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Asymptotic solution of 2D and 3D boundary integral equations arising in Fluid Mechanics and Electrostatics

A. Sellier

Abstract We present a systematic method to asymptotically expand, with respect to a small slenderness or thickness parameter, the solution of a wide class of boundary integral equations arising in Electrostatics and Fluid Mechanics. The adopted point of view permits us to bypass the tedious matching rules of the widely employed method of matched asymptotic expansions. If each step of the proposed procedure is described within a general framework, the paper also addresses applications to 2D and 3D problems. The 3D example not only briefly reports but also extends the results obtained elsewhere by the author. The whole 2D application to the potential flow around a thin aifoil is original. Finally, a special attention is paid to the case of a non-smooth 2D domain.

1 Introduction

This paper presents a general method to asymptotically invert, with respect to a small slenderness or thickness parameter ϵ, a large class of boundary integral equations encountered in different fields such as Electrostatics and potential flows. Henceforth, the dimension d belongs to $\{2,3\}$ and $\mathscr{A}_d$ designates a bounded, open and simply connected subset of $\mathbb{R}^d$. This domain $\mathscr{A}_d$, of smooth boundary $\partial\mathscr{A}_d$, admits rounded ends O and E and is either a 'straight' slender-body if $d=3$ or a thin profile if $d=2$ in the following sense (see respectively Figs. 1 and 2):

(i) For $d=3$ and cartesian co-ordinates (O,x'_1,x'_2,x'_3) such that $e_3 = OE/L$ then $e := \mathrm{Max}_{M\in\partial\mathscr{A}_3}[(x'_1)^2+(x'_2)^2]^{1/2} = \epsilon L$ where the slenderness ratio ϵ satisfies $0 \le \epsilon \ll 1$.

(ii) For $d=2$ and cartesian co-ordinates (O,x'_1,x'_2) such that $e_1 = OE/L$ then $e := \mathrm{Max}_{M\in\partial\mathscr{A}_2}|x'_2| = \epsilon L$ where the thickness ratio ϵ obeys $0 \le \epsilon \ll 1$.

For M belonging to $\partial\mathscr{A}_d$, the vector $\mathbf{n}(M)$ designates the unit outwarding normal. In Electrostatics or for potential flow problems, one often looks for a perturbation potential ϕ solution to an elliptic and exterior boundary value problem consisting of the equations

$$\Delta\phi := \sum_{i=1}^{d}\frac{\partial^2\phi}{\partial x'_i\partial x'_i} = 0 \quad \text{in } \mathbb{R}^d\setminus(\mathscr{A}_d\cup\partial\mathscr{A}_d)\ , \tag{1.1}$$

$$\phi \sim C\log OM + O(1/OM) \text{ as } OM\to\infty \quad \text{if } d=2\ , \tag{1.2}$$

$$\phi \sim O(1/OM) \quad \text{as} \quad OM\to\infty \quad \text{if} \quad d=3\ , \tag{1.3}$$

together with the Neumann type boundary condition

$$\nabla\phi\cdot\mathbf{n} = g' \text{ on } \partial\mathscr{A}_d \quad \text{with} \quad \int_{\partial\mathscr{A}_d} g'\,\mathrm{d}s = 0\ , \tag{1.4}$$

or the Dirichlet type boundary condition, for function g and constants a and b,

$$\phi = g + a \text{ on } \partial\mathscr{A}_d \quad \text{and} \quad b = \int_{\partial\mathscr{A}_d}\nabla\phi\cdot\mathbf{n}\,\mathrm{d}s\ . \tag{1.5}$$

The constant C arising in behavior (1.2) actually obeys $2\pi C = \int_{\partial\mathscr{A}_2}\nabla\phi\cdot\mathbf{n}\,\mathrm{d}s$. In Eqs. (1.4) and (1.5), the functions g' and g are given (for instance $g' = -\nabla\phi_\infty\cdot\mathbf{n}$ and $g = -\phi_\infty$ where ϕ_∞ denotes the applied and undisturbed potential). In addition, in Eq. (1.5) a or b is prescribed (see Sect. 3). Note that for $d=2$ the Neumann type problems (1.1), (1.2) and (1.4) lacks another condition to be well-posed (see Sect. 4). As the small parameter ϵ goes to zero, the domain $\mathscr{A}_d$ collapses to the segment $[OE]$ and the derivation of the asymptotic expansion of unknown function ϕ, in terms of this small slenderness or thickness ratio ϵ, has received a considerable attention in the past decades. Nowadays, a theoretical approach of this question remains of interest for at least two reasons: the numerical treatments experience troubles in the limit $\epsilon\to 0$ and it is worth deriving simple asymptotic models. To achieve this task, one may think about the well-known method of matched asymptotic expansions (Van Dyke 1975). This method approximates ϕ in two domains: the inner one where $\delta(M) = O(e)$, if $\delta(M)$ stands for the distance from point M to the segment $[OE]$, and the outer one where $\delta(M) = O(L)$. The procedure requires to match, at each order, the derived asymptotic estimates. For details the reader is referred (among others) to Van Dyke (1954, 1956) if $d=2$ and Van Dyke (1959), Tuck (1964) and Euvrard (1983) if $d=3$. However, this approach yields more and more cumbersome calculations when enforcing the matching rules at high orders. This increasing complexity explains why the method actually only provides the very first orders of approximation. A different point of view, first alluded to by Landweber (1951), has motivated many works in the field. It consists in building the func-

A. Sellier
Ladhyx, Ecole Polytechnique, 91128 Palaiseau cedex, France
e-mail: sellier@ladhyx.polytechnique.fr

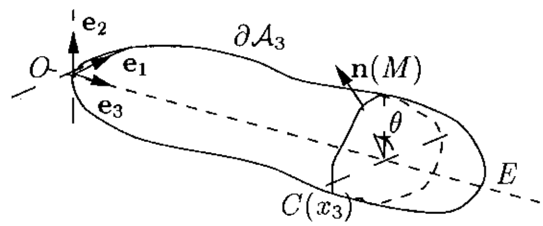

Fig. 1. A slender body and our notations

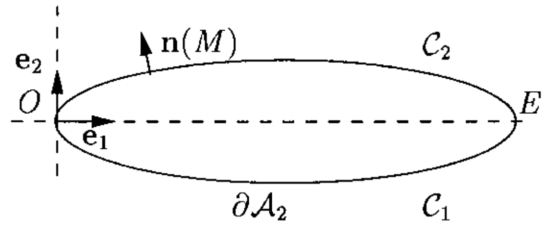

Fig. 2. A thin profile and our notations

tion ϕ outside $\mathscr{A}_d$ by spreading singularities distributed along a part of the segment $[OE]$. The boundary condition (1.4) or (1.5) imposes a one-dimensional integral equation (depending on the small parameter ϵ) for the unknown strength and extend of these singularities. Since he employed the method of matched asymptotic expansions to deal with this integral equation, Moran (1963) applied this procedure up to the second order only. Solutions up to any order were later derived by Geer and Keller (1968), Geer (1974, 1975) by using a systematic expansion of the one-dimensional integral equation. This expansion was established by Handelsman and Keller (1967a, b). This treatment avoids the matching rules previously alluded to. Unfortunately, this attractive approach presents at least three drawbacks:

(i) For $d = 3$, it only addresses slender bodies of revolution.
(ii) As recently outlined by Cade (1994), the one-dimensional integral equation under consideration does not, in general, admit a solution, i.e. is ill-posed.
(iii) The quantities of physical interest are related to the value of ϕ or its derivatives on the boundary $\partial\mathscr{A}_d$. Thus, additional calculations are needed in approximating these quantities.

The aim of this work is to present a general treatment free from the previous objections. The paper is organized as follows. Section 2 details, within a general framework, the different steps to achieve. It highlights the ability of the concept of integration in the finite part-sense of Hadamard to bypass the mathematical difficulties of the problem. The proposed procedure is illustrated in Sects. 3 and 4, respectively for a three-dimensional Dirichlet type problem arising in Electrostatics and the two-dimensional Neumann type problem pertaining to the potential flow past a thin (but smooth) airfoil. Our formulation is given in Sect. 2 for a smooth boundary $\partial\mathscr{A}_d$. However, in Sect. 5 we also show how it works for a thin airfoil with a cusped trailing edge E. Finally, a few concluding remarks close the paper in Sect. 6.

2
A general treatment

In this section we present, within a general framework, the typical steps of our approach. Henceforth, indices k or l belong to $\{1,2\}$ whilst indices i or j belong to $\{1,2,3\}$.

2.1
The "well-posed" boundary integral equation

The very first task consists in reducing the initial elliptic boundary value problem to a well-posed boundary integral equation. As soon as there exists a fundamental Green solution for the initial system, this integral equation is actually obtained by employing the well-known Green's representation formula. It often takes the form of the following general and linear Fredholm integral equation

$$\lambda q(M) + pv \int_{\partial\mathscr{A}_d} q(P)K_d(P,M)\mathrm{d}s = T(M) \quad \text{on} \quad \mathscr{A}_d \tag{2.1}$$

for the unknown density q and given constant λ, function T and kernel K_d. One speaks of integral equation of the first kind if $\lambda = 0$. In this case, the kernel K_d is regular on $\mathscr{A}_d$ and the symbol pv is omitted. For $\lambda \neq 0$, (2.1) becomes an integral equation of the second kind and our kernel K_d may be weakly-singular for $M = P$. In these latter circumstances, the pv symbol designates the Cauchy principal value (see, for $d = 3$, Kupradze 1963) with

$$pv \int_{\partial\mathscr{A}_d} q(P)K_d(P,M)\mathrm{d}s := \lim_{\mu\to 0,\mu>0} \int_{\partial\mathscr{A}_d\setminus v_\mu(M)} q(P)K_d(P,M)\mathrm{d}s \tag{2.2}$$

where the removed neighborhood $v_\mu(M)$ is defined as $v_\mu(M) = \{P \in \partial\mathscr{A}_d; MP < \mu\}$. In many cases, the kernel K_d turns out to be a finite sum of typical kernels K'_d of the following form

$$K'_d(P,M) = \delta_{2d}a_0(M)\log[PM] + a_1(M)\left\{\prod_{l=1}^{d}[x'_l(P) - x'_l(M)]^{i_l}\right\}/PM^{E+i_1+\cdots+i_d} \tag{2.3}$$

where δ_{ij} denotes the Kronecker delta and the positive integers i_l and positive or negative integer $E \leq d-1$ are given. If $E < d-1$, the kernel K'_d is regular whilst for $E = d-1$ with assume that $i_1 + \cdots + i_d = 1$ and K'_d becomes weakly-singular for $P = M$. This decomposition of K_d not only holds for the kernels arising in Electrostatics and potential flows but also for those encountered in 2D or 3D Stokes flow and Elasticity problems (think about the usual fundamental Oseen and Kelvin tensors). For many applications, the boundary integral equation (2.1) is "well-posed" in the following sense: the existence of solutions q may be established in adequate functional spaces and one also knows the associated eigenspace. Indeed, the solution to Eq. (2.1) is not always unique (for instance, in 3D Stokes flow, for a prescribed velocity T and zero vector λ on $\partial\mathscr{A}_3$, the vector q is obtained up to a multiple of the normal vector $\mathbf{n}$). Since "well-posed", the integral equation (2.1) is free from the objections of Cade (1994). Moreover, for $d = 3$ it holds for a body which is not necessary of revolution about the axis Ox'_3. In addition, in many applications the density q is directly related to the physical quantities of interest. Thus, the integral formulation (2.1) is highly preferable to the one-dimensional integral handled by Handelsman and Keller (1967a, b) and others. Since the domain of integration in Eq. (2.1) is the whole boundary $\partial\mathscr{A}_d$, this approach might however yield technical difficulties. This key point is addressed in the following Subsections 2.2 and 2.3.

2.2
Asymptotic expansion of the integral equation

Let us introduce the set $\mathscr{F}$ of positive functions f whose square f^2 is defined, analytic and of unit magnitude on $[0, 1]$ and can be expanded in power-series about the end-points as follows:

$$f^2(x) = \sum_{n\geq 1} c_n^0 x^n \quad \text{for } x \to 0^+ \ , \tag{2.4}$$

$$f^2(x) = \sum_{n\geq 1} c_n^1 (1-x)^n \quad \text{for } x \to 1^- \ , \tag{2.5}$$

with $c_1^0 > 0$ and $c_1^1 > 0$. For f belonging to $\mathscr{F}$, the notation $f^{(n)}$ designates the derivative of order n. In order to rewrite Eq. (2.1) in terms of the small parameter ϵ, we locate each point of $\partial\mathscr{A}_d$ by new and non-dimensional coordinates. Two cases occur:

(i) If $d = 2$, we set $x'_1 = Lx_1, x'_2 = ex_2$. The closed path $\partial\mathscr{A}_2$ is shared into its upper side $\mathscr{C}_2$ and lower side $\mathscr{C}_1$ (see Fig. 2) such that there exist functions $f_2, f_1 \leq f_2$ of $\mathscr{F}$ with

$$x_2 = f_k(x_1); q(M) = q_k(x_1) \quad \text{if } M(x_1, x_2) \in \mathscr{C}_k \ . \tag{2.6}$$

We set $x_l(P) = x_l^P$ and apply, for our typical kernel K'_2, the change of scales $(x'_1, x'_2) \to (x_1, x_2)$ and the shift $x_1^P = x_1 + u$ to the integral arising in Eq. (2.1). Hence, one is led to the following types of integrals

$$J_{kl}^0(\epsilon) = \int_{-x_1}^{1-x_1} v_k(u + x_1) \log[u^2 + \epsilon^2 h_{kl}^2(u)] du \ , \tag{2.7}$$

$$J_{kl}^1(\epsilon) = pv \int_{-x_1}^{1-x_1} \frac{v_k(u + x_1) u^{i_1} h_{kl}^{i_2}(u) du}{[u^2 + \epsilon^2 h_{kl}^2(u)]^{(E+i_1+i_2)/2}} \ , \tag{2.8}$$

if the new functions v_k and k_{kl} obey

$$v_k(x_1) = q_k(x_1)\{1 + [\epsilon f_k^{(1)}(x_1)]^2\}^{1/2} \ , \tag{2.9}$$

$$h_{kl}(u) = f_k(u + x_1) - f_l(x_1) \ . \tag{2.10}$$

(ii) If $d = 3$, we set $x'_3 = Lx_3, x'_2 = ex_2, x'_1 = ex_1$ and introduce non-dimensional cylindrical coordinates (r, θ, x_3) such that $r^2 = x_1^2 + x_2^2$. Then, the boundary $\partial\mathscr{A}_3$ admits the equation $r = f(\theta, x_3)$ where the function $f(\theta, .)$ belongs to $\mathscr{F}$. If $f_v^1 = \partial^1 f / \partial v$ for $v \in \{\theta, x_3\}$ and

$$s_\epsilon = \{1 + (f^{-1} f_\theta^1)^2 + (\epsilon f_{x_3}^1)^2\}^{1/2} \ , \tag{2.11}$$

it follows that, for $x_i(P) = x_i^P$,

$$ds = eL[fs_\epsilon](\theta_P, x_3^P) d\theta_P \, dx_3^P \ . \tag{2.12}$$

Owing to Eq. (2.12) and for $x_3^P = x_3 + u$, the integral occurring in Eq. (2.1) obeys, for K'_3 given by Eq. (2.3),

$$e^{E-1} pv \int_{\partial\mathscr{A}_3} q(P) K'_3(P, M) ds$$
$$= a_1(M) \epsilon^{E+i_1+i_2-1} pv \int_0^{2\pi} J_3(\epsilon) d\theta_P \ , \tag{2.13}$$

$$J_3(\epsilon) = \int_{-x_1}^{1-x_1} \frac{[x_1^P - x_1]^{i_1} [x_2^P - x_2]^{i_2} v_3(u + x_3) du}{u^{-i_3}[u^2 + \epsilon^2 h_3^2(u)]^{(E+i_1+i_2+i_3)/2}} \ , \tag{2.14}$$

where $v_3(x_3) = efs_\epsilon q(\theta_P, x_3)$ and

$$h_3(u) = h_3(\theta_P, u + x_3, \theta, x_3)$$
$$= \{f^2(\theta_P, x_3 + u) + f^2(\theta, x_3)$$
$$- 2\cos(\theta_P - \theta) f(\theta, x_3) f(\theta_P, x_3 + u)\}^{1/2} \ . \tag{2.15}$$

Clearly, the next step consists in building the asymptotic expansions of integrals $J_{kl}^0(\epsilon), J_{kl}^1(\epsilon)$ and $J_3(\epsilon)$ with respect to the small parameter ϵ. This task is not at all trivial and concentrates the Mathematical difficulties of our approach. Contrary to the cases of $J_{kl}^0(\epsilon)$ and $J_{kl}^1(\epsilon)$, the asymptotic estimate of $J_3(\epsilon)$ may be singular, i. e. it may happen that $|J_3(\epsilon)| \to \infty$ as $\epsilon \to 0$ (for instance, if $i_1 = i_2 = i_3 = 0$ and $E = 2$ the integral $J_3(0)$ becomes hypersingular). It is possible, through tedious calculations, to approximate the quantities $J_{kl}^0(\epsilon), J_{kl}^1(\epsilon)$ and $J_3(\epsilon)$ by employing the method of matched asymptotic expansions. Here, we rather invoke a systematic and powerful formula established in Sellier (1996). This formula makes use of the fruitful concept of integration in the finite part sense of Hadamard and addresses a large class of regular, weakly or strongly singular integrals depending upon a small parameter. The next Subsection 2.3 introduces this concept and provides the general asymptotic expansion of interest for our method.

2.3
A key result

Henceforth, C denotes the set of complex numbers. First, it is possible to define the Hadamard's finite part integration of specific complex functions (see also Hadamard 1932 and Schwartz 1966).

Definition 1 For any complex function v such that there exist $\eta > 0$, two complex families $(v_{pq}), (\alpha_p)$ and a complex function V fulfilling:

$$[v - V](\mu) = \sum_{p=0}^{P} \sum_{q=0}^{Q} v_{pq} \mu^{\alpha_p} \log^q \mu; \quad \mu \in]0, \eta[\ , \tag{2.16}$$

$$\mathrm{Re}(\alpha_P) < \cdots < \mathrm{Re}(\alpha_0) = 0; v_{00} = 0 \quad \text{if } \alpha_0 = 0 \ , \tag{2.17}$$

$$V(0) := \lim_{\mu \to 0} V(\mu) \text{ exists} \ , \tag{2.18}$$

the complex $V(0)$ is called the Hadamard's finite part of the quantity $v(\mu)$ and denoted by $fp[v(\mu)]$.

By employing the previous Definition 1, one may extend the usual concept of integration to strongly singular functions. These functions can exhibit singularities at a finite number of points and at infinity. If zero is the only authorized singularity, one for instance gets

Definition 2 If for $a < b$ and $u \in L^1_{loc}(]a,b[\setminus\{0\},C)$ there exist a strictly positive real η, four complex families $(u^1_{pq}), (u^2_{pq}), (\alpha^1_p), (\alpha^2_p)$ and two complex functions $U_1 \in L^1_{loc}(]a,b[,C)$ and $U_2 \in L^1_{loc}(]a,b[,C)$ such that

$$\mathrm{Re}(\alpha^l_P) < \cdots < \mathrm{Re}(\alpha^l_0) = -1; \quad l \in \{1,2\} \ , \tag{2.19}$$

$$u[(-1)^l x] = \sum_{p=0}^{P}\sum_{q=0}^{Q} u^l_{pq} x^{\alpha^l_p} \log^q x + U_l(x); \quad x \in]0,\eta[\ , \tag{2.20}$$

then:

$$fp\int_a^b u(x)\mathrm{d}x := fp\left[\int_a^{-\mu} + \int_\mu^b\right]u(x)\mathrm{d}x \ . \tag{2.21}$$

Of course, this definition applies to a function u regular on $]a,b[$ and if $u[(-1)^l x] = c(-1)^l/x + U_l(x)$ the integral (2.21) becomes the usual Cauchy principal value. The Definition 2 actually authorizes us to consider different strongly singular behaviors at the origin (on the right for $x > 0$ and on the left for $x < 0$). Through a simple shift, it also makes it possible to deal with a class of complex functions singular at $x_s \in \mathbb{R}\setminus\{0\}$. Moreover, if $u \in L^1_{loc}(]a,+\infty[\setminus\{0\},C)$ admits at infinity a singular behavior given by (2.20) for $l = 2$ and x replaced by $1/x$, then $fp\int_a^\infty u(x)\mathrm{d}x := fp[\int_a^{1/\mu} u(x)\mathrm{d}x]$.

For instance, we can apply Definition 2 to the following general integral

$$I(\epsilon) = \int_a^b g(u)L[u,\epsilon h(u)]\mathrm{d}u \tag{2.22}$$

if $-\infty < a < 0 < b < \infty$, g and h are smooth enough complex functions and $L[u,v]$ is a regular or singular "Q pseudo-homogeneous" kernel such that

$$L[tu,tv] = t^Q S(t)L[u,v], \qquad \text{for } t \neq 0 \tag{2.23}$$

with Q a positive or negative integer and $S(t) = 1$ or $S(t) = t/|t|$ on $\mathbb{R}\setminus\{0\}$. In addition, the kernel L may be singular only for $(u,v) = (0,0)$ and admits partial derivatives $\partial^n_2 L := \partial^n L/\partial v^n$ bounded for $(u,v) \neq (0,0)$. Observe that the asymptotic expansion of $J^0_{kl}(\epsilon)$ may be reduced to the one of $I(\epsilon)$ if we choose $L[u,v] = \log[u^2+v^2] - \log[u^2]$. This latter kernel indeed satisfies our assumptions with $Q = 0, S(t) = 1$ and bounded partial derivatives with respect to v for $(u,v) \neq (0,0)$. The remaining integrals $J^1_{kl}(\epsilon)$ and $J_3(\epsilon)$ also look like $I(\epsilon)$ (take as kernel $L[u,v] = (u^2+v^2)^{-(E+i_1+\cdots+i_d)/2}$).

When building the required asymptotic approximation of $I(\epsilon)$, as ϵ goes to zero, two different circumstances arise:

(i) The case $h(0) = 0$. Since our function h is smooth throughout $]a,b[$, the function $\epsilon h(u)/u$ remains small in this domain. Thus, one can approximate $L[1,\epsilon h(u)/u]$ by using a Taylor expansion of $L[1,v]$ with respect to v. More precisely, one may prove that, as $\epsilon \to 0$,

$$I(\epsilon) = \sum_{n=0}^{N}\frac{\partial^n_2 L[1,0]}{n!}\left[fp\int_a^b \frac{S(u)g(u)[h(u)]^n\,\mathrm{d}u}{u^{n-Q}}\right]\epsilon^n + O(\epsilon^{N+1}) \ . \tag{2.24}$$

Each term in Eq. (2.24) involves the functions g and h on the whole domain $]a,b[$: only outer terms appear in the asymptotic expansion of $I(\epsilon)$ as soon as $h(0) = 0$. Since (see Eq. (2.10) $h_{kk}(0) = 0$ (with no summation over k), this result immediately applies to integrals $J^0_{kk}(\epsilon)$ and $J^1_{kk}(\epsilon)$.

(ii) The case $h(0) \neq 0$. In those circumstances, it is more tedious to establish the asymptotic behavior of $I(\epsilon)$ and the reader is directed, for additional details, to Sellier (1996). This time inner terms (from the domain $|u| = O(\epsilon)$) occur in the result and the following theorem holds:

Theorem 1 For $h(0) > 0$ and smooth functions g, h and "Q pseudo-homogeneous" kernel L satisfying the previous assumptions, one gains, for $N \geq \mathrm{Max}[0, Q+1]$ and as $\epsilon \to 0$, the asymptotic estimate

$$\begin{aligned} I(\epsilon) = {} & \sum_{n=0}^{N}\frac{\partial^n_2 L[1,0]}{n!}\left[fp\int_a^b \frac{S(u)g(u)[h(u)]^n\,\mathrm{d}u}{u^{n-Q}}\right]\epsilon^n \\ & + \sum_{m=0}^{N-Q-1}\sum_{l=0}^{m}\sum_{i=0}^{m-l} C_1^{m,l,i}\epsilon^{Q+m+1} \\ & - [1 - S(-1)]\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{i=0}^{n-l}\sum_{j=0}^{l-Q-1} C_2^{n,l,i,j}\epsilon^n \log\epsilon \\ & + O(\epsilon^{N+1}\log\epsilon) \ , \end{aligned} \tag{2.25}$$

where $\sum_{j=0}^{J} a_j := 0$ if $J < 0$ and the quantities $C_1^{m,l,i}$ and $C_2^{n,l,i,j}$ are given by

$$C_1^{m,l,i} = \frac{g^{(l)}(0)c^i_{m-l-i}}{l!i!}\left[fp\int_{-\infty}^{\infty}\partial^i_2 L[t,h(0)]t^m\,\mathrm{d}t\right] \ , \tag{2.26}$$

$$C_2^{n,l,i,j} = \frac{[h(0)]^l g^{(j)}(0)c^i_{l-Q-j-1}}{l!i!j!}\partial^n_2 L[1,0] \ , \tag{2.27}$$

if $g^{(l)}$ denotes the derivative of order l and the coefficients c^i_p are deduced from the behavior of h in vincinity of zero as follows

$$c^0_p = \delta_{p0}; \quad p \geq 0 \ , \tag{2.28}$$

$$\{[h(u) - h(0)]/u\}^i = \sum_{p\geq 0} c^i_p u^p, \text{ as } u \to 0 \ . \tag{2.29}$$

This key theorem provides the asymptotic expansion of $I(\epsilon)$ up to high orders of approximation without too much efforts. It also suggests several remarks:

1. Whilst the first sum on the right hand-side of Eq. (2.25) contains the "outer" terms the two other (and new) sums for $h(0) \neq 0$ only involve the functions g and h in vincinity of zero and are thereafter "inner" terms.
2. If $S(t) = 1$, then no logarithmic terms $\epsilon^n \log \epsilon$ occur in the asymptotic estimate.
3. Even if the initial integral $I(\epsilon)$ is regular, the integrals arising in our asymptotic formula are integrals in the finite part sense of Hadamard. In computing those integrals, one must be careful. For instance, the application of the change of scale $t = h(0)\xi$ to the integral occurring in definition (2.26) of coefficient $C_1^{m,l,i}$ may induce extra terms (see Sellier 1996 for details).

2.4 The asymptotic solution

By employing the previous material, we can asymptotically expand the initial boundary integral equation (2.1). In terms of our non-dimensional coordinates, the right-hand side $T(M)$ reads $T(M) = T(x_1, \epsilon f_k(x_1))$ for $M \in \mathscr{C}_k$ and $T(M) = T(x_3, \epsilon f(\theta, x_3))$ for $M \in \partial\mathscr{A}_3$. Thus, one easily expands $T(M)$ with respect to ϵ. The derived asymptotic approximation is in general free of logarithmic terms and reads

$$T(M) = \sum_{j=0}^{J} T_j(x_1, ..., x_3)\epsilon^{r_j} + o(\epsilon^{r_J}) \quad (2.30)$$

where the real family (r_j) obeys $r_0 < \cdots < r_J$. The integral arising on the left-hand side of Eq. (2.1) is treated as previously explained in Subsections 2.2 and 2.3. This yields the asymptotic expansion of the left-hand side, $\mathscr{I}[q]$, of Eq. (2.1). Such an estimate admits the general form

$$\mathscr{I}[q] = \sum_{i=3^{d-2}}^{d} \sum_{n=0}^{N} \sum_{m=0}^{1} \mathscr{L}^i_{nm}[v_i]\epsilon^{s_n} \log^m \epsilon + o(\epsilon^{s_N}) \quad (2.31)$$

with a real family (s_n) such that $s_0 < \cdots < s_N$. The combination of Eqs. (2.30) and Eq. (2.31) thereafter suggests to seek the unknown v_i in the form $v_i = \sum_{n\geq 0} \mu_n(\epsilon)v_i^n$ with $\mu_{n+1}(\epsilon) = o(\mu_n(\epsilon))$. The Sects. 3–5 will detail, for 2D and 3D problems, this basic step which deeply depends on the initial equation (2.1). For instance, the leading terms in Eq. (2.31) may either involve v_i on the whole boundary $\partial\mathscr{A}_d$ or only in vincinity of M. We successively detail this for $d = 2$ and $d = 3$.

1. If $d = 2$, one expands the integrals $J^0_{kl}(\epsilon)$ and $J^1_{kl}(\epsilon)$ by employing the general formulas (2.25)–(2.29). For instance, one deduces that $J^0_{kl}(\epsilon) = J^0[v_k] + O(\epsilon)$ with

$$J^0[v_k] = \int_0^1 v_k(x_1^P) \log[(x_1^P - x_1)^2]\mathrm{d}x_1^P \ . \quad (2.32)$$

Thus, as soon as the integral equation (2.1) only yields integrals similar to $J^0_{kl}(\epsilon)$, one determines the unknown (v_1^n, v_2^n) by solving (from top to bottom) a pyramidal set of integral equations $J^0[v_k^n] = g_n$. For $J^1_{kl}(\epsilon)$, one takes $L[u, v] = (u^2 + v^2)^{-(E+i_1+i_2)/2}$, i.e. $Q = -(E + i_1 + i_2)$. Since $L[1, 0] \neq 0$, the first outer term, J^1_{outer}, in (2.25) reads (take $n = 0$)

$$J^1_{\text{outer}} = fp \int_0^1 \frac{v_k(x_1^P)h_{kl}^{i_2}(x_1^P - x_1)S(x_1^P - x_1)}{(x_1^P - x_1)^{E+i_2}}\mathrm{d}x_1^P \ . \quad (2.33)$$

Note that J^1_{outer} may be small compared to the leading inner term, J^1_{inner}, given by (2.25). For instance, if $E = i_2 = 1$ and $i_1 = 0$ then $S(t) = 1$ and

$$J^1_{\text{inner}} = [v_k h_{kl}](x_1)\left\{fp \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{[t^2 + k^2_{kk}(0)]}\right\}\epsilon^{-1} \quad (2.34)$$

is non-zero for $k \neq l$ (remind definition (2.10)). Thus, if $K_2(P, M) = a_1(M)[x_2(P) - x_2'(M)]/PM^2$ the unknown functions v_1^n and v_2^n are obtained without solving any integral equation.

2. For $d = 3$, we set $I^3(\epsilon) = \int_0^{2\pi} J_3(\epsilon)\mathrm{d}\theta_P$ and also $v_3(x_3) = fs_0 t(\theta, x_3)$ (see the definition of v_3 right-after equality (2.14)). In the non-dimensional plane $x_1 - x_2$, we introduce the cross-section $C(x_3) = \{P \in \partial\mathscr{A}_3; x_3^P = x_3\}$. On this closed path $C(x_3)$, the line element $\mathrm{d}l_P$ reads $\mathrm{d}l_P = fs_0\,\mathrm{d}\theta_P$. Accordingly, the leading outer term arising in the asymptotic expansion of $I^3(\epsilon)$ is

$$I^3_{\text{outer}} = fp \int_0^1 \frac{F(x_3^P)S(x_3^P - x_3)}{(x_3^P - x_3)^{E+i_1+i_2}}\mathrm{d}x_3^P \quad (2.35)$$

where the quantity $F(x_3^P)$ is the following generalized moment of the unknown density t in the cross-section $C(x_3)$:

$$F(x_3^P) = \oint_{C(x_3^P)} \frac{[x_1^P - x_1]^{i_1} t(\theta_P, x_3^P)}{[x_2^P - x_2]^{-i_2}}\mathrm{d}l_P \ . \quad (2.36)$$

For $J_3(\epsilon)$, one has $Q = -(E + i_1 + i_2 + i_3)$. The leading inner term, $I^3_{\text{inner}}(\epsilon)$, may contains a logarithmic term (for Q even) and is either smaller or greater than I^3_{outer}. In general, it is however always possible to determine the unknown functions $v_3^n(\theta, x_3)$ by solving this time a pyramidal set of two-dimensional boundary integral equation on the closed path $C(x_3)$. Those boundary integral equations actually pertain to exotic two-dimensional boundary value problems since the density v_i^n along $C(x_3)$ may induce, in the non-dimensional plane $x_1 - x_2$, fields which does not vanish far from $C(x_3)$ (see the remarks taking place between equalities (3.15) and (3.16) in Sect. 3).

3 Treatment of a three-dimensional Dirichlet type problem

We assume that our slender-body $\mathscr{A}_3$ is perfectly conducting and embedded in an arbitrary electrostatic potential ϕ_∞. The perturbation potential, ϕ, is solution to Eqs. (1.1), (1.3) and the Dirichlet type boundary condition (1.5) with $g = -\phi_\infty$ and two different circumstances for the constant value a of the total electrostatic potential on $\partial\mathscr{A}_3$ and the total "charge" b of the body:

1. Case 1. The value of a is prescribed and b is unknown.
2. Case 2. The body is isolated. This time, b is given and a is unknown.

We seek the asymptotic expansion of a or b with respect to the slenderness parameter ϵ for our slender-body of general cross-section. Handelsman and Keller (1967b) solved this problem, for an axisymmetric slender-body only, by putting sources on the segment $[OE]$. The case of a general slender-body has been addressed by Sellier (1999). In order to illustrate our procedure, we briefly report but also extend the results derived in Sellier (1999) and refer the reader to this latter paper for additional details.

Here, we obtain the potential ϕ by spreading free electrostatic sources on $\partial\mathscr{A}_3$. If q designates the unknown source density, we immediately arrive at the following Fredholm boundary integral equation of the first kind

$$\int_{\partial\mathscr{A}_3} q(P)\mathrm{d}s/PM = a - \phi_\infty(M); \text{ on } \partial\mathscr{A}_3 \tag{3.1}$$

which admits, for given a and ϕ_∞, a unique solution $q = \mathscr{L}[a-\phi_\infty]$. Under this representation of ϕ, the constant b reads $b = -4\pi\int_{\partial\mathscr{A}_3}\mathscr{L}[a-\phi_\infty]\mathrm{d}s$. Accordingly,

$$\frac{b}{4\pi} = \int_{\partial\mathscr{A}_3}\mathscr{L}[\phi_\infty]\mathrm{d}s - a\int_{\partial\mathscr{A}_3}\mathscr{L}[1]\mathrm{d}s \ . \tag{3.2}$$

This relation (3.2) shows that one has only to determine the solution $q = \mathscr{L}[\phi_\infty]$ for any electrostatic potential ϕ_∞. However, since harmonic any potential ϕ_∞ admits near $\partial\mathscr{A}_3$ (see Geer 1976) the following behavior

$$\phi_\infty = a_0(\epsilon^2r^2, x_3) + \sum_{l\geq 1}\epsilon^l r^l\{a_l(\epsilon^2r^2,x_3)\cos l\theta + b_l(\epsilon^2r^2,x_3)\sin l\theta\} \tag{3.3}$$

with smooth enough functions a_0, a_l and b_l. By linearity and superposition, we thereafter restrict ourselves to the case

$$\phi_\infty = \phi_\infty^l := r^l\psi(\epsilon^2r^2,x_3)\cos l\theta; \quad l\geq 0 \tag{3.4}$$

where the function $\psi(u,x_3)$ admits partial derivatives $\partial_u^i\partial_{x_3}^j\psi$ of unit magnitude for $x_3 = O(1)$ and small values of u. Note that Sellier (1999) did not adopt this convenient choice of ϕ_∞. On $\partial\mathscr{A}_3$, the quantity ϵf is of order of ϵ and ϕ_∞^l thereafter reads

$$\phi_\infty^l = f^l(\theta,x_3)\psi(0,x_3)\cos l\theta + \epsilon^2 f^{l+2}(\theta,x_3)\partial_u^1\psi(0,x_3)\cos l\theta + O(\epsilon^4) \ . \tag{3.5}$$

We look for the source density $q_l = \mathscr{L}[\phi_\infty^l]$ such that

$$\mathscr{L}^{-1}[q_l] = \int_{\partial\mathscr{A}_3} q_l(P)\mathrm{d}s/PM = \phi_\infty^l, \quad \text{on } \partial\mathscr{A}_3 \ . \tag{3.6}$$

The above integral equation is similar to (2.1) with only one typical kernel $K_3^l(P,M) = PM^{-1}, E = 1, i_1 = i_2 = i_3 = 0, \lambda = 0$ and $T(M) = \phi_\infty^l(M)$. The approximation of T on $\partial\mathscr{A}_3$ has been given by (3.5). If we introduce the density $t(\theta,x_3)$ by $fs_0 t = eq_l fs_\epsilon$ and remind that $\mathrm{d}l_P = fs_0\,\mathrm{d}\theta_P$ on the closed path $C(x_3)$, a careful application of Eqs. (2.13)–(2.14) and Eqs. (2.25)–(2.29) yields the asymptotic estimate

$$\mathscr{L}^{-1}[q_l] = \mathscr{L}_0^{x_3}[t]\log\epsilon + \mathscr{L}_1^{\theta,x_3}[t] + \mathscr{L}_2^{\theta,x_3}[t]\epsilon^2\log\epsilon + \mathscr{L}_3^{\theta,x_3}[t]\epsilon^2 + O(fs_0 t\epsilon^4\log\epsilon) \ , \tag{3.7}$$

where the linear operators arising in (3.7) obey

$$\mathscr{L}_0^{x_3}[t] = -2\oint_{C(x_3)} t(P)\mathrm{d}l_P \ , \tag{3.8}$$

$$\mathscr{L}_1^{\theta,x_3}[t] = -2\oint_{C(x_3)} t(P)\log H(\theta_P,x_3,\theta,x_3)\mathrm{d}l_P - T_{x_3}\{\mathscr{L}_0^x[t]\} \ , \tag{3.9}$$

$$T_{x_3}\{\alpha(x)\} = \alpha(x_3)\log 2 + fp\int_0^1\frac{\alpha(x)\mathrm{d}x}{2|x-x_3|} \ , \tag{3.10}$$

$$\mathscr{L}_2^{\theta,x_3}[t] = \frac{\partial^2}{\partial x^2}\left\{\oint_{C(x)}\frac{H^2(\theta_P,x,\theta,x_3)}{2t^{-1}(P)}\mathrm{d}l_P\right\}_{x=x_3} \ , \tag{3.11}$$

$$\mathscr{L}_3^{\theta,x_3}[t] = (\tfrac{1}{2}-\log 2)\mathscr{L}_2^{\theta,x_3}[t] + fp\int_0^1\frac{\mathscr{L}_0^x[tH^2(\theta_P,x,\theta,x_3)]}{4|x-x_3|^3}\mathrm{d}x + \frac{\partial^2}{\partial x^2}\left\{\oint_{C(x)}\frac{t(P)[H^2\log H](\theta_P,x,\theta,x_3)}{2}\mathrm{d}l_P\right\}_{x=x_3} \ , \tag{3.12}$$

if $H = h_3$ is given by Eq. (2.15). Observe that $\mathscr{L}_0^{x_3}[t]$ only involves the function t on the cross-section $C(x_3)$: this (leading) quantity is a two-dimensional (inner) term. By requiring the function t respectively on the whole surface $\partial\mathscr{A}_3$ and on cross-sections nearby $C(x_3)$ the other terms $\mathscr{L}_1^{\theta,x_3}[t]$, $\mathscr{L}_3^{\theta,x_3}[t]$ and $\mathscr{L}_2^{\theta,x_3}[t]$ are strongly or weakly three-dimensional quantities. The combination of asymptotic formulas (3.5) and (3.7) yields the asymptotic expansion of t. In building this behavior of t, it is useful to determine, for given functions $a(x_3)$ and $b(\theta,x_3)$, the unknown functions $u(\theta,x_3)$ and $\mathscr{L}_0^{x_3}[v]$ such that

$$\mathscr{L}_0^{x_3}[u] = a(x_3); \mathscr{L}_1^{\theta,x_3}[u] = b(\theta,x_3) - \mathscr{L}_0^{x_3}[v] \ . \tag{3.13}$$

Owing to Eqs. (3.8)–(3.10), the basic problem (3.13) becomes

$$K^{x_3}[u] := \oint_{C(x_3)} u(P)\mathrm{d}l_P = -a(x_3)/2 \ , \tag{3.14}$$

$$L^{\theta,x_3}[u] := -\oint_{C(x_3)} u(P)\log\overline{PM}\mathrm{d}l_P = b'(\theta,x_3) = [b(\theta,x_3) - \mathscr{L}_0^{x_3}[v] + T_{x_3}\{a(x)\}]/2 \ . \tag{3.15}$$

In this definition of operator L^{θ,x_3}, the distance $\overline{PM}$ is given by $\overline{PM} = \{(x_1^P - x_1^M)^2 + (x_2^P - x_2^M)^2\}^{1/2}$. Our integral equation (3.15) seems to be the counter-part of a two-dimensional elliptic (Laplace) boundary value problem in the $x_1 - x_2$ plane. However, the total "charge" on $C(x_3)$ is non-zero as soon as $a(x_3) \neq 0$ and this actually prevents us from associating to Eq. (3.15) such a well-posed elliptic problem. If the diameter $\delta(x_3) = \text{Max}\{\overline{PM}$ for P and M belonging to $C(x_3)\}$ belongs to $]0,1[$ (this is always true through an adequate choice of our typical "radius" e), the integral equation of the first kind (3.15) admits a unique solution $u = \{L^{\theta,x_3}\}^{-1}[b']$ (see Hsiao and Wendland 1981). Accordingly, the systems (3.14) and (3.15) admits a unique solution u if and only if the functions a and b' fulfill the compatibility relation

$$K^{x_3}\left[\{L^{\theta,x_3}\}^{-1}[b']\right] = -a(x_3)/2 \ . \tag{3.16}$$

Moreover (see Giroire 1987), if u_1 is the solution of the equation $L^{\theta,x_3}[u] = 1$ then $c_1(x_3) := K^{x_3}[u_1] \neq 0$. Hence, the solutions u and $\mathscr{L}_0^{x_3}[v]$ of the initial problem (3.13) read

$$u = \{L^{\theta,x_3}\}^{-1}[b/2] - \frac{u_1}{2c_1(x_3)}\{a(x_3) + K^{x_3}(\{L^{\theta,x_3}\}^{-1}[b])\} \ , \tag{3.17}$$

$$\mathscr{L}_0^{x_3}[v] = S_{x_3}\{a(x)\} + \frac{K^{x_3}(\{L^{\theta,x_3}\}^{-1}[b])}{c_1(x_3)} \ , \tag{3.18}$$

where $u_1 = \{L^{\theta,x_3}\}^{-1}[1], c_1(x_3) = K^{x_3}[u_1]$ and

$$S_{x_3}\{\alpha(x)\} = T_{x_3}\{\alpha(x)\} + \alpha(x_3)/c_1(x_3) \ . \tag{3.19}$$

Observe that for $a(x_3) = 0$ and $b = b(x_3)$ the previous solutions take the pleasant forms: $u = 0, \mathscr{L}_0^{x_3}[v] = b(x_3)$. Moreover, if $b = 0$ then $u = -a(x_3)u_1/[2c_1(x_3)]$ and $\mathscr{L}_0^{x_3}[v] = S_{x_3}\{a(x)\}$. Accordingly, one obtains the following asymptotic expansion

$$fs_0t = \sum_{n=0}^{1}\sum_{m=-n}^{\infty}\frac{fs_0t_{n,m}\epsilon^{2n}}{[\log\epsilon]^m} + O(\epsilon^4\log^2\epsilon) \tag{3.20}$$

with family $(t_{0,m})$ given by relations

$$\mathscr{L}_0^{x_3}[t_{0,0}] = 0 \ , \tag{3.21}$$

$$\mathscr{L}_1^{\theta,x_3}[t_{0,0}] = f^l(\theta,x_3)\psi(0,x_3)\cos l\theta - \mathscr{L}_0^{x_3}[t_{0,1}] \ , \tag{3.22}$$

$$\mathscr{L}_1^{\theta,x_3}[t_{0,m}] = -\mathscr{L}_0^{x_3}[t_{0,m+1}] = 0; m \geq 1 \tag{3.23}$$

and family $(t_{1,m})$ obeying the equalities

$$\mathscr{L}_0^{x_3}[t_{1,-1}] = 0 \ , \tag{3.24}$$

$$\mathscr{L}_1^{\theta,x_3}[t_{1,-1}] = -\mathscr{L}_2^{\theta,x_3}[t_{0,0}] - \mathscr{L}_0^{x_3}[t_{1,0}] \ , \tag{3.25}$$

$$\mathscr{L}_1^{\theta,x_3}[t_{1,m}] = b_m(\theta,x_3) - \mathscr{L}_0^{x_3}[t_{1,m+1}]; \quad m \geq 0 \ , \tag{3.26}$$

$$b_m(\theta,x_3) = -\mathscr{L}_2^{\theta,x_3}[t_{0,m+1}] - \mathscr{L}_3^{\theta,x_3}[t_{0,m}] + \delta_{m0}f^{l+2}(\theta,x_3)\partial_u^1\psi(0,x_3)cosl\theta; \quad m \geq 0 \tag{3.27}$$

If one introduces the functions $v_l^0(\theta,x_3)$ and $d_l^0(x_3)$ by

$$L^{\theta,x_3}[v_l^0] = f^l(\theta,x_3)\cos l\theta; \quad d_l^0(x_3) = K^{x_3}[v_l^0] \tag{3.28}$$

it is straightforward to obtain the solution $(t_{0,m})$

$$t_{0,0} = \frac{\psi(0,x_3)}{2}[v_l^0 - \frac{d_l^0(x_3)}{c_1(x_3)}u_1] \ , \tag{3.29}$$

$$t_{0,m} = -\frac{S_{x_3}^{m-1}\{\psi(0,x)d_l^0(x)/c_1(x)\}}{2c_1(x_3)}u_1; \quad m \geq 1 \tag{3.30}$$

where $S_{x_3}^0\{\alpha(x)\} = 1$ and for $m \geq 1, S_{x_3}^m = S_{x_3} \circ S_{x_3}^{m-1}$. Note that for $l = 0, v_0^0 = u_1$ and $t_{0,0} = 0$. Moreover, for $m \geq 1$ each solution $t_{0,m}$ presents the same dependence as function u_1 upon the variable θ. The determination of $(t_{0,m})$ only requires to solve in each cross-section $C(x_3)$ the two-dimensional integral equations $L^{\theta,x_3}[u_1] = 1$ and, if $l \geq 1, L^{\theta,x_3}[v_l^0] = f^l(\theta,x_3)\cos l\theta$. The case of family $(t_{1,m})$ is more tricky. As the reader may check, equalities (3.24)–(3.27) yield

$$t_{1,-1} = \frac{v_l^1}{2} - \frac{d_l^1(x_3)}{2c_1(x_3)}u_1 \tag{3.31}$$

if the functions $v_l^1(\theta,x_3)$ and $d_l^1(x_3)$ obey

$$L^{\theta,x_3}[v_l^1] = -\mathscr{L}_2^{\theta,x_3}[t_{0,0}]; d_l^1(x_3) = K^{x_3}[v_l^1] \tag{3.32}$$

and, for $m \geq 0$, the relations

$$\mathscr{L}_0^{x_3}[t_{1,m+1}] = S_{x_3}\{\mathscr{L}_0^x[t_{1,m}]\} + \frac{K^{x_3}(\{L^{\theta,x_3}\}^{-1}[b_m])}{c_1(x_3)} \ , \tag{3.33}$$

$$L^{\theta,x_3}[t_{1,m}] = \frac{b_m}{2} - \frac{\mathscr{L}_0^{x_3}[t_{1,m}]}{2c_1(x_3)} - \frac{K^{x_3}(\{L^{\theta,x_3}\}^{-1}[b_m])}{2c_1(x_3)} \tag{3.34}$$

with b_m given, for $m \geq 0$, by equality (3.27) and

$$\mathscr{L}_0^{x_3}[t_{1,0}] = d_l^1(x_3)/c_1(x_3) \ . \tag{3.35}$$

Note that $t_{1,-1} = 0$ if $l = 0$. In getting the functions $t_{1,m}$, one needs to solve the integral equations (3.34) in each cross-section.

Our results (3.29)–(3.31) and (3.33)–(3.35) permit us to build the asymptotic estimate (3.20) for any imposed potential ϕ_∞^l. By authorizing such a general potential ϕ_∞, we have thereafter extended the results obtained in Sellier (1999). Observe that one can immediately approximate the total charge

$$\int_{\partial\mathscr{A}_3} q_l \, ds = L\int_0^1 K^{x_3}[t]dx_3 \tag{3.36}$$

and employ the link (3.2) to deal with both Cases 1 and 2. For further details and term-to-term comparisons with the exact solution in case of a slender ellipsoid, the reader is referred to Sellier (1999).

4
Treatment of a two-dimensional Neumann type problem

This Section considers the steady potential flow of an inviscid and homogeneous fluid around a thin airfoil $\mathscr{A}_2$. A special attention is paid to the pressure distribution.

4.1
The boundary integral equation

Far from $\mathscr{A}_2$ the flow is uniform with given pressure p_∞ and velocity $\mathbf{u}_\infty = Ue_1 + Ve_2$. The fluid velocity reads $\mathbf{u} = \mathbf{u}_\infty + \nabla\phi$ where ϕ denotes the perturbation potential. This function ϕ satisfies the equations (1.1) and (1.2) with $C = 0$ and the Neumann type boundary condition (1.4) for $g' = -\mathbf{u}_\infty \cdot \mathbf{n}$. In order to obtain a well-posed problem, we also impose the total circulation $\Gamma = \oint_{\partial\mathscr{A}_2} \mathbf{u} \cdot dl$. More precisely, we shall assume that

$$\Gamma/L = U\{c_0 + c_1\epsilon + O(\epsilon^2)\} + V\{d_0 + d_1\epsilon + O(\epsilon^2)\} \tag{4.1}$$

with $c_i = O(1)$ and $d_i = O(1)$. The fluid pressure p is deduced from ϕ by using the usual Bernoulli's theorem. More precisely, in the fluid and on $\partial\mathscr{A}_2$, the pressure coefficient C_p reads

$$C_p(M) := \frac{2[p(M) - p_\infty]}{\rho_\infty u_\infty^2} = 1 - \left[\frac{\mathbf{u}(M)}{u_\infty}\right]^2 \tag{4.2}$$

if ρ_∞ denotes the constant fluid density and $u_\infty = \{U^2 + V^2\}^{1/2}$. The potential function ϕ has been approximated with respect to the small thickness parameter ϵ by Geer (1974). However, this latter paper does not express the pressure on the airfoil. Here, we disregard the potential ϕ and directly build the asymptotic behavior of the fluid velocity on $\partial\mathscr{A}_2$.

For this purpose, it is convenient to obtain the perturbation velocity, $\mathbf{u}' = \nabla\phi$, by spreading either normal doublets or a vortex distribution over $\partial\mathscr{A}_2$ (see Dautray and Lions 1988). Thus, if $e_3 = e_1 \wedge e_2$, we set

$$\mathbf{u}'(M) = \frac{1}{2\pi}\left\{\int_{\mathscr{C}_2} q_2(P)e_3 \wedge \nabla_P[\log PM]dl_P + \int_{\mathscr{C}_1} q_1(P)e_3 \wedge \nabla_P[\log PM]dl_P\right\} . \tag{4.3}$$

In (4.3), q_k and dl_P denote the unknown vortex density and the arc length on curves $\mathscr{C}_k$ (dl_P is zero at O and increases on $\mathscr{C}_k$ as the point P moves from O to E). The relation (4.3) defines smooth, irrotational and solenoidal velocity fields $\mathbf{u}'_{\text{in}}$ and $\mathbf{u}'_{\text{ou}}$ respectively inside and outside $\partial\mathscr{A}_2$. More precisely (see Kellogg 1953) one obtains, for $k \in \{1, 2\}$,

$$[\mathbf{u}'_{\text{ou}} - \mathbf{u}'_{\text{in}}](M) = [q_k\mathbf{n}](M) \wedge e_3; \text{ on } \mathscr{C}_k . \tag{4.4}$$

Thus, $\mathbf{u}'_{\text{in}} \cdot \mathbf{n} = \mathbf{u}'_{\text{ou}}.\mathbf{n} = -\mathbf{u}_\infty \cdot \mathbf{n}$ on $\partial\mathscr{A}_2$ and this implies that $\mathbf{u}'_{\text{in}} = -\mathbf{u}_\infty$ in $\mathscr{A}_2$. Consequently, on $\partial\mathscr{A}_2$ the fluid velocity $\mathbf{u}$ and the pressure coefficient C_p read

$$\mathbf{u} = [q_k\mathbf{n}] \wedge e_3; C_p = 1 - \left[\frac{q_k}{u_\infty}\right]^2, \quad \text{on } \mathscr{C}_k . \tag{4.5}$$

Moreover, the flow-tangency condition (1.4) yields the boundary integral equation

$$2\pi\mathbf{u}_\infty \cdot \mathbf{n}(M) = pv\int_{\mathscr{C}_1} q_1(P)\frac{e_3 \wedge PM}{PM^2} \cdot \mathbf{n}(M)dl_P + pv\int_{\mathscr{C}_2} q_2(P)\frac{e_3 \wedge PM}{PM^2} \cdot \mathbf{n}(M)dl_P; \text{ on } \partial\mathscr{A}_2 . \tag{4.6}$$

Depending on the location of M, the first or the second integral in (4.6) reduces to a regular integration. One has to solve the problem consisting of (4.6) and (4.1) with the relation $\Gamma = \int_{\mathscr{C}_1} q_1(P)dl_P + \int_{\mathscr{C}_2} q_2(P)dl_P$. Clearly, (4.6) looks like (2.1) with a kernel satisfying (2.3).

4.2
The asymptotic solution

Since, for $M \in \mathscr{C}_k$, the normal vector $\mathbf{n}$ reads

$$\mathbf{n}(M) = (-1)^k \frac{[e_2 - \epsilon f_k^{(1)}(x_1)e_1]}{\{1 + [\epsilon f_k^{(1)}(x_1)]^2\}^{1/2}} \tag{4.7}$$

the boundary integral equation (4.6) becomes (we keep for functions v_k the definitions (2.9)).

$$pv\int_0^1 \frac{\{(x_1 - x_1^P) + \epsilon^2 f_k^{(1)}(x_1)[f_k(x_1) - f_k(x_1^P)]\}dx_1^P}{\pi q_k^{-1}(x_1^P)\{(x_1 - x_1^P)^2 + \epsilon^2[f_k(x_1) - f_k(x_1^P)]^2\}} + \int_0^1 \frac{\{(x_1 - x_1^P) + \epsilon^2 f_{3-k}^{(1)}(x_1)[f_k(x_1) - f_{3-k}(x_1^P)]\}dx_1^P}{\pi q_{3-k}^{-1}(x_1^P)\{(x_1 - x_1^P)^2 + \epsilon^2[f_k(x_1) - f_{3-k}(x_1^P)]^2\}}$$
$$= 2[V - \epsilon U f_k^{(1)}(x_1)]; \quad 0 < x_1 < 1, k \in \{1, 2\} . \tag{4.8}$$

Hence, the asymptotic expansion of (4.6) rests on the asymptotic approximation of the following integral

$$I_{\epsilon,x_1}^{h,a}[g] = \frac{1}{\pi}pv\int_{-x_1}^{1-x_1} \frac{g(u + x_1)[a\epsilon^2 h(u) - u]}{u^2 + \epsilon^2 h^2(u)}du \tag{4.9}$$

for real values of a and smooth enough functions g and h. Indeed, for $0 < x_1 < 1$, (4.8) also reads

$$I_{\epsilon,x_1}^{h_1,a_1}[v_2] + I_{\epsilon,x_1}^{h_2,a_2}[v_1] = 2[V - \epsilon U f_2^{(1)}(x_1)] , \tag{4.10}$$

$$I_{\epsilon,x_1}^{h_3,a_3}[v_2] + I_{\epsilon,x_1}^{h_4,a_4}[v_1] = 2[V - \epsilon U f_1^{(1)}(x_1)] , \tag{4.11}$$

provided one successively chooses

$$a_1 = a_2 = f_2^{(1)}(x_1); \quad a_3 = a_4 = f_1^{(1)}(x_1) , \tag{4.12}$$

$$h_1(u) = f_2(x_1) - f_2(x_1 + u) , \tag{4.13}$$

$$h_2(u) = f_2(x_1) - f_1(x_1 + u) , \tag{4.14}$$

$$h_3(u) = f_1(x_1) - f_2(x_1 + u) , \tag{4.15}$$

$$h_4(u) = f_1(x_1) - f_1(x_1 + u) . \tag{4.16}$$

Note that $h_1(0) = h_4(0) = 0$ whilst $h_2(0) > 0$ and $h_3(0) < 0$. If $g_m = \text{Max}_{[0,1]}|g|$, a careful application of our systematic formula (2.25) yields

$$I_{\epsilon,x_1}^{h,a}[g] = -L_{x_1}[g] + I_{1,x_1}^{h,a}[g]\epsilon + I_{2,x_1}^{h,a}[g]\epsilon^2 + O(g_m\epsilon^3) , \tag{4.17}$$

with, if $\Delta_0 := 0$ and $\Delta_y := y/|y|$ for non-zero y,

$$L_{x_1}[g] = \frac{1}{\pi} pv \int_0^1 \frac{g(x)\mathrm{d}x}{x - x_1} \ , \tag{4.18}$$

$$I^{h,a}_{1,x_1}[g] = \Delta_{h(0)}\{[a + h^{(1)}(0)]g + h(0)g^{(1)}\}(x_1) \ , \tag{4.19}$$

$$I^{h,a}_{2,x_1}[g] = fp \int_0^1 \frac{[a(x - x_1) + h(x - x_1)]\mathrm{d}x}{\pi[g(x)h(x - x_1)]^{-1}(x - x_1)^3} \ . \tag{4.20}$$

Thus, the leading term $-L_{x_1}[g]$ does not depend on function h and real a. Like $I^{h,a}_{2,x_1}[g]$, it is an outer term whereas $I^{h,a}_{1,x_1}[g]$ is an inner quantity. Finally, the functions v_1 and v_2 obey the asymptotic problem

$$\begin{aligned} &- L_{x_1}[v_1 + v_2] + I^{h_2,a_2}_{1,x_1}[v_1]\epsilon + I^{h_1,a_1}_{2,x_1}[v_2]\epsilon^2 \\ &\quad + I^{h_2,a_2}_{2,x_1}[v_1]\epsilon^2 + O(\epsilon^3\bar{v}) = 2[V - \epsilon U f_2^{(1)}(x_1)] \ , \end{aligned} \tag{4.21}$$

$$\begin{aligned} &- L_{x_1}[v_1 + v_2] + I^{h_3,a_3}_{1,x_1}[v_2]\epsilon + I^{h_3,a_3}_{2,x_1}[v_2]\epsilon^2 \\ &\quad + I^{h_4,a_4}_{2,x_1}[v_1]\epsilon^2 + O(\epsilon^3\bar{v}) = 2[V - \epsilon U f_1^{(1)}(x_1)] \ , \end{aligned} \tag{4.22}$$

$$\int_0^1 [v_1 + v_2](x_1)\mathrm{d}x_1 = \Gamma/L \ , \tag{4.23}$$

where $\bar{v}$ designates the typical magnitude of v_1 and v_2 and Γ/L is given by Eq. (4.1). A glance at Eqs. (4.21)–(4.23) suggests to look for the solution g of the following integral problem $R_r[s]$

$$\int_0^1 g(x)\mathrm{d}x = r; \quad L_{x_1}[g] = s(x_1) \text{ in }]0,1[\ , \tag{4.24}$$

for given real number r and function s. Standard textbooks devoted to integral equations (see Muskhelishvili 1958; Kanwal 1971; Zabreyko 1975) only address the case of functions s and g obeying a Lipschitz condition respectively on $[0,1]$ and on $[\eta, 1-\eta]$ (where $0 < \eta < 1$) with the condition: $|h(x)| \leq A|x - c|^\alpha$ for some $0 < \alpha < 1$ and $A > 0$ near $c \in \{0,1\}$. Unfortunately, we also need to consider functions s such that $s(x) \sim s'_c|c - x|^{-1/2}$ near $c \in \{0,1\}$. By extending the works of Schröder (1938) and Söhngen (1939), the Appendix therefore defines an adequate set, $E_{[0,1]}$, of functions s fulfilling weaker assumptions. For $s \in E_{[0,1]}$, the unique solution $g = R_r^{-1}[s]$ satisfies (see the Appendix)

$$\frac{\pi\sqrt{x_1}g(x_1)}{\{1 - x_1\}^{-1/2}} = pv \int_0^1 \frac{\sqrt{x(1-x)}}{x_1 - x} s(x)\mathrm{d}x + r \ . \tag{4.25}$$

The inspection of Eqs. (4.21)–(4.23) and our result (4.25) suggest to seek each solution v_k in the following form

$$\begin{aligned} v_k = {} & U\{u_0^k + u_1^k\epsilon + u_2^k\epsilon^2 + +h_E O(\epsilon^3)\} \\ & + V\{v_0^k + v_1^k\epsilon + v_2^k\epsilon^2 + h_E O(\epsilon^3)\} \end{aligned} \tag{4.26}$$

with $h_E(x_1) = [x_1(1 - x_1)]^{-1/2}$. By re-introducing those approximations into Eqs. (4.21) and (4.22), one gets the incoming pyramidal set of one-dimensional integral equations

$$L_{x_1}[u_0^1 + u_0^2] = 0; \quad L_{x_1}[v_0^1 + v_0^2] = -2 \ , \tag{4.27}$$

$$\begin{aligned} L_{x_1}[u_1^1 + u_1^2] &= I^{h_2,a_2}_{1,x_1}[u_0^1] + 2f_2^{(1)}(x_1) \\ &= I^{h_3,a_3}_{1,x_1}[u_0^2] + 2f_1^{(1)}(x_1) \ , \end{aligned} \tag{4.28}$$

$$L_{x_1}[v_1^1 + v_1^2] = I^{h_2,a_2}_{1,x_1}[v_0^1] = I^{h_3,a_3}_{1,x_1}[v_0^2] \ , \tag{4.29}$$

$$\begin{aligned} L_{x_1}[w_2^1 + w_2^2] &= I^{h_2,a_2}_{1,x_1}[w_1^1] + I^{h_1,a_1}_{2,x_1}[w_0^2] + I^{h_2,a_2}_{2,x_1}[w_0^1] \\ &= I^{h_3,a_3}_{1,x_1}[w_1^2] + I^{h_3,a_3}_{2,x_1}[w_0^2] + I^{h_4,a_4}_{2,x_1}[w_0^1]; \\ & w \in \{u, v\} \ . \end{aligned} \tag{4.30}$$

Note that

$$I^{h_2,a_2}_{1,x_1}[w] = I^{h_3,a_3}_{1,x_1}[w] = [(f_2 - f_1)w]^{(1)}(x_1) \ . \tag{4.31}$$

Since $f_2 - f_1$ vanishes for $x_1 = 0$ and $x_1 = 1$ the Eqs. (4.28) and (4.29) thereafter yield $u_0^2 - u_0^1 = 2$ and $v_0^2 = v_0^1$. By invoking (4.27) and the additional conditions (4.1) and (4.23), it follows that

$$u_0^k(x_1) = (-1)^k + \frac{c_0}{2\pi\sqrt{x_1(1 - x_1)}} \ , \tag{4.32}$$

$$v_0^k(x_1) = \frac{d_0 + (1 - 2x_1)\pi}{2\pi\sqrt{x_1(1 - x_1)}} \ . \tag{4.33}$$

Those solutions pertain to a zero-thickness aerofoil and thereafter do not depend upon the shape functions f_k. The functions $u_1^1 + u_1^2$ and $v_1^1 + v_1^2$ are given by Eq. (4.25), respectively for $r = c_1$ and $r = d_1$. Moreover, our definition (4.20) of operator $I^{h,a}_{2,x_1}$ shows that, for $k \in \{1, 2\}$,

$$\begin{aligned} &[I^{h_{2+k},a_{2+k}}_{2,x_1} - I^{h_k,a_k}_{2,x_1}][w] \\ &= \frac{\mathrm{d}}{\mathrm{d}x_1}\left\{pf \int_0^1 \frac{[2f_k(x) - f_2(x_1) - f_1(x_1)]w(x)\mathrm{d}x}{2\pi(x_1 - x)^2\{[f_2 - f_1](x_1)\}^{-1}}\right\} \ . \end{aligned} \tag{4.34}$$

Thus, for $w \in \{u, v\}$, the relations (4.30) yield

$$\begin{aligned} [w_1^2 - w_1^1](x_1) = {} & -pf \int_0^1 \frac{[f_1 w_0^1 + f_2 w_0^2](x)\mathrm{d}x}{\pi(x_1 - x)^2} \\ & + \frac{[f_1 + f_2](x_1)}{2\pi} pf \int_0^1 \frac{[w_0^1 + w_0^2](x)\mathrm{d}x}{(x_1 - x)^2} \ . \end{aligned} \tag{4.35}$$

By employing the previous solutions (4.32) and (4.33), one finally obtains, for $w \in \{u, v\}$,

$$2\pi w_1^k(x_1)$$
$$= (-1)^{k-1} pf \int_0^1 \frac{[f_1 w_0^1 + f_2 w_0^2](x)}{(x_1 - x)^2} dx$$
$$+ \frac{[f_1 + f_2](x_1)}{2(-1)^k} pf \int_0^1 \frac{[w_0^1 + w_0^2](x)}{(x_1 - x)^2} dx + c(w) h_E(x_1)$$
$$+ h_E(x_1) pv \int_0^1 \frac{\sqrt{x(1-x)}}{x_1 - x} [(f_2 - f_1) w_0^2 + 2f_w]^{(1)}(x) dx \quad (4.36)$$

where $c(u) = c_1, f_u = f_1, c(v) = d_1$ and $f_v = 0$. The computation of u_2^k requires to detail the term $O(g_m \epsilon^3)$ in Eq. (4.17). The calculations are let to the reader. Finally, from Eq. (4.26) we deduce the pressure coefficient on $\partial\mathscr{A}_2$ by employing Eq. (4.5).

4.3
Comparison with an exact solution

This Subsection compares the proposed results with the asymptotic estimate of an exact solution available in case of a thin elliptic aerofoil. More precisely, $\partial\mathscr{A}_2$ admits the equations

$$(2x'_1/L - 1)^2 + (x'_2/e)^2 = 1; \quad x_2^2 = 4x_1(1 - x_1) \ . \quad (4.37)$$

We set $X = x'_1 - L/2, Y = x'_2$ and $\mathbf{u} \cdot e_k = \bar{u}_k$. If $Z = X + iY$ and $W(Z) = \bar{u}_1 - i\bar{u}_2$ denote the complex variable and velocity, one obtains, on $\partial\mathscr{A}_2$,

$$(-1)^k v_k = \bar{u}_1 + \epsilon f_k^{(1)} \bar{u}_2 = \mathrm{Re}\{W(Z)[1 + i\epsilon f_k^{(1)}]\} \ . \quad (4.38)$$

Here we gain the complex velocity W by using the well-known Joukowsky's conformal mapping $Z = F(z)$ (see for instance Von Mises and Friedrichs 1971)

$$F(z) = z + [L^2 - 4e^2]/[16z]; \quad z = re^{i\theta} \ . \quad (4.39)$$

Such a mapping indeed transforms the circle of radius $R = (L + 2e)/4$ into our ellipse $\partial\mathscr{A}_2$. The flow about this circle admits, for a free velocity $\mathbf{u}_\infty = u_\infty[\cos\alpha e_1 + \sin\alpha e_2]$ of angle of attack α and a prescribed circulation Γ, the complex velocity

$$w(z) = u_\infty \left[e^{-i\alpha} - \frac{R^2}{r^2} e^{i(\alpha - 2\theta)} \right] + \frac{i\Gamma e^{-i\theta}}{2\pi r}, r \geq R \ . \quad (4.40)$$

On $\partial\mathscr{A}_2$ we have $X = L\cos\theta/2, Y = e\sin\theta$ and the required complex velocity $W(\theta) = w(z)dF/dZ$ is

$$W(\theta) = \left\{ u_\infty[e^{-i\alpha} - e^{i(\alpha - 2\theta)}] + \frac{2i\Gamma e^{-i\theta}}{\pi(1 + 2\epsilon)L} \right\} \times \left\{ 1 - \frac{1 - 2\epsilon}{1 + 2\epsilon} e^{-2i\theta} \right\}^{-1} \ . \quad (4.41)$$

Since $\cos\theta = 2x - 1$ and $f_k(x_1) = 2(-1)^k \sqrt{x_1(1 - x_1)}$, the exact solutions v_k read

$$v_k(x_1) = u_\infty[1 + 2\epsilon] \left\{ \frac{[1 - 2x_1]\sin\alpha}{\sqrt{x_1(1 - x_1)}} + (-1)^k \cos\alpha \right\} + \frac{\Gamma}{2\pi L\sqrt{x_1(1 - x_1)}}; \quad 0 < x_1 < 1 \ . \quad (4.42)$$

Our previous results (4.32), (4.33) and (4.36) perfectly agree with these solutions (one has only to employ Eq. (4.42) successively for $\alpha = 0$ and $\alpha = \pi/2$).

5
Case of a non-smooth boundary

This last Section addresses the case of a non-smooth boundary $\partial\mathscr{A}_2$. In order to highlight the encountered troubles, the problem treated in Sect. 4 is handled this time for a thin-airfoil $\mathscr{A}_2$ of cusped trailing edge E.

5.1
The asymptotic solution

This time, the boundary $\partial\mathscr{A}_2$ is smooth except at its cusped trailing edge E. More precisely, f_k^2 is smooth in $]0, 1[$, admits near zero the behavior (2.4) (with $c_1^0 > 0$) and satisfies $f_k^2(x_1) \sim b_n(1 - x_1)^n$ as $x_1 \to 1^-$ with $n \geq 2$. For such a profile it is not possible any more to impose the circulation Γ. Instead, one applies the Kutta–Joukowsky condition which specifies that the fluid velocity is finite at the cusped trailing edge (see Ashley and Landhal 1965; Moran 1984 and Anderson 1991). Again, we adopt the representation (4.3) and the definitions (9) of v_k. Since $f_1^{(1)}(1) = f_2^{(1)}(1)$, the Kutta–Joukowsky condition (i.e. the requirement of a finite velocity at E) becomes

$$[q_1 + q_2](1) = [v_1 + v_2](1) = 0 \ . \quad (5.1)$$

The equalities (4.2)–(4.20) still hold and the functions v_k obey Eqs. (4.21) and Eq. (4.22) with Eq. (4.23) replaced by Eq. (5.1). Accordingly, the basic problem $R_r[s]$ (see Eq. (4.24)) is replaced by the problem $S[s]$

$$g(1) = 0; L_{x_1}[g] = s(x_1) \text{ in }]0, 1[\ . \quad (5.2)$$

If $s \in E_{[0,1]}$ with $s'_1 = 0$ (see the Appendix), the unique solution to this problem is

$$S^{-1}[s](x_1) = \frac{1}{\pi}\sqrt{\frac{1 - x_1}{x_1}} pv \int_0^1 \sqrt{\frac{x}{1 - x}} \frac{s(x)}{x_1 - x} dx \quad (5.3)$$

with the useful relation

$$\int_0^1 S^{-1}[s](x)dx = -\int_0^1 \sqrt{\frac{x}{1 - x}} s(x)dx \ . \quad (5.4)$$

The solution (5.3) is bounded as x goes to 1 but may not vanish here. The value of $S^{-1}[s](1)$ deeply depends on the given function s (see the Appendix). Again we seek v_k in the form (4.26) with, this time, $h_E(x_1) = x_1^{-1/2}$. The equalities (4.27)–(4.31) remain valid. Hence, one obtains

$$u_0^k(x_1) = (-1)^k; v_0^k(x_1) = \{(1 - x_1)/x_1)\}^{1/2} \ . \quad (5.5)$$

For our cusped trailing edge, the reader may check from Eq. (5.5) that the right-hand sides s of Eqs. (4.28) and

(4.29) belong to $E_{[0,1]}$ with $s_1' = 0$. This permits us to use once more Eq. (5.3) in determining $u_1^1 + u_1^2$ and $u_1^1 + u_1^2$ and to obtain

$$2\pi u_1^k(x_1) = (-1)^{k+1} pf \int_0^1 \frac{[f_2 - f_1](x)}{(x_1 - x)^2} \mathrm{d}x + \sqrt{\frac{1-x_1}{x_1}} pv \int_0^1 \sqrt{\frac{x}{1-x}} \frac{[f_1 + f_2]^{(1)}(x)}{x_1 - x} \mathrm{d}x \ , \tag{5.6}$$

$$2\pi v_1^k(x_1) = (-1)^{k+1} pf \int_0^1 \sqrt{\frac{1-x}{x}} \frac{[f_1 + f_2](x)}{(x_1 - x)^2} \mathrm{d}x + \sqrt{\frac{1-x_1}{x_1}} pv \int_0^1 \sqrt{\frac{x}{1-x}} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f_2 - f_1}{\sqrt{x/(1-x)}}\right] \frac{\mathrm{d}x}{x_1 - x} \ . \tag{5.7}$$

Among the quantities of practical interest, it is worth giving the total force $\mathbf{R} = \rho_\infty \Gamma e_3 \wedge \mathbf{u}_\infty$ acting on the airfoil. Since

$$\Gamma = \int_0^1 [v_1(x_1) + v_2(x_1)] \mathrm{d}x_1 \ , \tag{5.8}$$

we only need to compute the quantities $w_i^1 + w_i^2$ for $w \in \{u, v\}$ and $i \in \{0, 1, 2\}$. Actually, Eq. (5.4) clearly indicates that one has only to calculate, by using (4.19), (4.20) and (5.5) the right-hand sides of Eqs. (4.28)–(4.30). Curtailing the details, we obtain

$$\Gamma/L = \pi V - \epsilon \int_0^1 \sqrt{\frac{x}{1-x}} [U(f_1 + f_2) + \frac{V(f_2 - f_1)}{\sqrt{x/(1-x)}}]^{(1)}(x) \mathrm{d}x - \epsilon^2 \int_0^1 \sqrt{\frac{x}{1-x}} [U g_u + V g_v]^{(1)}(x) \mathrm{d}x + O(\epsilon^3) \ , \tag{5.9}$$

where, if $w \in \{u, v\}$,

$$g_w(x_1) = ([f_2 - f_1] w_1^2)(x_1) + \sum_{k=1}^{2} pf \int_0^1 \frac{[f_1(x_1) - f_k(x)]^2 w_0^k(x) \mathrm{d}x}{\pi (x_1 - x)^2} \ . \tag{5.10}$$

Note that the term of order of ϵ^2 in Eq. (5.9) requires to calculate the solutions w_1^k by using Eqs. (5.6) and (5.7). The determination of this term by our method extends the formula proposed by Geer (1974) for Γ/L up to order ϵ only.

5.2 Remarks on the asymptotic solution near a sharp trailing edge

One may wonder whether our solutions u_1^k and v_1^k remain valid for a sharp trailing edge (if $f_1^{(1)}(1) - f_2^{(1)}(1) \neq 0$). Here we should impose $q_1(1) = q_2(1) = 0$ instead of Eq. (5.1). We try once more to expand v_k as in Eq. (4.26) with $h_E(x_1) = x_1^{-1/2}$. Under those assumptions, we again arrive at Eqs. (5.5)–(5.7). Note that the last terms arising on the right-hande sides of Eqs. (5.6)–(5.7) vanish as $x_1 \to 1^-$. The remaining terms are treated by noting that, for $a \in \{0, 1/2\}$ and g smooth enough near 1,

$$pf \int_0^1 \left[\frac{1-x}{x}\right]^a \frac{g(x) \mathrm{d}x}{(x_1 - x)^2} = [g A_a^{(1)} + g^{(1)} A_a](x_1) + \int_0^1 \left[\frac{1-x}{x}\right]^a \frac{g(x) - g(x_1) - g^{(1)}(x_1)(x - x_1)}{(x_1 - x)^2} \mathrm{d}x \tag{5.11}$$

with, through elementary algebra,

$$A_a(x_1) = pf \int_0^1 \frac{[1-x]^a \mathrm{d}x}{x^a [x - x_1]} \tag{5.12}$$

$$A_0(x_1) = \log\left[\frac{1-x_1}{x_1}\right] ; A_{1/2}(x_1) = -\pi \ . \tag{5.13}$$

Thus, as $x_1 \to 1^-$, one obtains $v_1^{\pm}(x) = O(1)$ and

$$2\pi u_1^k(x_1) \sim (-1)^k [f_1 - f_2]^{(1)}(1) \log(1 - x_1) \ . \tag{5.14}$$

Hence, the Poincaré expansion (4.26) fails for a sharp trailing edge E as soon as $U \neq 0$ since ϵu_1^k becomes greater than u_0^k near E ($\epsilon u_1^k(x_1)$ actually becomes of unit magnitude for $1 - x_1 = O(e^{-1/\epsilon})$). Such troubles have been also encountered by the method of matched asymptotic expansions (see Van Dyke 1975). In case of a sharp trailing edge, Hoogstraten (1967) resorted to the Lighthill 's method of strained coordinates (see Lighthill 1949) to obtain a first-order approximation to the potential function free from this drawback. Sheer (1971) later derived the solution up to any order when the thin airfoil presents both sharp leading and trailing edges, is symmetric with respect to the axis Ox_1' and $\mathbf{u}_\infty = U\mathbf{e}_1$. If $\mathbf{u}_t$ denotes the (tangential) velocity on the profile, Sheer (1971) actually proved in this case that $\log|\mathbf{u}_t|$ (not $\mathbf{u}_t$ itself) admits a Poincaré type expansion uniformly valid in $]0, 1[$. It would be nice to extend Sheer's results to the case of a non-symmetric thin-airfoil admitting a rounded leading end O and a sharp trailing edge E. As previously explained, the present method is not the right one for such a challenge. To the author's knowledge, this question is still unsolved.

6 Conclusions

This paper presents a general method to built asymptotic solutions of a wide class of boundary value problems admitting a fundamental Green solution. Contrary to other methods, such as the method of matched asymptotic expansions, our approach provides, without too much efforts, the solution at high orders of approximation and for non-axisymmetric slender bodies. This general scheme is also likely to apply to 2D and 3D elasticity problems.

Finally, one should note that the case of a slender-body of curved (not straight) centre-line would probably require a different treatment. In such circumstances, one may employ the method of matched asymptotic expansions.

Appendix

We denote by $E_{]0,1[}$ the set of complex functions s, defined on $]0,1[$ and such that $s(x) = R_s(x) + s_0 \log x + s_1 \log(1-x) + s_0' x^{-1/2} + s_1' \times (1-x)^{-1/2}$ where s_0, s_0', s_1 and s_1' are complex values and the complex function R_s fulfills the next assumptions:

1. There exist $\eta > 0$ with R_s bounded on $[0,\eta]$ and $[1-\eta, 1]$ and also four strictly positive real values $C_0, C_1, \alpha_0 < 1$ and $\alpha_1 < 1$ such that $|R_s(x) - R_s(y)| < C_0|x-y|^{\alpha_0}$ and $|R_s(x) - R_s(y)| < C_1|x-y|^{\alpha_1}$ respectively on $[0,\eta] \times [0,\eta]$ and $[1-\eta, 1] \times [1-\eta, 1]$.
2. $\int_0^1 R_s^2(x)\sqrt{x(1-x)}\mathrm{d}x$ and $\int_0^1 |R_s(x)|\mathrm{d}x$ exist.
3. $\forall x \in]0,1[$ there exist $0 < \eta_x < \mathrm{Min}(1-x, x), C_x > 0$ and $0 < \alpha_x < 1$ such that $\forall y \in [x-\eta_x, x+\eta_x]$ then $|R_s(y) - R_s(x)| < C_x|y-x|^{\alpha_x}$.

In this paper we need to solve the following integral equation

$$\frac{1}{\pi} pv \int_0^1 \frac{g(x)\mathrm{d}x}{x - x_1} = s(x_1) \quad \text{for } 0 < x_1 < 1 \tag{7.1}$$

for $s \in E_{]0,1[}$ and unknown function g not only obeying above assumption 3 (hölder and local condition on $]0,1[$ which give a sense to the integral on the left-hand side of Eq. (7.1)) but also such that $\int_0^1 g^2(x)\sqrt{x(1-x)}\mathrm{d}x$ exists. For $s \in E_{]0,1[}$ it is not legitimate to apply the results exhibited by standard textbooks (see Muskhelishvili 1958; Kanwal 1971; Zabreyko 1975) since given function s does not fulfill a Lipschitz condition on $[0,1]$ and "good" behaviors at end points 0 and 1. However, Schröder (1938) and Söhngen (1939) provided solutions when s obeys weaker assumptions which are true for $s \in E_{]0,1[}$. Application of a result derived by Schröder (1938, p 347) indeed yields the following solution

$$\frac{\pi g(x_1)}{\{x_1(1-x_1)\}^{-1/2}} = pv \int_0^1 \frac{s(x)\sqrt{x(1-x)}}{x_1 - x}\mathrm{d}x + C \ , \tag{7.2}$$

$$C = \int_0^1 g(x)\mathrm{d}x \ . \tag{7.3}$$

For $s_1' = 0$, we seek a solution g such that $|g(1)| < \infty$. If $I(x_1)$ denotes the first integral arising on the right-hand side of Eq. (7.2) then $I(x_1) = J(x_1) + K[s(x)](x_1)$ with

$$J(x_1) = \int_0^1 \frac{s(x)\sqrt{x}\mathrm{d}x}{\sqrt{1-x} + \sqrt{1-x_1}} \ , \tag{7.4}$$

$$\frac{K[s(x)](x_1)}{\sqrt{1-x_1}} = pv \int_0^1 \frac{s(x)\sqrt{x}}{x_1 - x}\mathrm{d}x \ . \tag{7.5}$$

After some elementary algebra, one can prove (for $s_1' = 0$) that

$$\lim_{x_1 \to 1^-} K[s(x)](x_1) = 0 \ , \tag{7.6}$$

$$C = -\lim_{x_1 \to 1} J(x_1) = -\int_0^1 \sqrt{\frac{x}{1-x}} s(x)\mathrm{d}x \ . \tag{7.7}$$

Accordingly, the solution (7.2) is given by

$$\frac{\pi\sqrt{x_1}g(x_1)}{\sqrt{1-x_1}} = O_{x_1}[s] = pv \int_0^1 \sqrt{\frac{x}{1-x}} \frac{s(x)}{x_1 - x}\mathrm{d}x \ . \tag{7.8}$$

This form (7.8) does not imply that $g(1) = 0$!. For instance, observe that

$$pv \int_0^1 \frac{\mathrm{d}x}{x - x_1} = \log\left[\frac{1-x_1}{x_1}\right] = S(x_1) \tag{7.9}$$

with $S \in E_{]0,1[}$ and $S_1' = 0$. The reader may separately check that

$$pv \int_0^1 \sqrt{\frac{x}{1-x}} \frac{\log[(1-x)/x]}{x_1 - x}\mathrm{d}x = \pi^2 \sqrt{\frac{x_1}{1-x_1}} \ . \tag{7.10}$$

Thus, if $s = S$ the formula (7.8) indeed gives the constant solution $g(x) = \pi$. Thereafter, it seems quite useful to determine what kind of condition bearing on s of $E_{]0,1[}$ with $s_1' = 0$ makes the solution (7.8) vanish at the end point 1. In previous property 1, assume that $\alpha_1 > 1/2$. The relation

$$O_{x_1}[s] = [R_s(x_1) + (s_0 + s_1)\log x_1] O_{x_1}[1] + \pi^2 s_1 \sqrt{\frac{x_1}{1-x_1}} + O_{x_1}[\bar{s}(x_1, .)] \tag{7.11}$$

where the operator O_{x_1} is given by Eq. (7.8) and

$$\bar{s}(x_1, x) = R_s(x) - R_s(x_1) + (s_0 + s_1)\log(x/x_1) \ , \tag{7.12}$$

clearly shows that the solution (7.8) is such that $g(1) = \pi s_1$. Thus, the value of $g(1)$ is only governed by the logarithmic term $s_1 \log(1-x)$ of the given function s (as soon as $s_1' = 0$ and $\alpha_1 > 1/2$.)

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