1^c\sin 3\varphi - A_1^s\cos 3\varphi) + \frac{1}{2r_o^3}(A_1^c\sin\varphi - A_1^s\cos\varphi) - \frac{1}{r_o^2}\log r_o\left(A_1^c A_1^s\cos 2\varphi + \frac{1}{2}(A_1^{s2}-A_1^{c2})\sin 2\varphi\right). \quad (40)$$

Thus we can obtain the second approximation of Ω^i by substituting Eq. (40) into Eq. (19). Since $a_o=O(\varepsilon)$ in Eq. (30), we can also obtain the second approximation of Ω^o from Eq. (19). Furthermore, we can obtain the second-order stream functions for the outer and inner regions. From the requirement of the matching, we finally have (the detailed derivation is omitted due to the limitation of pages)

$$V_o \approx \frac{1}{2\pi\varepsilon^{1/2}}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega^o\cos\theta dRd\theta - \frac{A_1^c}{4}\log\varepsilon + \frac{1}{32}\varepsilon\log^2\varepsilon A_1^s + O(\varepsilon\log\varepsilon), \quad (41)$$

$$-U_o \approx \frac{1}{2\pi\varepsilon^{1/2}}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega^o\sin\theta dRd\theta - \frac{A_1^s}{4}\log\varepsilon - \frac{1}{32}\varepsilon\log^2\varepsilon A_1^c + O(\varepsilon\log\varepsilon). \quad (42)$$

Here, the first approximation of Ω^o is given from Eq. (30)

$$\Omega^o \approx \frac{\varepsilon^{1/2}}{2}\beta_o R_o\,\mathrm{Pf}\int_0^\infty \frac{1}{(1+t^2)^{1/2}}\exp(\beta_o C R_o^2) \times(A_1^c A + A_1^s B)dt \equiv \varepsilon^{1/2}(A_1^s\Omega_1^s + A_1^c\Omega_1^c). \quad (43)$$

Substituting Eq. (43) into Eqs. (41), (42), we have A_1^s and A_1^c:

$$A_1^s \approx -4\left[U_o(T^c-\log\varepsilon) + V_o\left(T^s - \frac{1}{32}\varepsilon\log^2\varepsilon\right)\right] \Big/\Big[(\log\varepsilon - S^s)(\log\varepsilon - T^c) - \left(T^s-\frac{1}{32}\varepsilon\log^2\varepsilon\right)\left(S^c+\frac{1}{32}\varepsilon\log^2\varepsilon\right)\Big], \quad (44)$$

$$A_1^c \approx 4\left[V_o(S^s-\log\varepsilon) + U_o\left(S^c + \frac{1}{32}\varepsilon\log^2\varepsilon\right)\right] \Big/\Big[(\log\varepsilon - S^s)(\log\varepsilon - T^c) - \left(T^s-\frac{1}{32}\varepsilon\log^2\varepsilon\right)\left(S^c+\frac{1}{32}\varepsilon\log^2\varepsilon\right)\Big], \quad (45)$$

where

$$S^c = \frac{2}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega_1^s\cos\theta dRd\theta, \quad (46)$$

$$S^s = \frac{2}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega_1^s\sin\theta dRd\theta, \quad (47)$$

$$T^c = \frac{2}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega_1^c\cos\theta dRd\theta, \quad (48)$$

$$T^s = \frac{2}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \Omega_1^c\sin\theta dRd\theta. \quad (49)$$

Substituting Ω_1^s and Ω_1^c defined by Eq. (43) into Eqs.

(46) - (49), we finally arrive at (see Appendix 4) :

$$\frac{S^s}{\beta_o}=-\gamma+\frac{2}{3}\log 2,\ \frac{S^c}{\beta_o}=\frac{8}{9}\pi+\frac{3^{1/2}}{3}, \tag{50}$$

$$\frac{T^s}{\beta_o}=\frac{1}{3^{1/2}}-\frac{4}{9}\pi,\ \frac{T^c}{\beta_o}=-\gamma+\frac{5}{3}\log 2. \tag{51}$$

Following the above-mentioned procedure, we can obtain $A_2^{c,s}$:

$$\begin{aligned}A_2^c\approx\,&1+\frac{1}{4}A_1^cU_o\varepsilon\log\varepsilon+\frac{\varepsilon}{4}(\log\varepsilon+2)\\&\times(A_1^sV_o-A_1^cU_o)-\frac{\varepsilon}{16}(\log^2\varepsilon+2)A_1^sA_1^c\\&-\frac{1}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\frac{\Omega^o}{R}\cos 2\theta dRd\theta,\end{aligned} \tag{52}$$

$$\begin{aligned}A_2^s\approx\,&-\frac{1}{4}A_1^sU_o\varepsilon\log\varepsilon+\frac{\varepsilon}{4}(\log\varepsilon-2)\\&\times(A_1^cV_o+A_1^sU_o)+\frac{\varepsilon}{32}(\log^2\varepsilon-2)(A_1^{s2}-A_1^{c2})\\&-\frac{1}{\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\frac{\Omega^o}{R}\cos 2\theta dRd\theta.\end{aligned} \tag{53}$$

5. Aerodynamic Forces

We consider the aerodynamic forces acting on a circular cylinder. Taking into account no-slip condition, we easily obtain the pressure forces:

$$\begin{aligned}X_p&=\mu U_c\int_0^{2\pi}\sin\theta\nabla^2u_sd\theta,\\Y_p&=-\mu U_c\int_0^{2\pi}\cos\theta\nabla^2u_sd\theta,\end{aligned} \tag{54}$$

where u_s is the tangential component of velocity on the surface of the cylinder, and (X_p, Y_p) is the pressure force. Friction forces are also obtained as

$$\begin{aligned}X_f&=-\mu U_c\int_0^{2\pi}\sin\theta\frac{\partial u_s}{\partial r}d\theta,\\Y_f&=\mu U_c\int_0^{2\pi}\cos\theta\frac{\partial u_s}{\partial r}d\theta.\end{aligned} \tag{55}$$

Thus, total forces (F_X, F_Y) are given by

$$\begin{aligned}F_X&=\mu U_c\int_0^{2\pi}\sin\theta\frac{\partial^2 u_s}{\partial r^2}d\theta,\\F_Y&=-\mu U_c\int_0^{2\pi}\cos\theta\frac{\partial^2 u_s}{\partial r^2}d\theta.\end{aligned} \tag{56}$$

Since $\frac{\partial^2 u_s}{\partial r^2}=\left[\frac{\partial\Omega^i}{\partial r}-\Omega^i+\varepsilon V_o\sin\theta\Omega\right]\exp(-\varepsilon V_o(\sin\theta+U_o/\beta_o))$ on S, from Eq.(56), we finally obtain

$$\begin{aligned}F_X\approx\,&-\pi\mu U_c\Big(2A_1^s-\varepsilon\Big(\frac{\beta_o}{2}A_1^c+\frac{1}{4}A_1^c\\&+V_oA_2^c-2\frac{V_oU_o}{\beta_o}A_1^s\Big)\Big),\end{aligned} \tag{57}$$

$$\begin{aligned}F_Y\approx\,&\pi\mu U_c\Big(2A_1^c+\varepsilon\Big(\frac{\beta_o}{2}A_1^s+\frac{1}{4}A_1^s\\&+V_oA_2^s+2\frac{V_oU_o}{\beta_o}A_1^c\Big)\Big).\end{aligned} \tag{58}$$

The aerodynamic coefficients C_L and C_D are defined by

$$C_L=F_Y/(\pi\mu U_c),\ C_D=F_X/(\pi\mu U_c). \tag{59}$$

Figure 2 shows the coefficients A_1^s and A_1^c for various Re. Bretherton's results[2] are also shown in this figure as broken lines. We see that there is no essential difference between the present results and Bretherton's. The difference of A_1^c is due to the higher order, that is, the present one is given by a one cycle higher order approximation and we see that the effect of the higher order in the shear flow is much greater on the lift force than on the drag force.

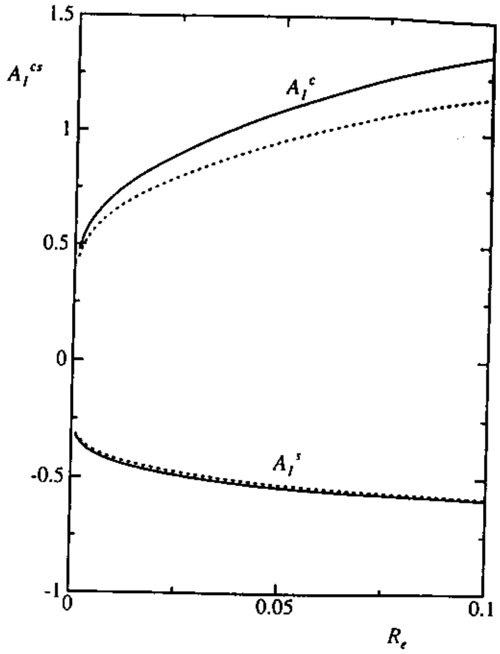

Fig. 2 Coefficients A_1^s and A_1^c in the case of $U=V=1/2^{1/2}$. Solid line: Eqs.(44), (45), Dotted line: Bretherton's results

6. Conclusions

The present paper investigates a circular cylinder in the combination of a uniform flow and a simple shear flow and the Reynolds number with respect to incoming flow velocity is assumed to be small. The governing integral expressions of the Oseen-type equations are constructed. The method of matched asymptotic expansions proposed by Kida and co-workers[(20)-(30)] is applied to the governing integral equations and the inner and outer solutions are obtained from the equations. The present paper reveals that the outer solutions given by Bretherton[(2)] can be obtained by the present approach without using his assumption (c) mentioned in section 1 and we can confirm that the assumptions (a) and (b) used by Bretherton are correct within the first approximation. The present approach is so systematic that it is shown to be easily extended to the higher order approximations. It is also shown that the lift force is much greater than the drag force in shear flows and the effect on aerodynamic forces due to a one cycle higher order approximation is very small for the drag force but slightly large for the lift force.

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Appendix 1

In order to solve Eq. (17), we introduce a new function $g_f(\vec{x}\,;\alpha)$ defined by the following expression:

$$G_f(\vec{x}-\vec{x}_o\,;\tilde{y}_o)$$
$$\equiv\int_0^\infty g_f(x_1-x_{o1}+\alpha\tilde{y},\ \tilde{y}-\tilde{y}_o\,;\alpha)d\alpha, \qquad (60)$$
$$g_f(\xi,\eta\,;\alpha)\to 0 \text{ as } |\xi|+|\eta|\to\infty \text{ or } \alpha\to\infty, \qquad (61)$$
$$g_f(\xi,\eta\,;\alpha)\to 0 \text{ as } \alpha\to 0. \qquad (62)$$

From the conditions in Eqs. (61), (62), we easily obtain

$$\int_0^\infty \frac{dg_f}{d\alpha}d\alpha=0. \qquad (63)$$

Therefore, we have

$$\int_0^\infty\left(\tilde{y}\frac{\partial g_f}{\partial\xi}+\frac{\partial g_f}{\partial\alpha}\right)d\alpha=0. \qquad (64)$$

Substituting Eq. (60) into Eq. (17) and using Eq. (64), we arrive at

$$\delta(\xi-\alpha\tilde{y}_o)\delta(\eta)=\int_0^\infty\left((1+\alpha^2)\frac{\partial^2 g_f}{\partial\xi^2}+\frac{\partial^2 g_f}{\partial\eta^2}\right.$$
$$\left.+2\alpha\frac{\partial^2 g_f}{\partial\xi\partial\eta}-2\varepsilon\beta_o\frac{\partial g_f}{\partial\alpha}-\varepsilon^2V_o^2 g_f\right)d\alpha, \qquad (65)$$

where $\xi=x_1-x_{o1}+\alpha\tilde{y}$ and $\eta=\tilde{y}-\tilde{y}_o$. Thus, we see that a solution of Eq. (65) is governed by the following differential equation:

$$(1+\alpha^2)\frac{\partial^2 g_f}{\partial\xi^2}+\frac{\partial^2 g_f}{\partial\eta^2}+2\alpha\frac{\partial^2 g_f}{\partial\xi\zeta\eta}-2\varepsilon\beta_o\frac{\partial g_f}{\partial\alpha}$$
$$-\varepsilon^2V_o^2g_f=\delta(\xi-\alpha\tilde{y}_o)\delta(\eta)\delta(\alpha). \qquad (66)$$

The solution of Eq. (66) is obtained using the Fourier transformation. The final expression is given by

$$g_f(\xi,\eta)=-\frac{1}{4\pi}\frac{1}{\alpha(1+\alpha^2/12)^{1/2}}\exp\left[-\frac{\varepsilon}{2\alpha}\frac{\beta_o}{1+\alpha^2/12}\right.$$
$$\left.\times\left(\xi^2+\left(1+\frac{\alpha^2}{3}\right)\eta^2-\alpha\xi\eta\right)-\frac{\alpha}{2\beta_o}\varepsilon V_o^2\right]. \qquad (67)$$

Changing the variable α to β by $\alpha=2\cdot 3^{1/2}/\beta$ and using $\xi=X+\alpha Y+\alpha\tilde{y}_o$ and $\eta=Y$, where $X=x_1-x_{o1}$ and $Y=x_2-x_{o2}$, the fundamental function G_f from Eqs. (60), (67) is given by Eq. (23).

No-shear-flow case, that is, $G\to 0$, corresponds to $\beta_o\to 0$ and $\beta_o\tilde{y}_o\to U_o$. As $\beta_o>0$, we change the integral variable β as $\xi=\beta\beta_o$. Then, G_f for $\beta_o\to 0$ becomes

$$G_f(X,Y\,;\tilde{y}_o)=-\frac{1}{4\pi}\int_0^\infty\frac{1}{\xi}\exp\left[-\frac{\varepsilon}{4\cdot 3^{1/2}}(X^2\right.$$
$$\left.+Y^2)\xi-\varepsilon 3^{1/2}\frac{V_o^2}{\xi}-\varepsilon XU_o-\varepsilon 3^{1/2}\frac{U_o^2}{\xi}\right]d\xi,$$
$$=-\frac{1}{2\pi}\exp(-\varepsilon U_oX)K_o(\varepsilon((X^2+Y^2)(V_o^2$$
$$+U_o^2))^{1/2}), \qquad (68)$$

where $K_o(x)$ is the modified Bessel function of zero-th order. Thus, we can derive the same fundamental solution in the case of flow past a body in a uniform flow as that given by Kida et al.[28],[29]

Appendix 2

In order to obtain the asymptotic expression of G_f for the inner and outer regions, we use the following theorem:

Theorem A: Let us consider the following integral:

$$P\equiv\int_0^\infty f(t)\exp\left(-\varepsilon\frac{a}{t}-\varepsilon bt\right)dt, \qquad (69)$$

where a and b are constants independent of ε. Suppose that the function f is sufficiently continuous, $|f|\to O(1/t)$ as $t\to\infty$ and $|f|\to O(1)$ as $t\to 0$, and for $t\in[0,\delta]$ where δ is an arbitrary small parameter with $\delta>\varepsilon_o$, where ε_o is some small value:

$$f=\sum_{n=0}f_nt^n\left(\sum_{m=0}^{N(n)}g_m^n\log^m t\right), \qquad (70)$$

where f_n and g_m^n are constant and $N(n)$ is an integer dependent on n. Then, we have the following asymptotic form with respect to ε for $\varepsilon\le\varepsilon_o$:

$$P=\mathrm{Pf}\int_0^\infty f(t)\exp(-\varepsilon bt)\sum_{n=0}\frac{1}{n!}\left(-\varepsilon\frac{a}{t}\right)^n dt$$
$$+\varepsilon af(0)(\gamma+\log(\varepsilon a)-1)+O(\varepsilon^2), \qquad (71)$$

where $\mathrm{Pf}\int(\cdot)dt$ denotes the Pf-integral, that is, the finite part of $\lim_{\Delta\to 0}\int_\Delta^\infty(\cdot)dt$ (see Sellier[31)-(33)]). □

Proof:

We introduce an arbitrary small parameter δ which is $\delta>\varepsilon_o$. Then, since $\exp\left(-\varepsilon\frac{a}{t}\right)=\sum_{m=0}^\infty\frac{1}{m!}\left(-\varepsilon\times\frac{a}{t}\right)^m$ for $t\ge\delta$, we can express P as

$$P=\int_\delta^\infty f(t)\exp(-\varepsilon bt)\sum_{n=0}^\infty\frac{1}{n!}\left(-\varepsilon\frac{a}{t}\right)^n dt$$
$$+\int_0^\delta f(t)\exp\left(-\varepsilon\frac{a}{t}-\varepsilon bt\right)dt,$$
$$=\mathrm{Pf}\int_0^\infty f(t)\exp(-\varepsilon bt)\sum_{n=0}^\infty\frac{1}{n!}\left(-\varepsilon\frac{a}{t}\right)^n dt$$
$$+\mathrm{Pf}\int_0^\delta f(t)\left[\exp\left(-\varepsilon\frac{a}{t}\right)\right.$$
$$\left.-\sum_{n=0}^\infty\frac{1}{n!}\left(-\varepsilon\frac{a}{t}\right)^n\right]\exp(-\varepsilon bt)dt. \qquad (72)$$

Here we note that P is independent of δ, that is, P is not a function of δ.

Let us estimate the second integral on the right-hand side of Eq. (72). We use the following relations:

$$K_k^l\equiv\mathrm{Pf}\int_0^\delta t^k\log^l t\,dt=\frac{\delta^{k+1}}{k+1}-\frac{l}{k+1}K_{k-1}^l$$
$$\text{for } k\ne -1, \qquad (73)$$
$$K_{-1}^l=\frac{1}{l+1}\log^{l+1}\delta \quad \text{for } l\ne -1, \qquad (74)$$
$$K_{-1}^{-1}=\mathrm{Pf}\int_{-\infty}^{\log\delta}\frac{1}{y}dy=\log(\log\delta). \qquad (75)$$

Deriving Eq. (75), the integral variable t was changed to y by $t=\exp(y)$. Then, we see that K_k^l=function of δ for any integer of k and l, where "function of δ"

denotes that a function is expressed as a sum of terms multiplied by a nonzero power of δ together with a power of $\log\delta$ or nonzero power of $\log\delta$. Since $\exp(-\varepsilon bt)=\sum_{n=1}\frac{1}{n!}(-\varepsilon bt)^n$ for $t\le\delta$, we have the following relation:

$$\mathrm{Pf}\int_0^\delta f(t)\exp(-\varepsilon bt)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\varepsilon\frac{a}{t}\right)^n dt$$
$$=\sum_n\sum_m^{N(n)}\sum_p\sum_q f_n g_m^n\frac{1}{p!q!}(-\varepsilon b)^p(-\varepsilon a)^q K_{n+p-q}^m$$
$$=\text{function of }\delta. \tag{76}$$

Let us consider the following integral:

$$L_k^l(\varepsilon a)\equiv\mathrm{Pf}\int_0^\delta t^k\log^l t\exp\left(-\varepsilon\frac{a}{t}\right)dt. \tag{77}$$

Here, we easily obtain

$$L_0^0(\varepsilon a)=\delta\exp\left(-\frac{\varepsilon a}{\delta}\right)-\varepsilon a$$
$$\times\text{finite part of }\lim_{\Delta\to 0}\left(E_i\left(-\frac{\varepsilon a}{\Delta}\right)-E_i\left(-\frac{\varepsilon a}{\delta}\right)\right)$$
$$\approx\varepsilon a(\gamma+\log\varepsilon+\log a-1)+O(\varepsilon^2)$$
$$+\text{function of }\delta, \tag{78}$$

where $E_i(-x)=-\int_x^\infty\frac{\exp(-x)}{x}dx$. From integration by parts, we have

$$L_0^n(\varepsilon a)=-\varepsilon a(-1)^n n!$$
$$-\varepsilon a\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!}L_{-1}^{n-r}(\varepsilon a)$$
$$+\text{function of }\delta. \tag{79}$$

Furthermore, we have the relation:

$$\frac{\partial}{\partial a}L_{k+1}^l(\varepsilon a)=-\varepsilon L_k^l(\varepsilon a). \tag{80}$$

Therefore, $L_0^n(\varepsilon a)$ becomes

$$L_0^n(\varepsilon a)=-\varepsilon a(-1)^n n!+a\frac{d}{da}L_0^n(\varepsilon a)$$
$$+a\sum_{r=1}^{n}(-1)^r\frac{n!}{(n-r)!}\frac{d}{da}L_0^{n-r}(\varepsilon a)$$
$$+\text{function of }\delta. \tag{81}$$

Thus, we obtain

$$L_0^1(\varepsilon a)=\varepsilon a+a\frac{d}{da}L_0^1(\varepsilon a)-\varepsilon a(\gamma+\log\varepsilon$$
$$+\log a)+\text{function of }\delta. \tag{82}$$

Since $L_0^1(0)=\int_0^\delta\log t\,dt=$ function of δ, we have

$$L_0^1(\varepsilon a)=\varepsilon a\log(\varepsilon a)\left(1-\gamma-\frac{1}{2}\log(a\varepsilon)\right)$$
$$+\text{function of }\delta. \tag{83}$$

Therefore, since $L_0^n(0)=$ function of δ, we easily obtain

$$L_0^n(\varepsilon a)=O_1(\varepsilon)+\text{function of }\delta, \tag{84}$$

where $O_1(\varepsilon)$ denotes the order of ε multiplied by a power of $\log\varepsilon$. Since $L_{k+1}^l(0)=$ function of δ, we have from Eq.(80)

$$L_m^n(\varepsilon a)=O_1(\varepsilon^{m+1})+\text{function of }\delta,$$
$$\text{for }n>0. \tag{85}$$

From the assumption of Theorem A, we have

$$\int_0^\delta f(t)\exp\left(-\varepsilon\frac{a}{t}\right)\exp(-\varepsilon bt)dt$$
$$=\sum_n\sum_{m=0}^{N(n)}\int_0^\delta f_n g_m^n t^n\log^m t$$
$$\times\sum_{p=0}^{\infty}\frac{1}{p!}(-\varepsilon b)^p t^p\exp\left(-\varepsilon\frac{a}{t}\right)dt,$$
$$=\sum_n\sum_{m=0}^{N(n)}\sum_{p=0}^{\infty}f_n g_m^n\frac{(-\varepsilon b)^p}{p!}L_{n+p}^m(\varepsilon a),$$
$$\approx\sum_{m=0}^{N(0)}f_0 g_m^0 L_0^m(\varepsilon a)+O_1(\varepsilon^2)$$
$$\approx f_0 g_0^0 L_0^0(\varepsilon a)+O_1(\varepsilon^2)=\varepsilon af(0)(\gamma+\log\varepsilon$$
$$+\log a-1)+O_1(\varepsilon^2). \tag{86}$$

Thus, we can estimate the second term on the right-hand side of Eq.(72) from Eqs.(76), (86) and we can arrive at Theorem A. We note that $N(0)=0$ from the assumption of this theorem. □

Appendix 3

Let us consider the asymptotic expansion of G_f for the inner region, say G_f^i. From Eq.(23), G_f^i is expressed as

$$G_f^i=-\frac{1}{4\pi}\mathrm{Pf}\int_0^\infty\frac{1}{(1+t^2)^{1/2}}\exp\left[-\varepsilon bt-\varepsilon\frac{a}{t}-\varepsilon c\right.$$
$$\left.+\varepsilon\frac{c}{1+t^2}-\varepsilon d\frac{t}{1+t^2}\right]dt, \tag{87}$$

where a, b, c and d are independent of t and of $O(1)$ with respect to ε:

$$a=\frac{3^{1/2}}{\beta_o}V_o^2,\ b=\frac{\beta_o}{4\cdot 3^{1/2}}(X^2+Y^2),$$
$$c=\frac{\beta_o}{2}(XY+2Y\bar{y}_o),$$
$$d=3^{1/2}\beta_o\left(Y\bar{y}_o+\bar{y}_o^2+\frac{Y^2}{4}-\frac{X^2}{12}\right).$$

From Theorem A (see Appendix 2), G_f^i is expanded as

$$G_f^i=-\frac{1}{4\pi}\mathrm{Pf}\int_0^\infty\frac{1}{(1+t^2)^{1/2}}\exp(-\varepsilon bt)\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$\times\left[-\varepsilon\frac{a}{t}-\varepsilon c+\varepsilon\frac{c}{1+t^2}-\varepsilon d\frac{t}{1+t^2}\right]^n dt$$
$$-\frac{1}{4\pi}\varepsilon a(\gamma+\log\varepsilon+\log a-1)+O_1(\varepsilon^2). \tag{88}$$

Here, we have to estimate the above Pf-integral. To do this, we use the following relations for $a_o>0$ (see 3.366 and 3.374 in Gradshteyn and Ryzhik[34]):

$$\int_0^\infty\frac{1}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$=-\frac{\pi}{2}[\boldsymbol{E}_o(a_o)+N_o(a_o)], \tag{89}$$
$$\int_0^\infty\frac{x}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$=\frac{\pi}{2}[\boldsymbol{H}_1(a_o)-N_1(a_o)]-1, \tag{90}$$

where N_n is the Bessel function and $\boldsymbol{H}_n$ and $\boldsymbol{E}_n$ are Struve functions and Weber's function, respectively. From Eqs.(89), (90), we easily obtain for $a_o\to 0$:

$$\int_0^\infty\frac{1}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$\approx-\gamma-\log\left(\frac{a_o}{2}\right)+a_o-\frac{a_o^2}{4}\left(\gamma+\log\frac{a_o}{2}+1\right). \tag{91}$$

$$\int_0^\infty \frac{x}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$\approx\frac{1}{a_o}-1+\frac{a_o}{2}\left(\gamma+\log\frac{a_o}{2}+\frac{3}{2}\right). \quad (92)$$

Using integration by parts, we easily obtain from Eqs. (91), (92):

$$\int_0^\infty \frac{1}{(1+x^2)^{3/2}}\exp(-a_o x)dx$$
$$\approx 1-a_o+\frac{a_o^2}{2}\left(\gamma+\log\frac{a_o}{2}+\frac{3}{2}\right), \quad (93)$$

$$\int_0^\infty \frac{1}{(1+x^2)^{5/2}}\exp(-a_o x)dx\approx\frac{2}{3}-\frac{1}{3}a_o, \quad (94)$$

$$\int_0^\infty \frac{1}{(1+x^2)^{7/2}}\exp(-a_o x)dx\approx\frac{8}{15}. \quad (95)$$

Differentiating these relations with respect to a_o, we arrive at

$$\int_0^\infty \frac{x^2}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$\approx\frac{1}{a_o^2}-\frac{1}{2}\left(\gamma+\log\left(\frac{a_o}{2}\right)+\frac{5}{2}\right), \quad (96)$$

$$\int_0^\infty \frac{x}{(1+x^2)^{3/2}}\exp(-a_o x)dx$$
$$\approx 1-a_o\left(\gamma+\log\left(\frac{a_o}{2}\right)+2\right), \quad (97)$$

$$\int_0^\infty \frac{x}{(1+x^2)^{5/2}}\exp(-a_o x)dx\approx\frac{1}{3}+\frac{1}{3}a_o. \quad (98)$$

Here, we use the relation:

$$\frac{\partial}{\partial a_o}\mathrm{Pf}\int_0^\infty \frac{1}{x(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$=-\mathrm{Pf}\int_0^\infty \frac{1}{(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$\approx\gamma+\log\frac{a_o}{2}-a_o,$$
$$\mathrm{Pf}\int_0^\infty \frac{1}{x(1+x^2)^{1/2}}dx$$
$$=-\text{finite part of }\lim_{\Delta\to 0}\log\left(\frac{1+(1+x^2)^{1/2}}{x}\right)\Big|_\Delta^\infty,$$
$$=\log 2.$$

Thus, we have

$$\mathrm{Pf}\int_0^\infty \frac{1}{x(1+x^2)^{1/2}}\exp(-a_o x)dx$$
$$\approx\log 2+a_o\left(\gamma+\log\left(\frac{a_o}{2}\right)-1\right)-\frac{1}{2}a_o^2. \quad (99)$$

Using Eqs. (91) - (99), Eq. (88) becomes

$$G_f^i\approx-\frac{1}{4\pi}\left[-\gamma-\log\left(\frac{\varepsilon b}{2}\right)+\varepsilon b-\varepsilon a\log 2\right.$$
$$\left.+\varepsilon c\left(\gamma+\log\left(\frac{\varepsilon b}{2}\right)+1\right)-\varepsilon d\right]$$
$$-\frac{1}{4\pi}\varepsilon a(\gamma+\log(\varepsilon a)-1).$$

Thus, we arrive at

$$G_f^i(X,Y;\tilde{y}_o)\approx\frac{1}{4\pi}\left[\left\{\gamma-3\log 2-\frac{1}{2}\log 3\right.\right.$$
$$\left.+\log\varepsilon+\log\beta_o+\log(X^2+Y^2)\right\}$$
$$\times\left\{1-\varepsilon\beta_o\frac{XY+2X\tilde{y}_o}{2}\right\}-\varepsilon\beta_o\frac{X^2+Y^2}{4\cdot 3^{1/2}}$$
$$+\varepsilon\frac{3^{1/2}}{\beta_o}V_o^2\log 2-\varepsilon\frac{3^{1/2}}{\beta_o}V_o^2\left(\gamma+\log\left(\frac{\varepsilon 3^{1/12}V_o^2}{\beta_o}\right)\right.$$
$$\left.-1\right)-\varepsilon\beta_o\frac{XY+2X\tilde{y}_o}{2}$$
$$\left.+\varepsilon\beta_o 3^{1/2}\left(Y\tilde{y}_o+\tilde{y}_o^2+\frac{Y^2}{4}-\frac{X^2}{12}\right)\right]+O(\varepsilon^2). \quad (100)$$

G_f for the outer variables, say G_f^o, is expressed as

$$G_f^o(\hat{X},\hat{Y};\hat{\tilde{Y}}_o)$$
$$=-\frac{1}{4\pi}\int_0^\infty f^o(\hat{X},\hat{Y};t)\exp\left(-\varepsilon\frac{3^{1/2}}{\beta_o t}V_o^2\right)dt, \quad (101)$$

where the outer variables are defined as $(\hat{X},\hat{Y})=\varepsilon^{1/2}(X,Y)$, and

$$f^o=\frac{1}{(1+t^2)^{1/2}}\exp\left[-\frac{\beta_o}{4\cdot 3^{1/2}}t(\hat{X}^2+\hat{Y}^2)\right.$$
$$-\frac{\beta_o}{2}\frac{t^2}{t^2+1}(\hat{X}\hat{Y}+2\hat{X}\hat{\tilde{Y}}_o)-3^{1/2}\beta_o\frac{t}{t^2+1}$$
$$\left.\times\left(\hat{Y}\hat{\tilde{Y}}_o+\hat{\tilde{Y}}_o^2+\frac{\hat{Y}^2}{4}-\frac{\hat{X}^2}{12}\right)\right]. \quad (102)$$

Then we have from Theorem A (see Appendix 2):

$$G_f^o\approx-\frac{1}{4\pi}\mathrm{Pf}\int_0^\infty f^o(\hat{X},\hat{Y};t)\left(1-\varepsilon\frac{3^{1/2}}{\beta_o t}V_o^2\right)dt$$
$$-\varepsilon\frac{1}{4\pi}\frac{3^{1/2}}{\beta_o}V_o^2\left(\gamma+\log\left(\frac{\varepsilon 3^{1/2}V_o^2}{\beta_o}\right)-1\right). \quad (103)$$

Thus, we arrive at

$$G_f^o(\hat{X},\hat{Y};\hat{\tilde{Y}}_o)\approx-\frac{1}{4\pi}\mathrm{Pf}\int_0^\infty\frac{1}{(1+\beta^2)^{1/2}}$$
$$\times\exp\left[-\frac{\beta_o}{4\cdot 3^{1/2}}\beta(\hat{X}^2+\hat{Y}^2)\right.$$
$$-\frac{\beta_o}{2}\frac{\beta^2}{\beta^2+1}(\hat{X}\hat{Y}+2\hat{X}\hat{\tilde{Y}}_o)$$
$$\left.-3^{1/2}\beta_o\frac{\beta}{\beta^2+1}\left(\hat{Y}\hat{\tilde{Y}}_o+\hat{\tilde{Y}}_o^2+\frac{\hat{Y}^2}{4}-\frac{\hat{X}^2}{12}\right)\right]$$
$$\times\left(1-\frac{\varepsilon}{\beta_o}\frac{3^{1/2}}{\beta}V_o^2+O(\varepsilon^2)\right)d\beta$$
$$-\frac{\varepsilon}{4\pi}\frac{3^{1/2}}{\beta_o}V_o^2\left(\gamma+\log\left(\frac{\varepsilon 3^{1/2}V_o^2}{\beta_o}\right)-1\right), \quad (104)$$

where $\hat{\tilde{Y}}_o=\hat{Y}_o+\varepsilon^{1/2}\dfrac{U_o}{\beta_o}$.

Appendix 4

Let us consider S^s defined by Eq. (47). Substituting Eq. (43) into Eq. (47), we have

$$\frac{S^s}{\beta_o}\approx\frac{2}{\pi}\int_0^{2\pi}\sin\theta d\theta\,\mathrm{Pf}\int_0^\infty\frac{B}{2(1+t^2)^{1/2}}dt$$
$$\times\mathrm{Pf}\int_0^\infty R\exp(CR^2)dR. \quad (105)$$

From the definition of Pf-integral, we obtain

$$\frac{S^s}{\beta_o}\approx\text{finite part of }\lim_{\Delta\to 0}\lim_{N\to\infty,\delta\to 0}\frac{2}{\pi}\int_0^{2\pi}\sin\theta d\theta$$
$$\times\int_\delta^N\frac{B}{2(1+t^2)^{1/2}}dt\int_\Delta^\infty R\exp(CR^2)dR. \quad (106)$$

We have to consider the order of limitation from the procedure of analysis.

The integration with respect to R is easily carried out:

$$\frac{S^s}{\beta_o}\approx-\text{finite part of }\lim_{\Delta\to 0}\lim_{N\to\infty,\delta\to 0}\frac{1}{2\pi}\int_0^{2\pi}\sin\theta d\theta$$

$$\times\int_{\delta}^{N}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}\exp(C\Delta^2)dt. \tag{107}$$

Here, we consider the following integral:

$$\begin{aligned}S_e &\equiv\int_{\delta}^{N}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}\exp(C\Delta^2)dt,\\ &=\int_{\delta}^{1/\delta_o}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}\exp(C\Delta^2)dt\\ &+\int_{1/\delta_o}^{N}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}\exp(C\Delta^2)dt,\\ &=\int_{\delta}^{\infty}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}dt\\ &+\int_{1/\delta_o}^{N}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}(\exp(C\Delta^2)-1)dt\\ &\quad\text{for } \Delta\to 0,\end{aligned} \tag{108}$$

where δ_o is a small parameter with $1\gg\delta_o\gg\Delta^2$.

For $t\gg 1$, $B\approx\frac{3^{1/2}}{6}t\sin\theta$, $C\approx-\frac{3^{1/2}}{12}t$ from the definitions in Eq. (30). Thus, the second integral of Eq. (108) becomes

$$\begin{aligned}&\int_{1/\delta_o}^{N}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}(\exp(C\Delta^2)-1)dt\\ &\approx-2\sin\theta\int_{1/\delta_o}^{N}\frac{1}{t}\left(\exp\left(-\frac{3^{1/2}}{12}t\Delta^2\right)-1\right)dt,\\ &\approx-2\sin\theta\left(E_1\left(-\frac{3^{1/2}}{12}\Delta^2N\right)-E_1\left(-\frac{3^{1/2}}{12}\Delta^2/\delta_o\right)\right.\\ &\left.-\log(\Delta^2N)+\log(\Delta^2/\delta_o)\right),\end{aligned} \tag{109}$$

where $E_1(-x)=-\int_x^{\infty}\frac{1}{x}\exp(-x)dx$.

From the order of limitation, first we have to take the step, $N\to\infty$, and second we have to take the step, $\Delta\to 0$. Thus, S_e becomes

$$\begin{aligned}S_e &=\int_{\delta}^{\infty}\frac{B}{C}\frac{1}{(1+t^2)^{1/2}}dt\\ &-2\sin\theta\left(-\gamma+2\log 2+\frac{1}{2}\log 3\right).\end{aligned} \tag{110}$$

Further, we consider the following integral:

$$\int_0^{2\pi}\frac{B}{C}\sin\theta d\theta=\frac{2\pi}{4+3t^2}(t^2+4-4t(1+t^2)^{1/2}). \tag{111}$$

From Eqs. (110), (111), we arrive at

$$\begin{aligned}\frac{S^s}{\beta_o} &=-\int_0^{\infty}\frac{1}{4+3t^2}\left(\frac{1}{t+(1+t^2)^{1/2}}+\frac{3}{(1+t^2)^{1/2}}\right)dt\\ &+2\log 2-\gamma+\frac{1}{2}\log 3.\end{aligned} \tag{112}$$

The first integral of the above equation is easily integrated and finally we arrive at the first equation of Eq. (50). These steps are applied to Eqs. (46) - (49), then we arrive at Eqs. (50), (51).

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