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Stokes flow past a slender particle

By A. Sellier

LADHYX, Ecole Polytechnique, 91128 Palaiseau Cedex, France

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This paper presents a systematic slender-body theory for a slender particle embedded in an arbitrary Stokes flow. Contrary to previous works, the body is not necessarily of revolution. The approach consists of gaining the surface stress acting on the particle by asymptotically solving, with respect to a slenderness ratio, a Fredholm boundary integral equation of the first kind. The procedure approximates integrals depending upon a small parameter by invoking a systematic formula. Special attention is paid to particles of elliptical cross-section and term-to-term comparisons are given for a slender ellipsoid embedded in a rather simple Stokes flow.

Keywords: Stokes flow; slender body; asymptotic approximation; singularities

1. Introduction

In predicting the main rheological properties of dilute suspensions (see, for example, Happel & Brenner 1973; Kim & Karrila 1991) it is of prime importance to consider the low-Reynolds-number flow about a single particle. As the particle is sometimes slender (for instance, a slender stiff fibre) many works expand, in terms of the small slenderness ratio ϵ, the physical quantities of interest such as drag force or torque acting on the body.

Some studies (see Cox 1970, 1971; Keller & Rubinow 1976; Johnson & Wu 1979; Johnson 1980) deal, by applying the method of matched asymptotic expansions (Van Dyke 1975), with a slender body of circular cross-section and of curved centreline (the line to which the body collapses as ϵ goes to zero). Most of the available studies address slender bodies of revolution by employing a method pioneered by Landweber (1951, 1959) and also applied in other fields (see Moran 1963; Handelsman & Keller 1967*b*; Geer 1974, 1975). This method consists of asymptotically solving, for unknown singularities (such as Stokeslets, rotlets) put on the centreline, the boundary integral equation imposed by the no-slip condition. This integral equation is expanded either by applying the method of matched asymptotic expansions or by invoking the treatment of Handelsman & Keller (1967*a*). For a slender body of revolution this procedure has been carried out for a uniform external flow (see Tuck 1964; Tillett 1970), for an external shear flow (see Cox 1971) and for a general external flow (see Geer 1976). At least for the inviscid flow about a particle, Cade (1994) recently proved that the derived boundary integral equation may be ill-posed. This feature casts some doubt on the validity of the procedure. To the author's knowledge, only Batchelor (1970) considers a straight (i.e. of straight centreline) slender body of arbitrary cross-section but for uniform or linear external flows. His work, which only gives the first-order solution, was extended to the motion of a slender particle near a plane fluid–fluid interface by Yang & Leal (1983).

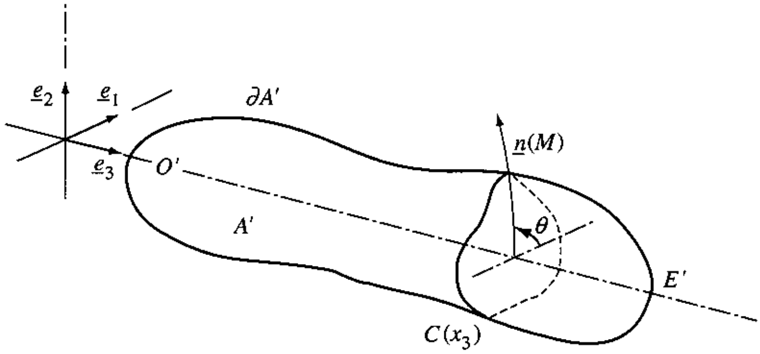

Figure 1.

This work presents a general theory valid, up to high orders, for a straight slender body of arbitrary cross-section embedded in a general Stokes flow. It rests on the asymptotic solution of a boundary integral that governs the surface stress exerted by the fluid on the particle. The 'boundary integral method' for linearized viscous flow was, for instance, employed by Youngren & Acrivos (1975) for the numerical calculation of the Stokes flow past a cylindrical or a spheroidal body, by Rallison & Acrivos (1978) for the prediction of the deformation and condition of break-up in shear of a liquid drop lying in a liquid of different viscosity and by Pozrikidis (1990) for the axisymmetric deformation of a red blood cell in uniaxial straining Stokes flow. This approach is free from the objections Cade (1994) alluded to.

The paper is organized as follows. The boundary integral equation is addressed in § 2. By using a systematic formula established in Sellier (1996), we asymptotically expand this integral equation in § 3. The asymptotic solution is built in § 4 while comparisons both with exact solutions and other works are given in § 5 for a slender body of elliptical cross-section. Finally, concluding remarks close the paper in § 6.

2. The boundary integral equation

The motionless, straight and rigid slender body $\mathcal{A}'$ (see figure 1) admits a smooth boundary $\partial\mathcal{A}'$ whose rounded ends are O' and E'. We adopt Cartesian coordinates (O', x_1', x_2', x_3') with $\boldsymbol{e}_3 := \boldsymbol{O'E'}/O'E'$.

This body disturbs the steady and smooth ambient Stokes flow $(\boldsymbol{u}^\infty, p^\infty)$ with $\boldsymbol{u}$ and p denoting the velocity and pressure disturbances. Under the tensor summation convention and the notation $\boldsymbol{a} = a_i\boldsymbol{e}_i$, these Stokes flows $(\boldsymbol{u}^\infty, p^\infty)$ and $(\boldsymbol{u}, p)$ obey (see Happel & Brenner 1973)

$$\mu\frac{\partial^2 u_i^\infty}{\partial x_j'\partial x_j'} = \frac{\partial p^\infty}{\partial x_i'}, \quad \frac{\partial u_i^\infty}{\partial x_i'} = 0, \quad x \in \mathbb{R}, \tag{2.1}$$

$$\mu\frac{\partial^2 u_i}{\partial x_j'\partial x_j'} = \frac{\partial p}{\partial x_i'}, \quad \frac{\partial u_i}{\partial x_i'} = 0, \quad x \in \mathbb{R}^3 \setminus (\mathcal{A}' \cup \partial\mathcal{A}'), \tag{2.2}$$

$$|\boldsymbol{u}| \to 0 \quad \text{and} \quad |p| \to 0 \quad \text{as } |\boldsymbol{x}'| \to \infty, \tag{2.3}$$

$$[\boldsymbol{u}^\infty + \boldsymbol{u}](M) = \boldsymbol{0} \quad \text{for } M \in \partial\mathcal{A}', \tag{2.4}$$

where μ denotes the fluid viscosity and (2.4) is the no-slip boundary condition.

If $\boldsymbol{f} = f_i\boldsymbol{e}_i$ designates the surface force exerted by the fluid on $\partial\mathcal{A}'$, the following integral representations hold (see Kim & Karrila 1991; Pozrikidis 1992):

$$u_i(M) = -\frac{1}{8\pi\mu}\int_{\partial\mathcal{A}'} f_j(P)G_{ij}(P,M)\,\mathrm{d}S'_P, \tag{2.5}$$

$$p(M) = -\frac{1}{4\pi}\int_{\partial\mathcal{A}'} \frac{f_j(P)\boldsymbol{PM}\cdot\boldsymbol{e}_j}{PM^3}\,\mathrm{d}S'_P, \tag{2.6}$$

where the fundamental Oseen tensor, (G_{ij}), reads

$$G_{ij}(P,M) = \delta_{ij}/PM + [\boldsymbol{PM}\cdot\boldsymbol{e}_i][\boldsymbol{PM}\cdot\boldsymbol{e}_j]/PM^3. \tag{2.7}$$

Thus, equations (2.2) and (2.3) are satisfied. As the field $\boldsymbol{u}$ given by (2.5) is continuous across $\partial\mathcal{A}'$ (see Pozrikidis 1992), the no-slip condition (2.4) yields the following Fredholm integral equation of the first kind:

$$u_i^\infty(M) = \frac{1}{8\pi\mu}\int_{\partial\mathcal{A}'} f_j(P)G_{ij}(P,M)\,\mathrm{d}S'_P, \quad M\in\partial\mathcal{A}', \quad i\in\{1,2,3\}. \tag{2.8}$$

Note that (2.8) is a Fredholm integral equation of the first kind for the unknown density $\boldsymbol{f}$ and a given ambient flow $\boldsymbol{u}^\infty$. For any point P (see Pozrikidis 1992), the Oseen tensor satisfies $\int_{\partial\mathcal{A}'} G_{ij}(P,M)n_i(M)\,\mathrm{d}S'_M = 0$ where (see figure 1) $\boldsymbol{n}(M)$ denotes the outward unit normal. Accordingly, the two following remarks arise.

(i) The left-hand side $\boldsymbol{u}^\infty$ of (2.8) must satisfy the compatibility relation

$$\int_{\partial\mathcal{A}'} \boldsymbol{u}^\infty(M)\cdot\boldsymbol{n}(M)\,\mathrm{d}S'_M = 0. \tag{2.9}$$

This is indeed the case (see (2.1)).

(ii) If $\boldsymbol{u}^\infty = \boldsymbol{0}$ then $\boldsymbol{f} = \boldsymbol{n}$ is the solution to (2.8).

Thus, under the compatibility condition (2.9) the integral equation (2.8) may admit several solutions. More precisely (see Dautray & Lions 1988), if $H^s(\partial\mathcal{A}')$ denotes the usual Sobolev space then as soon as the restriction $\boldsymbol{u}^\infty_{|\partial\mathcal{A}'}$ of $\boldsymbol{u}^\infty$ to $\partial\mathcal{A}'$ belongs to $(H^{1/2}(\partial\mathcal{A}'))^3$ and obeys (2.9), then (2.8) admits a solution in $(H^{-1/2}(\partial\mathcal{A}'))^3$ unique up to any constant multiple of the normal $\boldsymbol{n}$. The proof and additional information are available in Ladyzhenskaya (1969), Kress (1989) and Pozrikidis (1992). Hence, the integral equation (2.8) admits a general solution of the form

$$\boldsymbol{f}(P) = \boldsymbol{f}_\mathrm{s}(P) + \lambda\boldsymbol{n}(P), \tag{2.10}$$

where $\boldsymbol{f}_\mathrm{s}$ indicates any special solution and λ is an arbitrary real number. However, we retrieve the physical solution by taking into account the linearity of the problem: we set λ to zero in (2.10) and select the solution *linear* in our data $(\boldsymbol{u}^\infty, p^\infty)$ provided the associated incoming pressure p^∞ is also *linear* in $\boldsymbol{u}^\infty$.

3. Asymptotic expansion of the boundary-integral system

(a) *Dimensionless variables and general assumptions*

The typical length L and 'radius' e of our slender body $\mathcal{A}'$ read $L = O'E'$ and

$$e^2 := \max_{M\in\partial\mathcal{A}'}[(x'_1)^2 + (x'_2)^2]$$

and the slenderness ratio is $\epsilon := e/L \ll 1$. We introduce non-dimensional coordinates $(x_1, x_2, x_3) := (x'_1/e, x'_2/e, x'_3/L)$ and assume that the velocity $\boldsymbol{u}^\infty$ writes $\boldsymbol{u}^\infty(x'_1, x'_2, x'_3) = U_i(\epsilon x_1, \epsilon x_2, x_3)\boldsymbol{e}_i$ in a neighbourhood of $\partial\mathcal{A}'$. More precisely, there exist two integers $N \geqslant 1$, $R \geqslant 0$ and a real number $\delta > 1$ such that the derivatives

$$\partial^r_{x_3} U_i^{j,k}(X, Y, x_3) := \frac{\partial^{j+k+r} U_i}{\partial X^j \partial Y^k \partial x_3^r}(X, Y, x_3), \quad i \in \{1, 2, 3\}, \tag{3.1}$$

exist for positive integers j, k and r obeying $0 \leqslant j + k \leqslant N + 1$, $0 \leqslant r \leqslant R + 1$ and with (X, Y, x_3) belonging to the open set

$$\mathcal{O}'_{\delta\epsilon} := \{(X, Y, x_3); X^2 + Y^2 < \delta^2\epsilon^2 \text{ and } 1 - \delta < x_3 < \delta\}.$$

Since (2.8) is linear we restrict our attention to a velocity $\boldsymbol{u}^\infty$ such that in $\mathcal{O}'_{\delta\epsilon}$, and for $0 \leqslant j + k \leqslant N + 1$,

$$U_i^{j,k}(X, Y, x_3) = O(1), \quad i \in \{1, 2, 3\}. \tag{3.2}$$

In order to specify the surface $\partial\mathcal{A}'$ we also employ non-dimensional cylindrical coordinates (r, θ, x_3) with $r^2 = x_1^2 + x_2^2$. Then $\partial\mathcal{A}'$ is described by a positive single-valued and smooth-enough shape function $f(\theta, x_3)$ that vanishes for $x_3(1 - x_3) = 0$ and such that $r = f(\theta, x_3) = O(1)$ for $M(r, \theta, x_3) \in \partial\mathcal{A}'$. For almost any θ in $[0, 2\pi]$ we also require the square of the function f, denoted by $f^2(\theta, x_3)$, to be analytic with respect to x_3 in $[0, 1]$ and to admit the behaviours

$$f^2(\theta, x_3) = \sum_{n \geqslant 1} c_n(\theta) x_3^n, \qquad 2ff^1_{x_3} = \sum_{n \geqslant 1} n c_n(\theta) x_3^{n-1}, \quad x_3 \to 0^+, \tag{3.3}$$

$$f^2(\theta, x_3) = \sum_{n \geqslant 1} b_n(\theta)(1 - x_3)^n, \quad 2ff^1_{x_3} = -\sum_{n \geqslant 1} n b_n(\theta)(1 - x_3)^{n-1}, \quad x_3 \to 1^-, \tag{3.4}$$

where $0 < c_1(\theta) = O(1)$, $0 < b_1(\theta) = O(1)$ and $\partial^j_v f := \partial^j f/\partial v^j$ for $v \in \{\theta, x_3\}$.

(b) *The asymptotic form of the integral system*

By using the previous assumptions and expanding $u_i^\infty(M)$ in its Taylor polynomial expansion of order N, the left-hand side of (2.8) becomes

$$u_i^\infty(M) = \sum_{0 \leqslant n \leqslant N} \left[\sum_{j+k=n} a_i^{j,k}(x_3) x_1^j x_2^k\right] \epsilon^n + O(\epsilon^{N+1}), \quad i \in \{1, 2, 3\}, \tag{3.5}$$

where the new functions $a_i^{j,k}(x_3)$ obey

$$a_i^{j,k}(x_3) := U_i^{j,k}(0, 0, x_3)/[j!k!] = O(1). \tag{3.6}$$

By invoking the continuity equation (2.1) we deduce the following relations:

$$(p + 1)a_1^{p+1,q}(x_3) + (q + 1)a_2^{p,q+1}(x_3) + \partial^1_{x_3} a_3^{p,q}(x_3) = 0, \qquad \text{for } 0 \leqslant p + q = n \leqslant N - 1. \tag{3.7}$$

Since the integral occurring on the right-hand side, $rh_i(M)$, of (2.8) is regular we can apply to it the change of variables

$$P(x_1^P, x_2^P, x_3^P) = (f(\theta_P, x_3^P), \theta_P, x_3^P)$$

and Fubini's theorem (see Rudin 1966). If the functions s_ϵ and H obey

$$s_\epsilon = \{1 + (f^{-1} f_\theta^1)^2 + (\epsilon f_{x_3}^1)^2\}^{1/2}, \quad \mathrm{d}S'_P = eL[f s_\epsilon](\theta_P, x_3^P)\,\mathrm{d}\theta_P\,\mathrm{d}x_3^P, \tag{3.8}$$

$$H(\theta_P, x_3^P, \theta, x_3) = \{f^2(\theta_P, x_3^P) + f^2(\theta, x_3) - 2\cos(\theta_P - \theta) f(\theta, x_3) f(\theta_P, x_3^P)\}^{1/2}, \tag{3.9}$$

it follows, for $x_3 \in]0, 1[$ and $\theta \in [0, 2\pi]$ (see (2.7)), that

$$\begin{aligned} rh_i(\theta, x_3) = \frac{e}{8\pi\mu} \int_0^{2\pi} \Bigg[\int_0^1 \Bigg\{ & \frac{\delta_{ij}}{[(x_3^P - x_3)^2 + \epsilon^2 H^2(\theta_P, x_3^P, \theta, x_3)]^{1/2}} \\ & + \epsilon^{2-\delta_{i3}-\delta_{j3}} \frac{(x_i^P - x_i)(x_j^P - x_j)}{[(x_3^P - x_3)^2 + \epsilon^2 H^2(\theta_P, x_3^P, \theta, x_3)]^{3/2}} \Bigg\} \\ & \times [f_j f s_\epsilon](\theta_P, x_3^P)\,\mathrm{d}x_3^P \Bigg] \mathrm{d}\theta_P, \end{aligned} \tag{3.10}$$

with, for $M \in \partial\mathcal{A}'$, the links $x_1 = f(\theta, x_3)\cos\theta$ and $x_2 = f(\theta, x_3)\sin\theta$. Note that $rh_i(M)$ depends upon ϵ via $\boldsymbol{f} = f_i \boldsymbol{e}_i$, ϵH and also the function s_ϵ. Under the introduction of the functions $\boldsymbol{d}, h$ and linear operators $A_{\epsilon,h}^{x_3}$ and $B_{\epsilon,h}^{x_3}$ as

$$\boldsymbol{d}(P) := e[\boldsymbol{f} f s_\epsilon](P)/[8\pi\mu], \qquad h(u) = H(\theta_P, x_3 + u, \theta, x_3), \tag{3.11}$$

$$A_{\epsilon,h}^{x_3}[g] := \int_{-x_3}^{1-x_3} \frac{g(x_3 + u)\,\mathrm{d}u}{[u^2 + \epsilon^2 h^2(u)]^{1/2}}, \quad B_{\epsilon,h}^{x_3}[g] := \int_{-x_3}^{1-x_3} \frac{g(x_3 + u)\,\mathrm{d}u}{[u^2 + \epsilon^2 h^2(u)]^{3/2}}, \tag{3.12}$$

we obtain, for $x_3^P = x_3 + u$,

$$rh_i(\theta, x_3) = \int_0^{2\pi} \{A_{\epsilon,h}^{x_3}[d_i] + \epsilon^{2-\delta_{i3}-\delta_{j3}} B_{\epsilon,h}^{x_3}[(x_i^P - x_i)(x_j^P - x_j) d_j]\}\,\mathrm{d}\theta_P. \tag{3.13}$$

Thus, the asymptotic behaviour of $rh_i(\theta, x_3)$ is deduced from the asymptotic estimates of the integrals $A_{\epsilon,h}^{x_3}[g]$ and $B_{\epsilon,h}^{x_3}[g]$. By virtue of (3.9), $h > 0$ for $\theta_P \neq \theta$. Accordingly, $A_{\epsilon,h}^{x_3}[g]$ and $B_{\epsilon,h}^{x_3}[g]$ are regular integrals for $\epsilon > 0$ while $A_{0,h}^{x_3}[g]$ and $B_{0,h}^{x_3}[g]$ become hypersingular integrals (unless g vanishes together with certain derivatives at $u = x_3$). Hence, the asymptotic approximation of $A_{\epsilon,h}^{x_3}[g]$ and $B_{\epsilon,h}^{x_3}[g]$ with respect to ϵ is singular. This asymptotic behaviour cannot be obtained by the usual methods (see Bleistein & Handelsman 1975; Wong 1989; Estrada & Kanwal 1994), which address integrals such as

$$\int_a^b g(x) f(x/\epsilon)\,\mathrm{d}x.$$

For $A_{\epsilon,1}^{x_3}[g]$ a possible procedure has been used by Handelsman & Keller (1967*a*, *b*) but its extension to the present cases seems tedious. Here we employ a systematic formula whose establishment (see Sellier 1996) rests on the concept of integration in the finite-part sense of Hadamard (see Hadamard 1932; Schwartz 1966; Sellier 1994). This formula circumvents the tremendous matching rules required by the method of matched asymptotic expansions. It may also provide the asymptotic behaviour of $A^{x_3}_{\epsilon,h}[g]$ and $B^{x_3}_{\epsilon,h}[g]$ up to higher orders for smooth-enough functions g and h. Here, we restrict ourselves to the following estimates (see Appendix A):

$$A^{x_3}_{\epsilon,h}[g] = A^{x_3}_0[g]\log\epsilon + A^{\theta,x_3}_1[g] + A^{\theta,x_3}_2[g]\epsilon^2\log\epsilon + A^{\theta,x_3}_3[g]\epsilon^2 + O(M[g]\epsilon^4\log\epsilon), \tag{3.14}$$

$$B^{x_3}_{\epsilon,h}[g] = B^{\theta,x_3}_0[g]/\epsilon^2 + B^{\theta,x_3}_1[g]\log\epsilon + B^{\theta,x_3}_2[g] + O(M[g]\epsilon^2\log\epsilon), \tag{3.15}$$

where

$$M[g] := \max_{[0,1]}|g|$$

and the new linear operators A^{θ,x_3}_i or B^{θ,x_3}_i are given in Appendix A. Motivated by (3.13), we rather detail the operators

$$I^{\theta,x_3}_i[g] := \int_0^{2\pi} A^{\theta,x_3}_i[g]\,\mathrm{d}\theta_P \quad\text{and}\quad J^{\theta,x_3}_i[g] := \int_0^{2\pi} B^{\theta,x_3}_i[g]\,\mathrm{d}\theta_P.$$

These quantities read

$$I^{x_3}_0[g] = -2\int_0^{2\pi} g(\theta_P,x_3)\,\mathrm{d}\theta_P,\quad J^{\theta,x_3}_0[g] = 2\int_0^{2\pi}\frac{g(\theta_P,x_3)\,\mathrm{d}\theta_P}{H^2(\theta_P,x_3,\theta,x_3)}, \tag{3.16}$$

$$I^{\theta,x_3}_2[g] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_0^{2\pi}\frac{H^2(\theta_P,t,\theta,x_3)}{2g^{-1}(\theta_P,t)}\mathrm{d}\theta_P\right]_{t=x_3},\quad J^{x_3}_1[g] = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[I^t_0[g]]_{t=x_3}, \tag{3.17}$$

$$I^{\theta,x_3}_1[g] = -\left\{I^{x_3}_0[g]\log 2 + \mathrm{fp}\int_0^1\frac{I^t_0[g]\,\mathrm{d}t}{2|t-x_3|} + 2\int_0^{2\pi}\frac{\log H(\theta_P,x_3,\theta,x_3)}{g^{-1}(\theta_P,x_3)}\,\mathrm{d}\theta_P\right\}, \tag{3.18}$$

$$J^{\theta,x_3}_2[g] = -\,\mathrm{fp}\int_0^1\frac{I^t_0[g]\,\mathrm{d}t}{2|t-x_3|^3} - \int_0^{2\pi}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[(1+\log(\tfrac{1}{2}H(\theta_P,t,\theta,x_3)))g(\theta_P,t)]_{t=x_3}\,\mathrm{d}\theta_P, \tag{3.19}$$

$$I^{\theta,x_3}_3[g] = -\,\mathrm{fp}\int_0^1\left\{\frac{\int_0^{2\pi} g(\theta_P,t)H^2(\theta_P,t,\theta,x_3)\,\mathrm{d}\theta_P}{2|t-x_3|^3}\right\}\mathrm{d}t + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_0^{2\pi} g(\theta_P,t)\tfrac{1}{2}H^2(\theta_P,t,\theta,x_3)(\tfrac{1}{2}+\log(\tfrac{1}{2}H(\theta_P,t,\theta,x_3)))\,\mathrm{d}\theta_P\right]_{t=x_3}. \tag{3.20}$$

The symbol fp $\int$ denotes an integral in the finite-part sense of Hadamard. In (3.20) we switched the integration on θ and the derivation with respect to t. This is not possible for the operator J^{θ,x_3}_2 (see (3.19)). For $0 < x_3 < 1$, we introduce the non-dimensional cross-section $\mathrm{Cs}(x_3) = \{P \in \mathcal{A}'; x^P_3 = x_3\}$ and the closed path $C(x_3) := \partial\,\mathrm{Cs}(x_3)$. The quantities $I^{x_3}_0[g]$ and $J^{\theta,x_3}_0[g]$ only involve the function g on $\mathrm{Cs}(x_3)$ and are thereafter two-dimensional terms. The operators I^{θ,x_3}_2 and $J^{x_3}_1$ are weakly three dimensional in the sense that they require the values of g in a neighbourhood of the cross-section $\mathrm{Cs}(x_3)$. Finally, the terms I^{θ,x_3}_1, J^{θ,x_3}_2 and I^{θ,x_3}_3 are strongly three dimensional (the function g is taken on the whole surface $\partial\mathcal{A}'$). By virtue of our previous results the integral system (2.8) reads

$$\sum_{0\leqslant n\leqslant N}\left[\sum_{j+k=n} a_1^{j,k}(x_3)x_1^j x_2^k\right]\epsilon^n + O(\epsilon^{N+1}) = O(D_1\epsilon^4\log\epsilon; D_2\epsilon^4\log\epsilon; D_3\epsilon^3\log\epsilon)$$
$$+ I_0^{x_3}[d_1]\log\epsilon + I_1^{\theta,x_3}[d_1] + J_0^{\theta,x_3}[(x_1^P - x_1)^2 d_1 + (x_1^P - x_1)(x_2^P - x_2)d_2]$$
$$+ J_1^{x_3}[(x_1^P - x_1)(x_3^P - x_3)d_3]\epsilon\log\epsilon + J_2^{\theta,x_3}[(x_1^P - x_1)(x_3^P - x_3)d_3]\epsilon$$
$$+ \{I_2^{\theta,x_3}[d_1] + J_1^{x_3}[(x_1^P - x_1)^2 d_1] + J_1^{\theta,x_3}[(x_1^P - x_1)(x_2^P - x_2)d_2]\}\epsilon^2\log\epsilon$$
$$+ \{I_3^{\theta,x_3}[d_1] + J_2^{\theta,x_3}[(x_1^P - x_1)^2 d_1 + (x_1^P - x_1)(x_2^P - x_2)d_2]\}\epsilon^2, \tag{3.21}$$

$$\sum_{0\leqslant n\leqslant N}\left[\sum_{j+k=n} a_2^{j,k}(x_3)x_1^j x_2^k\right]\epsilon^n + O(\epsilon^{N+1}) = O(D_1\epsilon^4\log\epsilon; D_2\epsilon^4\log\epsilon; D_3\epsilon^3\log\epsilon)$$
$$+ I_0^{x_3}[d_2]\log\epsilon + I_1^{\theta,x_3}[d_2] + J_0^{\theta,x_3}[(x_2^P - x_2)^2 d_2 + (x_1^P - x_1)(x_2^P - x_2)d_1]$$
$$+ J_1^{x_3}[(x_2^P - x_2)(x_3^P - x_3)d_3]\epsilon\log\epsilon + J_2^{\theta,x_3}[(x_2^P - x_2)(x_3^P - x_3)d_3]\epsilon$$
$$+ \{I_2^{\theta,x_3}[d_2] + J_1^{x_3}[(x_2^P - x_2)^2 d_2] + J_1^{\theta,x_3}[(x_1^P - x_1)(x_2^P - x_2)d_1]\}\epsilon^2\log\epsilon$$
$$+ \{I_3^{\theta,x_3}[d_2] + J_2^{\theta,x_3}[(x_2^P - x_2)^2 d_2 + (x_1^P - x_1)(x_2^P - x_2)d_1]\}\epsilon^2, \tag{3.22}$$

$$\sum_{0\leqslant n\leqslant N}\left[\sum_{j+k=n} a_3^{j,k}(x_3)x_1^j x_2^k\right]\epsilon^n + O(\epsilon^{N+1}) = O(D_1\epsilon^3\log\epsilon; D_2\epsilon^3\log\epsilon; D_3\epsilon^2\log\epsilon)$$
$$+ \{I_0^{x_3}[d_3] + J_1^{x_3}[(x_3^P - x_3)^2 d_3]\}\log\epsilon + I_1^{\theta,x_3}[d_3] + J_2^{\theta,x_3}[(x_3^P - x_3)^2 d_3]$$
$$+ J_1^{x_3}[(x_3^P - x_3)(d_1(x_1^P - x_1) + d_2(x_2^P - x_2))]\epsilon\log\epsilon$$
$$+ J_2^{\theta,x_3}[(x_3^P - x_3)(d_1(x_1^P - x_1) + d_2(x_2^P - x_2))]\epsilon, \tag{3.23}$$

where $D_i = \max_{\partial\mathcal{A}'}[d_i], x_1^P = f(\theta_P, z_P)\cos\theta_P$ and $x_2^P = f(\theta_P, z_P)\sin\theta_P$.

4. Asymptotic solution

This section asymptotically inverts the previous system (3.21)–(3.23).

(*a*) *General properties of two basic integral problems*

If $M(x_1, x_2, x_3) \in \partial\mathcal{A}'$ we denote by $\boldsymbol{n}^0(M)$ the outward unit normal $C(x_3)$ at M and such that $\boldsymbol{n}^0(M)\cdot\boldsymbol{e}_3 = 0$. For $M \in C(x_3)$ it is useful to address the integral equations

$$L^{\theta,x_3}[u_3] := -\oint_{C(x_3)} u_3(P)\log[PM]\,\mathrm{d}l_P = b_3(M), \tag{4.1}$$

$$S_i^{\theta,x_3}(u_1;u_2) := \oint_{C(x_3)}\sum_{j=1}^{2} G_{ij}^0(P,M)u_j(P)\,\mathrm{d}l_P = b_i(M), \quad i\in\{1,2\}, \tag{4.2}$$

for given functions b_1, b_2 and b_3 and unknown functions u_3 and (u_1, u_2). In (4.2), the notation (G^0_{ij}) denotes the free-space but two-dimensional Oseen tensor defined, for $\boldsymbol{PM} \cdot \boldsymbol{e}_3 = 0$, by

$$G^0_{ij}(P, M) = -\delta_{ij} \log[PM] + [\boldsymbol{PM} \cdot \boldsymbol{e}_i][\boldsymbol{PM} \cdot \boldsymbol{e}_j]/PM^2, \quad (i,j) \in \{1,2\} \times \{1,2\}. \tag{4.3}$$

The boundary-integral equations of the first kind (4.1) and (4.2) are closely connected to the boundary integral formulation associated, respectively, to an exterior Dirichlet problem for the Laplace equation in the plane and to an exterior two-dimensional and Dirichlet–Stokes problem $(S_i^{\theta,x_3}(u_1;u_2)$, which is indeed proportional to the component $v_i(M)$ of the velocity induced at $M(\theta, x_3) \in C(x_3)$ by a distribution of two-dimensional Stokeslets over $C(x_3)$ and of density $(u_1, u_2))$. For mathematical aspects pertaining to these integral equations, the reader is directed to Fichera (1961) and to Hsiao & MacCamy (1973). Note that if $C(x_3) = \{P(x_1, x_2); x_1^2 + x_2^2 = 1\}$ then $L^{\theta,x_3}[1] = 0$. However, if the cross-section diameter $\delta(x_3) := \max\{PM \text{ for } P, M \in \mathrm{Cs}(x_3)\}$ is small enough (this is true for an adequate choice of 'radius' e), then (4.1) admits a unique solution $u = \{L^{\theta,x_3}\}^{-1}[b] \in H^{-1/2}(C(x_3))$ when $b \in H^{1/2}(C(x_3))$ (see Hsiao & Wendland 1977). According to Giroire (1987), if the operator K^{x_3} obeys

$$K^{x_3}[u] := \oint_{C(x_3)} u(P) \, \mathrm{d}l_P, \tag{4.4}$$

then $c_0(x_3) := K^{x_3}(\{L^{\theta,x_3}\}^{-1}[1]) \neq 0$. Moreover, the counterpart of the property of (G_{ij}) reads, for $M \in \mathrm{Cs}(x_3)$,

$$\oint_{C(x_3)} G^0_{ij}(P, M) n^0_j(P) \, \mathrm{d}l_P = 0, \quad i \in \{1, 2\}, \tag{4.5}$$

and (4.2) admits a sense only if (b_1, b_2) satisfies

$$\oint_{C(x_3)} [b_1 n^0_1 + b_2 n^0_2](M) \, \mathrm{d}l_M = 0. \tag{4.6}$$

In addition, $\boldsymbol{n}^0$ obeys (4.2) for $(b_1, b_2) = (0, 0)$. However, under the condition (4.6) the problem (4.2) admits a unique solution $\boldsymbol{u}^s$ *linear upon the data* (b_1, b_2) denoted $\boldsymbol{u}^s = (u^s_1; u^s_2) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[b_1; b_2]$. Throughout the paper we assume, for any cross-section $\mathrm{Cs}(x_3)$, that if

$$\boldsymbol{t}^1 := (t^1_1; t^1_2) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[1; 0], \quad \boldsymbol{t}^2 := (t^2_1, t^2_2) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[0; 1], \tag{4.7}$$

then $\Delta(x_3) := K^{x_3}[t^1_1] K^{x_3}[t^2_2] - K^{x_3}[t^2_1] K^{x_3}[t^1_2] \neq 0$.

(*b*) *Asymptotic behaviour of density* $\boldsymbol{d}$

In usual circumstances the flow $\boldsymbol{u}^\infty$ often admits a non-zero family

$$(a_i^{j,k}(x_3))_{0 \leqslant j+k \leqslant 1}$$

and the system (3.21)–(3.23) clearly suggests that $\boldsymbol{d} = d_i \boldsymbol{e}_i$ reads

$$d_i(\theta, x_3) = \sum_{n=0}^{1} \sum_{m=1-n}^{\infty} d^i_{n,m}(\theta, x_3) \epsilon^n [\log \epsilon]^{-m} + O(\epsilon^2 \log \epsilon), \quad i \in \{1, 2, 3\}. \tag{4.8}$$

Since $dl_P = [fs_0](P)\,\mathrm{d}\theta_P$ (see (3.8)) it follows that, for $g = fs_0u$,

$$J_i^{\theta,x_3}[(x_3^P - x_3)^2 d] = I_0^{x_3}[d] + \delta_{i,2} I_1^{\theta,x_3}[d], \quad i \in \{1,2\}, \tag{4.9}$$

$$I_0^{x_3}[g] = -2K^{x_3}[u], \quad I_1^{\theta,x_3}[g] = 2L^{\theta,x_3}[u] + 2V_{x_3}\{K^t[u]\}, \tag{4.10}$$

where the operator V_{x_3} obeys, for $x_3 \in]0,1[$ and a function $\alpha(t)$ defined on $]0,1[$,

$$V_{x_3}[\alpha(t)] := \alpha(x_3)\log 2 + \mathrm{fp}\int_0^1 \frac{\alpha(t)\,\mathrm{d}t}{2|t - x_3|}. \tag{4.11}$$

Accordingly, by setting $d_{n,m}^i = fs_0 t_{n,m}^i$ and substituting (4.8) into (3.21)–(3.23), we obtain, for the first-order terms,

$$K^{x_3}[t_{0,1}^1] = -\tfrac{1}{2}a_1^{0,0}(x_3), \quad K^{x_3}[t_{0,1}^2] = -\tfrac{1}{2}a_2^{0,0}(x_3), \quad K^{x_3}[t_{0,1}^3] = -\tfrac{1}{4}a_3^{0,0}(x_3), \tag{4.12}$$

$$S_j^{\theta,x_3}(t_{0,m}^1; t_{0,m}^2) = K^{x_3}[t_{0,m+1}^j] - V_{x_3}\{K^t[t_{0,m}^j]\} = b_{0,m}^j(x_3), \quad j \in \{1,2\} \text{ and } m \geqslant 1, \tag{4.13}$$

$$L^{\theta,x_3}[t_{0,m}^3] = K^{x_3}[t_{0,m+1}^3] - W_{x_3}\{K^t[t_{0,m}^3]\} = b_{0,m}^3(x_3), \quad m \geqslant 1, \tag{4.14}$$

with $W_{x_3}[\alpha(t)] := V_{x_3}[\alpha(t)] - \frac{1}{2}\alpha(x_3)$ and for the second-order approximation (under the definitions $d_{0,0}^i := 0$ and for $m \geqslant 0$)

$$K^{x_3}[t_{1,0}^1] = K^{x_3}[t_{1,0}^2] = K^{x_3}[t_{1,0}^3] = 0, \tag{4.15}$$

$$\begin{aligned} S_j^{\theta,x_3}(t_{1,m}^1; t_{1,m}^2) &= K^{x_3}[t_{1,m+1}^j] - V_{x_3}\{K^t[t_{1,m}^j]\} + \tfrac{1}{2}\{\delta_{m,0}a_j^{1,0}(x_3)x_1 + \delta_{m,0}a_j^{0,1}(x_3)x_2 \\ &\quad - J_1^{x_3}[(x_j^P - x_j)(x_3^P - x_3)d_{0,m+1}^3] \\ &\quad - (1 - \delta_{m,0})J_2^{\theta,x_3}[(x_j^P - x_j)(x_3^P - x_3)d_{0,m}^3]\} \\ &= b_{1,m}^j(\theta, x_3), \quad j\{1,2\}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} L^{\theta,x_3}[t_{1,m}^3] &= K^{x_3}[t_{1,m+1}^3] - W_{x_3}\{K^t[t_{1,m}^3]\} + \tfrac{1}{4}\{\delta_{m,0}a_3^{1,0}(x_3)x_1 + \delta_{m,0}a_3^{0,1}(x_3)x_2 \\ &\quad - J_1^{x_3}[(x_3^P - x_3)(d_{0,m+1}^1(x_1^P - x_1) + d_{0,m+1}^2(x_2^P - x_2))] \\ &\quad - (1 - \delta_{m,0})J_2^{\theta,x_3}[(x_3^P - x_3)(d_{0,m}^1(x_1^P - x_1) + d_{0,m}^2(x_2^P - x_2))]\} \\ &= b_{1,m}^3(\theta, x_3). \end{aligned} \tag{4.17}$$

Hence, at each order $n \in \{0,1\}$, the functions $t_{n,m}^3$ and $(t_{n,m}^1; t_{n,m}^2)$, respectively, obey a 'Laplace-type' integral equation (4.1) and a two-dimensional 'Stokes-type' integral system (4.2) with data $b_i(M)$ depending, through $(a_i^{j,k}(x_3))$, on the external flow and also on the previous orders. If (4.6) holds for any couple $[b_{n,m}^1; b_{n,m}^2]$ the system (4.12)–(4.17) can be progressively solved from top to bottom. This task may be numerically achieved (with especial care for stability reasons (see Hsiao 1986)). However, we calculate the terms $b_{m,n}^i$ in order to check the condition (4.6) and to give, in § 5, analytical comparisons.

Owing to the definitions of operators K^{x_3}, V_{x_3} and W_{x_3}, each term $b_{0,m}^i$ indeed only depends on x_3. Thus, (4.6) holds for $[b_{0,m}^1; b_{0,m}^2]$ and the solution $(t_{0,m}^1; t_{0,m}^2)$ of (4.13) writes, for $m \geqslant 1$,

$$t_{0,m}^j(\theta, x_3) = b_{0,m}^1(x_3)t_j^1(\theta, x_3) + b_{0,m}^2(x_3)t_j^2(\theta, x_3), \quad j \in \{1,2\}. \tag{4.18}$$

This latter equality gives $K^{x_3}[t^j_{0,m}]$ in terms of $[b^1_{0,m}; b^2_{0,m}]$. By replacing each $b^j_{0,m}$ in (4.13), we not only deduce the link between $K^{x_3}[t^j_{0,m+1}]$ and $K^{x_3}[t^j_{0,m}]$ but also, since $c_0(x_3) \neq 0$, the couple $[b^1_{0,m}; b^2_{0,m}]$. More precisely, the solution reads

$$(K^{x_3}[t^1_{0,m}]; K^{x_3}[t^2_{0,m}]) = -\tfrac{1}{2}\boldsymbol{B}^{m-1}_{x_3}(a^{0,0}_1(t); a^{0,0}_2(t)), \quad m \geqslant 1, \tag{4.19}$$

$$(b^1_{0,m}(x_3); b^2_{0,m}(x_3)) = \boldsymbol{A}_{x_3}(K^t[t^1_{0,m}]; K^t[t^2_{0,m}]), \quad m \geqslant 1, \tag{4.20}$$

with

$$\boldsymbol{B}^0_{x_3}(\alpha_1(t); \alpha_2(t)) := (\alpha_1(x_3); \alpha_2(x_3)), \quad \boldsymbol{B}^k_{x_3} := \boldsymbol{B}_{x_3} \circ \boldsymbol{B}^{k-1}_{x_3}$$

if $k \geqslant 1$, and for

$$(v_1(x_3); v_2(x_3)) = \boldsymbol{A}_{x_3}(\alpha_1(t); \alpha_2(t)) \quad \text{and} \quad (w_1(x_3); w_2(x_3)) = \boldsymbol{B}_{x_3}(\alpha_1(t), \alpha_2(t)),$$

then

$$\Delta(x_3)v_1(x_3) = K^{x_3}[t^2_2]\alpha_1(x_3) - K^{x_3}[t^2_1]\alpha_2(x_3), \quad w_1(x_3) = v_1(x_3) + V_{x_3}[\alpha_1(t)], \tag{4.21}$$

$$\Delta(x_3)v_2(x_3) = K^{x_3}[t^1_1]\alpha_2(x_3) - K^{x_3}[t^1_2]\alpha_1(x_3), \quad w_2(x_3) = v_2(x_3) + V_{x_3}[\alpha_2(t)]. \tag{4.22}$$

If $u_0 := \{L^{\theta,x_3}\}^{-1}[1]$ recall that $c_0(x_3) = K^{x_3}[u_0] \neq 0$. Consequently, $t^3_{0,m} = b^3_{0,m}u_0$ and $K^{x_3}[t^3_{0,m}] = b^3_{0,m}c_0(x_3)$. Thanks to (4.14) it follows that, for $T^0_{x_3}[\alpha(t)] := \alpha(x_3)$,

$$K^{x_3}[t^3_{0,m}] = -\tfrac{1}{4}T^{m-1}_{x_3}[a^{0,0}_3(t)], \quad t^3_{0,m}(\theta, x_3) = \frac{K^{x_3}[t^3_{0,m}]}{c_0(x_3)}u_0(\theta, x_3), \quad m \geqslant 1, \tag{4.23}$$

$$T_{x_3}[\alpha(t)] = \left\{\log 2 - \tfrac{1}{2} + \frac{1}{c_0(x_3)}\right\}\alpha(x_3) + \mathrm{fp}\int_0^1 \frac{\alpha(t)\,\mathrm{d}t}{2|t - x_3|}. \tag{4.24}$$

In view of (4.18)–(4.24) the first-order functions $t^i_{0,m}$ are inductively deduced by inverting for any cross-section $\mathrm{Cs}(x_3)$ the three integral problems

$$L^{\theta,x_3}[u_0] = 1, \quad \boldsymbol{S}^{\theta,x_3}[t^1_1; t^1_2] = [1; 0], \quad \boldsymbol{S}^{\theta,x_3}[t^2_1; t^2_2] = [0; 1]. \tag{4.25}$$

Note that $V_{x_3}[\alpha(t)], W_{x_3}[\alpha(t)]$ or $T_{x_3}[\alpha(t)]$ involve the values of α throughout $]0, 1[$. Hence, except for

$$(t^1_{0,1}; t^2_{0,1}) \quad \text{and} \quad t^3_{0,1},$$

which are solely respectively related to

$$(a^{0,0}_1(x_3); a^{0,0}_2(x_3)) \quad \text{and} \quad a^{0,0}_3(x_3),$$

each function $t^i_{0,m}(\theta, x_3)$ is a fully three-dimensional correction of the cross-section approximation but depends on each other's cross-section $\mathrm{Cs}(t)$ only via a global quantity: $K^t[t^i_{0,m}]$.

The previous first-order solution authorizes us to handle the equations (4.15)–(4.17). For brevity we only report the solution (see Appendix B for details). For $(t^1_{1,0}; t^2_{1,0}; t^3_{1,0})$ one has to solve, for any cross-section $\mathrm{Cs}(x_3)$, the following well-posed pure shear-flow problems:

$$L^{\theta,x_3}[u_1] = x_1 = f(\theta, x_3)\cos\theta, \quad \boldsymbol{\tau}^1 = (\tau^1_1; \tau^1_2) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[x_2; 0], \tag{4.26}$$

$$L^{\theta,x_3}[u_2] = x_2 = f(\theta,x_3)\sin\theta, \quad \boldsymbol{\tau}^2 = (\tau_1^2;\tau_2^2) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[0;x_1], \tag{4.27}$$

and the 'extensional-type' problem

$$\boldsymbol{\tau}^3 = (\tau_1^3;\tau_2^3) = \{\boldsymbol{S}^{\theta,x_3}\}^{-1}[x_1;-x_2]. \tag{4.28}$$

Under the notation $\partial_x\{\alpha(t)\} := [\mathrm{d}\alpha/\mathrm{d}t](x)$, we obtain

$$t_{1,0}^j(\theta,x_3) = \sum_{l=1}^{2} F_0^l(x_3)t_j^l(\theta,x_3) + C_{2,0}^1(x_3)\tau_j^1(\theta,x_3) + C_{1,0}^2(x_3)\tau_j^2(\theta,x_3) + C_{1,0}^1(x_3)\tau_j^3(\theta,x_3), \tag{4.29}$$

$$K^{x_3}[t_{1,1}^j] = F_0^j(x_3) - \partial_{x_3}\{K^t[x_j^P t_{0,1}^3]\}, \quad j\in\{1,2\}, \tag{4.30}$$

$$t_{1,0}^3(\theta,x_3) = F_0^3(x_3)u_0(\theta,x_3) + C_{1,0}^3(x_3)u_1(\theta,x_3) + C_{2,0}^3(x_3)u_2(\theta,x_3), \tag{4.31}$$

$$K^{x_3}[t_{1,1}^3] = F_0^3(x_3) - \partial_{x_3}\tfrac{1}{2}\{K^t[x_1^P t_{0,1}^1 + x_2^P t_{0,1}^2]\}, \tag{4.32}$$

where the functions $C_{l,0}^i$ only depend upon the family $(a_i^{j,k}(x_3))_{j+k=1}$ and the first-order solution $(t_{0,1}^1;t_{0,1}^2;t_{0,1}^3)$ in the following way:

$$\left.\begin{aligned} 2C_{1,0}^1(x_3) &= a_1^{1,0}(x_3) - 2\partial_{x_3}\{K^t[t_{0,1}^3]\}, \\ 2C_{2,0}^2(x_3) &= a_2^{0,1}(x_3) - 2\partial_{x_3}\{K^t[t_{0,1}^3]\}, \end{aligned}\right\} \tag{4.33}$$

$$2C_{1,0}^2(x_3) = a_2^{1,0}(x_3), \quad 4C_{1,0}^3(x_3) = a_3^{1,0}(x_3) - 2\partial_{x_3}\{K^t[t_{0,1}^1]\}, \tag{4.34}$$

$$2C_{2,0}^1(x_3) = a_1^{0,1}(x_3), \quad 4C_{2,0}^3(x_3) = a_3^{0,1}(x_3) - 2\partial_{x_3}\{K^t[t_{0,1}^2]\}, \tag{4.35}$$

while the remaining functions F_m^i obey

$$(F_0^1(x_3);F_0^2(x_3)) = -\boldsymbol{A}_{x_3}(C_{2,0}^1(t)K^t[\tau_1^1] + C_{1,0}^2(t)K^t[\tau_1^2] + C_{1,0}^1(t)K^t[\tau_1^3]; C_{2,0}^1(t)K^t[\tau_2^1] + C_{1,0}^2(t)K^t[\tau_2^2] + C_{1,0}^1(t)K^t[\tau_2^3]), \tag{4.36}$$

$$c_0(x_3)F_0^3(x_3) = -\sum_{j=1}^{2} C_{j,0}^3(x_3)K^{x_3}[u_j]. \tag{4.37}$$

For $m \geqslant 1$ the determination of $t_{1,m}^i$ requires additional efforts since each term $b_{1,m}^j$ involves for $j\in\{1,2\}$ the new operator L_1^{θ,x_3} defined as

$$L_1^{\theta,x_3}[u] := \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}t}[(fs_0u)(\theta_P,t)\log H(\theta_P,t,\theta,x_3)]_{t=x_3}\,\mathrm{d}\theta_P. \tag{4.38}$$

More precisely, for $m\geqslant 1$ and $j\in\{1,2\}$ we obtain

$$b_{1,m}^j(\theta,x_3) = F_m^j(x_3) + a_m^j(\theta,x_3), \quad a_m^j = \sum_{l=1}^{2} C_{l,m}^j(x_3)x_l + L_1^{\theta,x_3}[(x_j^P - x_j)t_{0,m}^3], \tag{4.39}$$

$$C_{l,m}^j(x_3) = \delta_{j,l}\partial_{x_3}\{\tfrac{1}{2}K^t[t_{0,m}^3] + W_t\{K^v[t_{0,m}^3]\} - K^t[t_{0,m+1}^3]\}. \tag{4.40}$$

Thus, we write, for $m\geqslant 1$,

$$t_{1,m}^j = \sum_{l=1}^{2} F_m^l(x_3)t_j^l + r_j^{0,m}(\theta,x_3), \quad j\in\{1,2\}, \tag{4.41}$$

$$t_{1,m}^3 = F_m^3(x_3)u_0 + \sum_{l=1}^{2}\{C_{l,m}^3(x_3)u_l + v_l\partial_{x_3}b_{0,m}^l + b_{0,m}^l(x_3)w_l\}, \tag{4.42}$$

where the occurring functions v_l, w_l and $C_{l,m}^3$ obey, for $l \in \{1,2\}$,

$$2L^{\theta,x_3}[v_l] = -L^{\theta,x_3}\left[\sum_{j=1}^{2}(x_j^P - x_j)t_j^l\right], \quad 2L^{\theta,x_3}[w_l] = L_1^{\theta,x_3}\left[\sum_{j=1}^{2}(x_j^P - x_j)t_j^l\right], \tag{4.43}$$

$$2C_{l,m}^3(x_3) = \partial_{x_3}\{V_t[K^v[t_{0,m}^l]] - K^t[t_{0,m+1}^l]\} = -\partial_{x_3}b_{0,m}^l \tag{4.44}$$

and the function $\boldsymbol{r}^{0,m} = (r_1^{0,m}; r_2^{0,m})$ obeys (use (4.5) and the link $K^{x_3}[t_{0,m+1}^3] = T_{x_3}\{K^t[t_{0,m}^3]\}$)

$$S_j^{\theta,x_3}[r_1^{0,m}; r_2^{0,m}] = a_m^j = \partial_{x_3}\left\{\left[\tfrac{1}{2} - \frac{1}{c_0(t)}\right]K^t[t_{0,m}^3]x_j\right\} + L_1^{\theta,x_3}[(x_j^P - x_j)t_{0,m}^3]. \tag{4.45}$$

Finally (see Appendix B for details), the terms F_m^i and $K^{x_3}[t_{1,m}^i]$ are inductively obtained by invoking conditions (4.30), (4.31) and the relations, for $j \in \{1,2\}$ and $m \geqslant 1$,

$$\begin{aligned} c_0(x_3)F_m^3(x_3) = {} & K^{x_3}[t_{1,m}^3] \\ & - \sum_{l=1}^{2}\{C_{l,m}^3(x_3)K^{x_3}[u_l] + K^{x_3}[v_l]\partial_{x_3}b_{0,m}^l + b_{0,m}^l(x_3)K^{x_3}[w_l]\}, \end{aligned} \tag{4.46}$$

$$\begin{aligned} K^{x_3}[t_{1,m+1}^3] = {} & -\sum_{l=1}^{2}\frac{\{C_{l,m}^3(x_3)K^{x_3}[u_l] + K^{x_3}[v_l]\partial_{x_3}b_{0,m}^l + b_{0,m}^l(x_3)K^{x_3}[w_l]\}}{c_0(x_3)} \\ & + T_{x_3}\{K^t[t_{1,m}^3]\} - \partial_{x_3}\tfrac{1}{2}\{V_t(K^v[x_1^P t_{0,m}^1 + x_2^P t_{0,m}^2]) \\ & + K^t[x_1^P t_{0,m+1}^1 + x_2^P t_{0,m+1}^2]\}, \end{aligned} \tag{4.47}$$

$$(F_m^1(x_3); F_m^2(x_3)) = \boldsymbol{A}_{x_3}(K^t[t_{1,m}^1] - K^t[r_1^{0,m}]; K^t[t_{1,m}^2] - K^t[r_2^{0,m}]), \tag{4.48}$$

$$\begin{aligned} K^{x_3}[t_{1,m+1}^j] = {} & F_m^j(x_3) + V_{x_3}\{K^t[t_{1,m}^j]\} \\ & - \partial_{x_3}\{K^t[x_j^P t_{0,m+1}^3] - V_t(K^v[x_j^P t_{0,m}^3])\}. \end{aligned} \tag{4.49}$$

The equalities (4.45)–(4.49) outline the efforts spent in getting the second-order solutions $t_{1,m}^i$. It sometimes occurs that $(a_i^{0,0}(x_3)) = (0)$. In such circumstances, the first-order solutions $t_{0,m}^i$ vanish and many simplifications occur in (4.46)–(4.49). More precisely, we obtain, for $m \geqslant 1$ and $j \in \{1,2\}$,

$$t_{1,m}^j(\theta, x_3) = \sum_{l=1}^{2}F_m^l(x_3)t_j^l(\theta, x_3), \quad t_{1,m}^3(\theta, x_3) = \frac{K^{x_3}[t_{1,m}^3]}{c_0(x_3)}u_0(\theta, x_3), \tag{4.50}$$

$$K^{x_3}[t_{1,m}^3] = -\tfrac{1}{4}T_{x_3}^{m-1}\left\{\frac{K^t[u_1]}{c_0(t)}a_3^{1,0}(t) + \frac{K^t[u_2]}{c_0(t)}a_3^{0,1}(t)\right\}, \tag{4.51}$$

$$(K^{x_3}[t^1_{1,m+1}]; K^{x_3}[t^2_{1,m+1}]) = \boldsymbol{B}^m_{x_3}(K^t[t^1_{1,1}]; K^t[t^2_{1,1}]), \tag{4.52}$$

$$(F^1_m(x_3); F^2_m(x_3)) = \boldsymbol{A}_{x_3}(K^t[t^1_{1,m}]; K^t[t^2_{1,m}]), \tag{4.53}$$

with

$$K^{x_3}[t^j_{1,1}] = F^j_0(x_3)$$

given by (4.36), (4.37). The above solution (4.50)–(4.53) is similar to the general first-order solution (4.18)–(4.23) with $(b^1_{0,m}; b^2_{0,m})$ replaced by $(F^1_{0,m}; F^2_{0,m})$.

(*c*) *The next order estimate in specific cases*

Throughout this subsection $(a^{0,0}_i(x_3)) = (0)$. Since the leading solution (4.50)–(4.53) may vanish (think about a quadratic velocity $\boldsymbol{u}^\infty$) it is worth giving the higher corrections. These are achieved by detailing the term $O(D_3\epsilon^2 \log\epsilon)$ on the right-hand side of (3.23). By invoking our systematic formula (see equation (A 1) in Appendix A) the remainder $O(M[g]\epsilon^2\log\epsilon)$ in (3.15) reads

$$O(M[g]\epsilon^2\log\epsilon) = \mathcal{L}[g]\epsilon^2\log\epsilon + O(M[g]\epsilon^3), \tag{4.54}$$

where $\mathcal{L}$ denotes a linear operator. Thus, the system (3.21)–(3.23) suggests setting

$$d_i(\theta, x_3) = \sum_{n=1}^{2}\sum_{m=1-n}^{\infty}[f s_0 t^i_{n,m}](\theta, x_3)\epsilon^n[\log\epsilon]^{-m} + O(\epsilon^3\log^2\epsilon), \quad i \in \{1,2,3\}. \tag{4.55}$$

For $d^3_{1,-1} := 0$ one deduces, for $j \in \{1,2\}$ and $m \geqslant -1$,

$$K^{x_3}[t^1_{2,-1}] = K^{x_3}[t^3_{2,-1}] = K^{x_3}[t^3_{2,-1}] = 0, \tag{4.56}$$

$$\begin{aligned}
S^{\theta,x_3}_j&(t^1_{2,m}; t^2_{2,m}) \\
&= K^{x_3}[t^j_{2,m+1}] - V_{x_3}\{K^t[t^j_{2,m}]\} \\
&\quad + \tfrac{1}{2}\bigg(\delta_{m,0}\sum_{l+k=2} a^{l,k}_j(x_3)x^l_1 x^k_2 - J^{x_3}_1[(x^P_j - x_j)(x^P_3 - x_3)d^3_{1,m+1}] \\
&\qquad - J^{\theta,x_3}_2[(x^P_j - x_j)(x^P_3 - x_3)d^3_{1,m}]\bigg) = b^j_{2,m}(\theta, x_3),
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
L^{\theta,x_3}&[t^3_{2,m}] \\
&= K^{x_3}[t^3_{2,m+1}] - W_{x_3}\{K^t[t^3_{2,m}]\} \\
&\quad + \tfrac{1}{4}\bigg(\delta_{m,0}\sum_{l+k=2} a^{l,k}_3(x_3)x^l_1 x^k_2 \\
&\qquad - J^{x_3}_1[(x^P_3 - x_3)(d^1_{1,m+1}(x^P_1 - x_1) + d^2_{1,m+1}(x^P_2 - x_2))] \\
&\qquad - (1-\delta_{m,-1})J^{\theta,x_3}_2[(x^P_3 - x_3)(d^1_{1,m}(x^P_1 - x_1) + d^2_{0,m}(x^P_2 - x_2))]\bigg) \\
&= b^3_{2,m}(\theta, x_3).
\end{aligned} \tag{4.58}$$

The equations (4.56)–(4.58) are similar to (4.15)–(4.17) with functions $b^i_{2,0}$ involving quadratic terms $x_1^l x_2^{2-l}$, $l \in \{1,2\}$. Thus, the treatment looks like the approach detailed in Appendix B. If

$$(a_i^{j,k}(x_3))_{j+k=1} = (0)$$

the solutions $(t^i_{2,m})$ take a simple form:

$$(t^i_{1,m}) = (0), \qquad t^1_{2,-1} = t^2_{2,-1} = t^3_{2,-1} = 0$$

and, if the new functions $u_3, u_4, u_5, \boldsymbol{r}^1$, $\boldsymbol{r}^2, \boldsymbol{r}^3$ and $\boldsymbol{r}^4$ obey the integral equations

$$L^{\theta,x_3}[u_3] = x_1^2, \quad L^{\theta,x_3}[u_4] = x_1 x_2, \quad L^{\theta,x_3}[u_5] = x_2^2, \quad \boldsymbol{S}^{\theta,3}[r_1^1; r_2^1] = [x_1^2; 0], \tag{4.59}$$

$$\boldsymbol{S}^{\theta,3}[r_1^2; r_2^2] = [0; x_1^2], \quad \boldsymbol{S}^{\theta,3}[r_1^3; r_2^3] = [x_1^2 - 2x_1x_2; 0], \quad \boldsymbol{S}^{\theta,3}[r_1^4; r_2^4] = [-2x_1x_2; x_2^2], \tag{4.60}$$

then the solution is given, for $m = 0$, by

$$t^j_{2,0} = \sum_{l=1}^{2} G_0^l(x_3) t_j^l + \tfrac{1}{2}\{a_1^{0,2}(x_3) r_j^1 + a_2^{2,0}(x_3) r_j^2 + a_1^{2,0}(x_3) r_j^3 + a_2^{0,2}(x_3) r_j^4\}, \tag{4.61}$$

$$(G_0^1(x_3); G_0^2(x_3)) = (K^{x_3}[t^1_{2,1}]; K^{x_3}[t^2_{2,1}]) = -\boldsymbol{A}_{x_3}(\gamma_0^1(t); \gamma_0^2(t)), \tag{4.62}$$

$$2\gamma_0^j(t) = a_1^{0,2}(t) K^t[r_j^1] + a_2^{2,0}(t) K^t[r_j^2] + a_1^{2,0}(t) K^t[r_j^3] + a_2^{0,2}(t) K^t[r_j^4], \tag{4.63}$$

$$t^3_{2,0} = G_0^3(x_3) u_0 + \tfrac{1}{4}\{a_3^{2,0}(x_3) u_3 + a_3^{1,1}(x_3) u_4 + a_3^{0,2}(x_3) u_5\}, \tag{4.64}$$

$$4c_0(x_3) G_0^3(x_3) = -(a_3^{2,0}(x_3) K^{x_3}[u_3] + a_3^{1,1}(x_3) K^{x_3}[u_4] + a_3^{0,2}(x_3) K^{x_3}[u_5]), \tag{4.65}$$

and, for $m \geqslant 1$, by the following equations, similar to (4.50)–(4.53):

$$t^j_{2,m}(\theta, x_3) = \sum_{l=1}^{2} G_m^l(x_3) t_j^l(\theta, x_3), \quad t^3_{2,m}(\theta, x_3) = \frac{K^{x_3}[t^3_{2,m}]}{c_0(x_3)} u_0(\theta, x_3), \tag{4.66}$$

$$K^{x_3}[t^3_{2,m}] = -\tfrac{1}{4} T_{x_3}^{m-1}\left\{\frac{K^t[u_3]}{c_0(t)} a_3^{2,0}(t) + \frac{K^t[u_4]}{c_0(t)} a_3^{1,1}(t) + \frac{K^t[u_5]}{c_0(t)} a_3^{0,2}(t)\right\}, \tag{4.67}$$

$$(K^{x_3}[t^1_{2,m+1}]; K^{x_3}[t^2_{2,m+1}]) = \boldsymbol{B}^m_{x_3}(K^t[t^1_{2,1}]; K^t[t^2_{2,1}]), \tag{4.68}$$

$$(G_m^1(x_3); G_m^2(x_3)) = \boldsymbol{A}_{x_3}(K^t[t^1_{2,m}]; K^t[t^2_{2,m}]). \tag{4.69}$$

(d) *Asymptotic behaviour of global quantities*

Global quantities of interest are the following moments:

$$M_i^n(l_1, l_2, l_3) = \int_{\partial \mathcal{A}'} f_i(M) [x_1']^{l_1} [x_2']^{l_2} [x_3']^{l_3} \, \mathrm{d}S'_M, \quad l_1 + l_2 + l_3 = n \geqslant 0. \tag{4.70}$$

In contrast to other studies (see Geer 1976) we directly gain the asymptotic behaviour of $M_i^n(l_1, l_2, l_3)$ by using the solutions (4.8) or (4.55). For instance, the total force $\boldsymbol{F}$ acting on the body reads, for the usual case handled in § 4 *b*,

$$\boldsymbol{F} = 8\pi\mu L \sum_{i=1}^{3}\left\{\sum_{n=0}^{1} \sum_{m=1-n}^{\infty} \left(\int_0^1 K^t[t^i_{n,m}] \, \mathrm{d}t\right) \epsilon^n / [\log \epsilon]^m + O(\epsilon^2 \log \epsilon)\right\} \boldsymbol{e}_i, \tag{4.71}$$

and, for the exotic circumstances addressed in § 4 *c*,

$$\boldsymbol{F} = 8\pi\mu L \sum_{i=1}^{3} \left\{ \sum_{n=1}^{2} \sum_{m=1-n}^{\infty} \left(\int_0^1 K^t[t^i_{n,m}]\,\mathrm{d}t \right) \epsilon^n / [\log\epsilon]^m + O(\epsilon^3 \log^2 \epsilon) \right\} \boldsymbol{e}_i. \quad (4.72)$$

Observe that for $a_i^{0,0}$ non-zero the leading term arising in the asymptotic estimate of F_i (see (4.71)) becomes

$$-2^{2-\delta_{i,3}}\mu\pi L \int_0^1 a_i^{0,0}(t)\,\mathrm{d}t/\log\epsilon,$$

i.e. does not depend on the body shape once (L, e) is given.

5. Application to particles of elliptical cross-section

This whole section addresses analytical comparisons with previous works or exact solutions. For a slender body the available estimates (except Batchelor 1970) concern the circular cross-section case, whereas analytical results exist for the elliptical particle embedded in a uniform or linear external flow (see Jeffery 1922). Thus, we restrict ourselves to a body of elliptical cross-section: the boundary $C(x_3)$ of the non-dimensional cross-section $\mathrm{Cs}(x_3)$ is an ellipse, $\mathcal{E}(x_3)$, of equation

$$x_1^2 + \frac{x_2^2}{\eta^2} = h^2(x_3), \quad (5.1)$$

where h is a smooth function such that $h(0) = h(1) = 0$ and η is constant. By replacing $(\eta, \boldsymbol{e}_1, \boldsymbol{e}_2)$ by $(1/\eta, \boldsymbol{e}_2, \boldsymbol{e}_1)$ it is possible to assume that $0 < \eta \leqslant 1$. In this case, we obtain $f(\theta, x_3) = h(x_3) g_\eta(\theta)$ with

$$g_\eta(\theta) = \left\{ \frac{1 + \tan^2\theta}{1 + \tan^2\theta/\eta^2} \right\}^{1/2}, \quad \frac{[fs_0](\theta, x_3)}{h(x_3)(1 + \tan^2\theta)} = \frac{\{1 + \tan^2\theta/\eta^4\}^{1/2}}{\{1 + \tan^2\theta/\eta^2\}^{3/2}}. \quad (5.2)$$

The elliptical cross-section authorizes us to invert the encountered boundary integral equations and to recover the circular cross-section by setting $\eta = 1$.

(*a*) *First-order solution*

We apply (4.18)–(4.24) with solutions $u_0, \boldsymbol{t}^1$ and $\boldsymbol{t}^2$ given in Appendix D. If $a_1 = 1/(\eta + 1)$, $a_2 = \eta a_1$, $a_3 = -\frac{1}{2}$ and the new operator $O_{x_3}(a)$ obeys

$$O_{x_3}(a)[\alpha(t)] = \{a - \log[\tfrac{1}{4}(\eta + 1)h(x_3)]\}\alpha(x_3) + \mathrm{fp}\int_0^1 \frac{\alpha(t)\,\mathrm{d}t}{2|t - x_3|}, \quad (5.3)$$

then the solution $d^i_{0,m} = fs_0 t^i_{0,m}$ reads, for $i \in \{1, 2, 3\}$ and $m \geqslant 1$,

$$d^i_{0,m}(\theta, x_3) = \frac{g_\eta^2(\theta)}{2\pi\eta} K^{x_3}[t^i_{0,m}] = -\frac{g_\eta^2(\theta)}{4\pi\eta} \frac{1}{2^{\delta_{i,3}}} O_{x_3}^{m-1}(a_i)[a_i^{0,0}(t)], \quad (5.4)$$

if we set $O^0_{x_3}(a) = id$. As soon as $a_i^{0,0}(t) = U_i t^n$ with $n \in \{0, 1\}$, this result becomes

$$d^i_{0,1} = -\frac{g_\eta^2(\theta)}{4\pi\eta} \frac{U_i x_3^n}{2^{\delta_{i,3}}}, \quad d^i_{0,2} = -\frac{g_\eta^2(\theta)}{4\pi\eta} \frac{U_i}{2^{\delta_{i,3}}} \{x_3^n[W(x_3) + a_i] + \delta_{1,n}(\tfrac{1}{2} - x_3)\}, \quad (5.5)$$

$$d_{0,3}^{i} = -\frac{g_\eta^2(\theta)}{4\pi\eta}\frac{U_i}{2^{\delta_{i,3}}}\left\{[W(x_3)+a_i]^2 + \int_{-x_3}^{1-x_3}\frac{[W(x_3+u)-W(x_3)]\,\mathrm{d}u}{2|u|}\right\}, \quad n=0, \tag{5.6}$$

$$d_{0,3}^{i} = -\frac{g_\eta^2(\theta)}{4\pi\eta}\frac{U_i}{2^{\delta_{i,3}}}\left\{a_i - \tfrac{1}{2} + \tfrac{1}{2}\left[W(x_3) + \int_{-x_3}^{1-x_3} W(x_3+u)\,\mathrm{sgn}(u)\,\mathrm{d}u\right]\right.$$
$$\left. + x_3\left[W(x_3) + (a_i - 1 + W(x_3))^2 + \int_{-x_3}^{1-x_3}\frac{[W(x_3+u)-W(x_3)]\,\mathrm{d}u}{2|u|}\right]\right\},$$
$$n=1, \tag{5.7}$$

where the function W depends on the shape function h and obeys

$$2W(x_3) = \log\left[\frac{16x_3(1-x_3)}{(\eta+1)^2h^2(x_3)}\right]. \tag{5.8}$$

If $a_i^{0,0}(t) = U_i$, the total force $\boldsymbol{F}$ reads (see (4.71)) as

$$F_i = -\frac{4\pi\mu L U_i}{2^{\delta_{i,3}}}\left\{(\log\epsilon)^{-1} + \left[a_i + \int_0^1 W(t)\,\mathrm{d}t\right](\log\epsilon)^{-2}\right.$$
$$\left. + \left[\int_0^1\left\{(W(t)+a_i)^2 + \int_{-t}^{1-t}\frac{[W(t+u)-W(t)]\,\mathrm{d}u}{2|u|}\right\}\mathrm{d}t\right](\log\epsilon)^{-3} + O(\log\epsilon)^{-4}\right\}. \tag{5.9}$$

For a circular cross-section ($\eta = 1$, $a_1 = a_2 = -a_3 = \frac{1}{2}$) the above behaviour agrees with Geer (1976). Moreover, for the ellipsoidal slender body $h^2(x_3) = 4x_3(1-x_3)$ (see Appendix C) and for $a_i^{0,0}(t) = U_i$ or $a_2^{0,0}(t) = w_2 t$ the results (5.4) and (5.5), (5.7), respectively, become

$$d_{0,m}^{i} = -\frac{g_\eta^2(\theta)}{4\pi\eta}\frac{U_i}{2^{\delta_{i,3}}}\left\{\log\left(\frac{2}{\eta+1}\right) + a_i\right\}^{m-1}, \quad \text{for } m \geqslant 1, \tag{5.10}$$

$$d_{0,1}^{2} = -\frac{g_\eta^2(\theta)w_2x_3}{4\pi\eta}, \quad d_{0,2}^{2} = -\frac{g_\eta^2(\theta)w_2}{4\pi\eta}\left\{\left[\log\left(\frac{2}{\eta+1}\right) - \frac{1}{\eta+1}\right]x_3 + \tfrac{1}{2}\right\}, \tag{5.11}$$

$$d_{0,3}^{2} = -\frac{g_\eta^2(\theta)w_2}{4\pi\eta}\left\{\left[\log\left(\frac{2}{\eta+1}\right) - \frac{1}{\eta+1}\right]^2 x_3 + \log\left(\frac{2}{\eta+1}\right) + \frac{\eta-1}{2(\eta+1)}\right\}. \tag{5.12}$$

These results agree perfectly with estimates of the exact solution (see Appendix C) when choosing $p_0(\epsilon) = o(\epsilon)$ as ϵ goes to zero, $w = w_1 = w_2 = w_3 = 0$, for (5.10) and, for (5.11), (5.12), $U_1 = U_2 = U_3 = w = w_1 = w_3 = 0$.

(b) *Second-order solution*

The use of (4.27)–(4.29) yields the solutions $d_{1,m}^{i} = fs_0t_{1,m}^{i}$ and Appendix D provides the solutions $\boldsymbol{\tau}^1, \boldsymbol{\tau}^2$ and $\boldsymbol{\tau}^3$. One obtains $K^{x_3}[t_{1,1}^{3}] = 0$,

$$K^{x_3}[t_{1,1}^{1}] = K^{x_3}[t_{1,1}^{2}] = \tfrac{1}{4}\partial_{x_3}a_3^{0,0}$$

and

$$d_{1,0}^{1} = \frac{g_\eta^2(\theta)}{8\pi\eta}\{[((1/\eta)+2)a_1^{0,1}(x_3) + a_2^{1,0}(x_3)]x_2 + (1+\eta)^2[a_1^{1,0}(x_3) + \tfrac{1}{2}\partial_{x_3}a_3^{0,0}]x_1\}, \tag{5.13}$$

$$d^2_{1,0} = \frac{g^2_\eta(\theta)}{8\pi}\left\{[a^{0,1}_1(x_3) + (1 + (2/\eta))a^{1,0}_2(x_3)]x_1 - [a^{1,0}_1(x_3) + \tfrac{1}{2}\partial_{x_3}a^{0,0}_3]\frac{(1+\eta)^2 x_2}{\eta^3}\right\}, \tag{5.14}$$

$$d^3_{1,0} = \frac{(\eta+1)g^2_\eta(\theta)}{8\pi\eta}\{[a^{1,0}_3(x_3) + \partial_{x_3}a^{0,0}_1]x_1 + [a^{0,1}_3(x_3) + \partial_{x_3}a^{0,0}_2]x_2/\eta\}. \tag{5.15}$$

The results (5.13)–(5.15) agree with the behaviour of the exact solution for given (w_1, w_3) and $U_1 = U_2 = U_3 = w_2 = 0$ with $p_0(\epsilon) = o(\epsilon)$. After some algebra we get, for the elliptical cross-section, $K^{x_3}[t^i_{1,m}] = F^i_m(x_3) = 0$ and

$$S_j[r^{0,m}_1; r^{0,m}_2] = \partial_{x_3}\left\{\left[\tfrac{1}{2} - \frac{\eta^{j-1}}{\eta+1}\right]K^t[t^3_{0,m}]x_j\right\}, \quad j \in \{1,2\}, \quad m \geqslant 1. \tag{5.16}$$

Thus, $2(\eta+1)r^{0,m}_j = (\eta-1)\partial_{x_3}\{K^t[t^3_{0,m}]\}\tau^3_j$ and, for $m \geqslant 1$,

$$d^j_{1,m} = \frac{(-1)^j(1-\eta^2)g^2_\eta(\theta)x_j}{8\pi\eta^{2j-1}}\partial_{x_3}\{K^t[t^3_{0,m}]\}, \quad j \in \{1,2\}, \tag{5.17}$$

$$d^3_{1,m} = -\frac{g^2_\eta(\theta)}{2\pi\eta}\{x_1\partial_{x_3}\{K^t[t^1_{0,m}]\} + x_2\partial_{x_3}\{K^t[t^2_{0,m}]\}\}. \tag{5.18}$$

Recall that $K^t[t^i_{0,m}]$ depends on $a^{0,0}_i$ through the relation (5.4). If $a^{0,0}_i(t) = U_i$ and $w_i = w = 0$, we get $d^i_{1,m} = 0$ for $m \geqslant 0$. This matches the asymptotic behaviour of the exact solution $\boldsymbol{d}^e$ provided we choose this time $p_0(\epsilon) = o(\epsilon)$. Since $K^t[t^i_{1,m}] = 0$,

$$F_i(\epsilon) = \sum_{m=1}^{\infty} F^m_i[\log\epsilon]^{-m} + O(\epsilon^2\log\epsilon).$$

For a circular cross-section, $d^j_{1,m}$ vanishes. The author has also checked that the asymptotic behaviour of the exact solution (handled in Appendix B for $w \neq 0$ and $w_i = U_i = 0$) perfectly agrees with results (5.17) and (5.18) provided we take for $p_0(\epsilon)$ the following behaviour:

$$p_0(\epsilon) = \frac{\epsilon w\mu}{2\eta a_1}\left[1 - \eta^2 + 2\eta - \frac{1+\eta^2}{2\log\epsilon} + o\left(\frac{1}{\log\epsilon}\right)\right]. \tag{5.19}$$

6. Concluding remarks

The surface stress $\boldsymbol{f}$ may be split into its normal and tangential parts, respectively denoted by $\boldsymbol{f}^{\mathrm{n}}$ and $\boldsymbol{f}^{\mathrm{t}}$, with

$$\boldsymbol{f}^{\mathrm{n}}(P) = [\boldsymbol{f}(P)\cdot\boldsymbol{n}](P)\boldsymbol{n}(P), \quad \boldsymbol{f}(P) = \boldsymbol{f}^{\mathrm{n}}(P) + \boldsymbol{f}^{\mathrm{t}}(P). \tag{6.1}$$

Under our notation the vector $\boldsymbol{n}(M)$ obeys

$$[fs_\epsilon\boldsymbol{n}](M) = [\cos\theta + \sin\theta f^{-1}f^1_{x_3}]e_1 + [\sin\theta - \cos\theta f^{-1}f^1_{x_3}]e_2 - \epsilon f^1_{x_3}e_3. \tag{6.2}$$

Thus, the normal component $f^{\mathrm{n}}(P) = [\boldsymbol{f}^{\mathrm{n}}.\boldsymbol{n}](P)$ of the surface stress becomes

$$f^{\mathrm{n}}(P) = \frac{8\pi\mu\{[\cos\theta + \sin\theta f^{-1}f^1_{x_3}]d_1 + [\sin\theta - \cos\theta f^{-1}f^1_{x_3}]d_2 - \epsilon f^1_{x_3}d_3\}}{\epsilon[fs_\epsilon]^2}. \tag{6.3}$$

Accordingly, the term $fs_\epsilon f^{\mathrm{n}}$ does not admit a uniform asymptotic estimate for $x_3 \in]0,1[$. The same remark also holds for the tangential counterpart $fs_\epsilon \boldsymbol{f}^{\mathrm{t}}$.

By re-introducing the solution (4.8) into (2.5), (2.6) one may deduce asymptotic estimates of the flow $(\boldsymbol{u}, p)$ outside the body.

The present theory applies to the case of a non-circular cross-section and, if necessary, up to high orders with respect to ϵ. It thereafter requires calculations that may appear somewhat cumbersome. However, recall that the method of 'matched asymptotic expansions' is likely to induce much more tedious calculations when enforcing, for high orders, the matching rules.

Appendix A. Getting asymptotic estimates (3.14) and (3.15)

This appendix presents our systematic formula and details the operators A_i^{θ,x_3} and B_i^{θ,x_3}. According to Sellier (1996, theorem 16) if $x_3 \in]0,1[$ and $K(u,v)$ is a 'Q pseudo-homogeneous' kernel such that $K(\alpha u, \alpha v) = \mathrm{sgn}(\alpha)|\alpha|^Q K(u,v)$ with Q a strictly negative integer, then for g and h smooth enough, respectively, near x_3 and zero and a positive integer N the following asymptotic estimate holds, as $\epsilon \to 0^+$:

$$
\begin{aligned}
\mathrm{fp}\int_{-x_3}^{1-x_3} & g(x_3+u)K[u,\epsilon h(u)]\,\mathrm{d}u \\
&= \sum_{n=0}^{N} \frac{\partial_2^n K(1,0)}{n!}\left[\mathrm{fp}\int_{-x_3}^{1-x_3} \frac{\mathrm{sgn}(u) g(x_3+u)\,\mathrm{d}u}{u^{n-Q}[h(u)]^{-n}}\right]\epsilon^n \\
&\quad + \sum_{m=0}^{N-Q-1}\sum_{l=0}^{m}\sum_{i=0}^{m-l} \frac{g^{(l)}(x_3)a^i_{m-l-i}}{l!i!}\left[\mathrm{fp}\int_{-\infty}^{\infty} \partial_2^i K[t,h(0)]t^m\,\mathrm{d}t\right]\epsilon^{Q+m+1} \\
&\quad - 2\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{i=0}^{n-l}\sum_{j=0}^{l-Q-1} \frac{g^{(j)}(x_3)[h(0)]^l}{l!i!j!} a^i_{l-Q-j-1}\partial_2^n K(1,0)\epsilon^n \log\epsilon + o(\epsilon^N),
\end{aligned} \tag{A 1}
$$

where $\mathrm{fp}\int$ denotes an integration in the finite-part sense of Hadamard,

$$
\partial_2^n K(u,v) = \frac{\partial^n K}{\partial v^n}, \qquad g^{(n)}(t) = \frac{\mathrm{d}^n g}{\mathrm{d}t^n}
$$

and the coefficients a_p^i obey $a_p^0 = \delta_{p0}$ and for $i \geqslant 1$

$$
\{[h(u)-h(0)]/u\}^i = \sum_p a_p^i u^p, \quad \text{as } u \to 0. \tag{A 2}
$$

By applying (A 1) successively to

$$
K(u,v) = [u^2+v^2]^{-1/2} \qquad \text{with } Q=-1,
$$

or

$$
K(u,v) = [u^2+v^2]^{-3/2} \qquad \text{with } Q=-3,
$$

we build the required behaviours (3.14), (3.15). In computing the right-hand side of (A 1), note that the change of scale $t = h(0)x$ may induce extra terms when handling

$$\mathrm{fp}\int_{-\infty}^{\infty} \partial_2^i K[t, h(0)]t^m \, \mathrm{d}t$$

(see Sellier 1996). For instance, if $K(u,v) = [u^2+v^2]^{-3/2}$, $(i,m) = (0,2)$ and $h(0) > 0$, we obtain

$$\mathrm{fp}\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{[t^2+h^2(0)]^{3/2}} = \mathrm{fp}\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{[x^2+1]^{3/2}} - 2\log[h(0)]. \tag{A 3}$$

Finally, (A 1) yields (3.14), (3.15) with the following definitions:

$$A_0^{x_3}[g] = -2g(x_3), \quad B_0^{\theta,x_3}[g] = 2g(x_3)/h^2(0), \tag{A 4}$$

$$A_2^{\theta,x_3}[g] = \frac{\mathrm{d}^2}{\mathrm{d}u^2}[g(x_3+u)\tfrac{1}{2}h^2(u)]_{u=0}, \quad B_1^{\theta,x_3}[g] = -\frac{\mathrm{d}^2}{\mathrm{d}u^2}[g(x_3+u)]_{u=0}, \tag{A 5}$$

$$A_1^{\theta,x_3}[g] = 2\{\log 2 - \log h(0)\}g(x_3) + \mathrm{fp}\int_0^1 \frac{g(t)\,\mathrm{d}t}{|t-x_3|}, \tag{A 6}$$

$$B_2^{x_3}[g] = \mathrm{fp}\int_0^1 \frac{g(t)\,\mathrm{d}t}{|t-x_3|^3} - \frac{\mathrm{d}^2}{\mathrm{d}u^2}[(1+\log(\tfrac{1}{2}h(u)))g(x_3+u)]_{u=0}, \tag{A 7}$$

$$A_3^{\theta,x_3}[g] = -\,\mathrm{fp}\int_0^1 \frac{g(t)h^2(t-x_3)\,\mathrm{d}t}{2|t-x_3|^3} + \frac{\mathrm{d}^2}{\mathrm{d}u^2}[\tfrac{1}{2}g(x_3+u)h^2(u)(\tfrac{1}{2}+\log(\tfrac{1}{2}h(u)))]_{u=0}. \tag{A 8}$$

Appendix B.

In calculating the terms $b_{1,m}^i(\theta,x_3)$ and $b_{2,m}^i(\theta,x_3)$, respectively, for $m \geqslant 0$ and $m \geqslant -1$ we remark that, if $g = fs_0u$ and $j \in \{1,2\}$, then

$$J_1^{x_3}[(x_j^P - x_j)(x_3^P - x_3)g] = -2\partial_{x_3}\{K^t[x_j^P u]\} + 2x_j\partial_{x_3}\{K^t[u]\}, \tag{B 1}$$

$$\begin{aligned} J_2^{\theta,x_3}[(x_j^P - x_j)(x_3^P - x_3)g] = {} & 2\partial_{x_3}[V_t\{K^v[x_j^P u]\}] - 2x_j\partial_{x_3}[V_t\{K^v[u]\}] \\ & - 2\int_0^{2\pi} \partial_{x_3}[(x_j^P - x_j)g(\theta_P,t)\log H(\theta_P,t,\theta,x_3)]\,\mathrm{d}\theta_P. \end{aligned} \tag{B 2}$$

Here $\partial_x\{\alpha(t)\} := [\mathrm{d}\alpha/\mathrm{d}t]_{t=x}$ and it has been noticed that

$$\mathrm{fp}\int_0^1 \frac{\alpha(t)\,\mathrm{sgn}(t-x_3)\,\mathrm{d}t}{(t-x_3)^2} = \partial_{x_3}\left\{2\alpha(t) + \mathrm{fp}\int_0^1 \frac{\alpha(v)\,\mathrm{d}v}{|v-t|}\right\}. \tag{B 3}$$

Thus, if one sets $d_{0,0}^i = d_{1,-1}^i := 0$ it follows, for $i \in \{1,2,3\}$, that

$$b_{1,m}^i = F_m^i(x_3) + \sum_{l=1}^{2} C_{l,m}^i(x_3)x_l + R^i[d_{0,m}^1; d_{0,m}^2; d_{0,m}^3], \tag{B 4}$$

$$b_{2,m}^i = G_m^i(x_3) + \sum_{l=1}^{2} D_{l,m}^i(x_3)x_l + \delta_{m,0}\sum_{j+k=2} \frac{a_i^{j,k}(x_3)}{2^{1+\delta_{i,3}}}x_1^j x_2^k + R^i[d_{1,m}^1; d_{1,m}^2; d_{1,m}^3], \tag{B 5}$$

with functions $C^i_{l,m}$ or $D^i_{l,m}$ given by (4.33)–(4.35), (4.42) and (4.43) or (B 15) and (B 16), functions F^i_m and G^i_m to be determined and the remaining terms

$$R^i[d^1_{j,m}; d^2_{j,m}; d^3_{j,m}]$$

being available from the previous order solution and such that

$$R^j[u_1;u_2;u_3] = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}t}[(x^P_j - x_j)u_3(\theta_P,t) \times \log H(\theta_P,t,\theta,x_3)]_{t=x_3}\,\mathrm{d}\theta_P, \quad j\in\{1,2\}, \tag{B 6}$$

$$2R^3[u_1;u_2;u_3] = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\sum_{j=1}^{2}(x^P_j - x_j)u_j(\theta_P,t)\right]\log H(\theta_P,t,\theta,x_3)\right\}_{t=x_3}\mathrm{d}\theta_P. \tag{B 7}$$

The compatibility relation (4.6) requires

$$T_{j,m}(x_3) := \oint_{C(x_3)} [b^1_{j,m}n^0_1 + b^2_{j,m}n^0_2](M)\,\mathrm{d}l_M = 0 \quad \text{for } j\in\{1,2\}.$$

Case $j=1$, $m=0$. If $E_i(x_3) = F^i_0(x_3) - K^{x_3}[t^i_{1,1}]$, one finds that

$$2E_3(x_3) = \partial_{x_3}\{K^t[x^P_1 t^1_{0,1} + x^P_2 t^2_{0,1}]\}, \quad E_i(x_3) = \partial_{x_3}\{K^t[x^P_i t^3_{0,1}]\}, \quad i\in\{1,2\}. \tag{B 8}$$

If $S(x_3)$ denotes the area of the cross-section $\mathrm{Cs}(x_3)$, (B 4) yields

$$T_{1,0}(x_3) = \sum_{i=1}^{2}\sum_{j=1}^{2}\oint_{C(x_3)} C^i_{j,0}(x_3)x^P_j n^0_i(P)\,\mathrm{d}l_P = S(x_3)[C^1_{1,0}(x_3) + C^2_{2,0}(x_3)] = 0. \tag{B 9}$$

The last equality is ensured by (4.33)–(4.35), the value of $K^{x_3}[t^3_{0,1}]$ given by (4.12) and the basic link (3.7) for $(p,q) = (0,0)$. Inspection of (B 4) suggests the introduction of (see (4.26)–(4.28)) the functions $u_1, u_2, \boldsymbol{\tau}^1, \boldsymbol{\tau}^2$ or $\boldsymbol{\tau}^3$ and thereby to cast $(t^1_{1,0}; t^2_{1,0})$ and $t^3_{1,0}$, respectively, into the forms (4.29) and (4.31). By combining (4.29), (4.31) with the condition $K^{x_3}[t^i_{1,0}] = 0$ we deduce F^i_0 and also, through the previous definition of E_i, the value of $K^{x_3}[t^i_{1,1}]$ (see (4.30), (4.32) and results (4.36), (4.37)).

Case $j=1$, $m\geqslant 1$. The reader may check that $C^1_{2,m}(x_3) = C^2_{1,m}(x_3) = 0$ and

$$C^1_{1,m}(x_3) = C^2_{2,m}(x_3) = -\partial_{x_3}\{K^t[t^3_{0,m+1}] - W_t\{K^v[t^3_{0,m}]\} - K^t[t^3_{0,m}/2]\}. \tag{B 10}$$

By gathering (B 4), (B 5) and (B 9) we obtain

$$T_{1,m}(x_3) = 2S(x_3)C^1_{1,m}(x_3) + \int_0^{2\pi} \partial_{x_3}\{d^3_{0,m}(\theta_P,t)\phi(\theta_P,t,\theta,x_3)\}\,\mathrm{d}\theta_P, \tag{B 11}$$

with, for $P(\theta_P,t), M(\theta,x_3) = (x_1,x_2)\in\mathrm{Cs}(x_3)$ and $H = H(\theta_P,t,\theta,x_3)$,

$$\phi = \oint_{C(x_3)} \sum_{j=1}^{2} n^0_j(M)(x^P_j - x_j)\log[H]\,\mathrm{d}l_M = -S(x_3) - 2\int_{\mathrm{Cs}(x_3)} \log[H]\,\mathrm{d}S_M. \tag{B 12}$$

For (B 12) recall that $x_1 = f(\theta, x_3)\cos\theta$, $x_2 = f(\theta, x_3)\sin\theta$. Accordingly,

$$T_{1,m}(x_3) = 2S(x_3)C^1_{1,m}(x_3) - S(x_3)\partial_{x_3}\{K^t[t^3_{0,m}]\} - 2\int_{\mathrm{Cs}(x_3)} \frac{\mathrm{d}}{\mathrm{d}t}[\psi^M_t(d^3_{0,m})]\,\mathrm{d}S_M, \tag{B 13}$$

with, for t near x_3, if $M(\theta, x_3)$ is inside $\mathrm{Cs}(x_3)$ then M is inside $\mathrm{Cs}(t)$ and

$$\psi^M_t(d^3_{0,m}) := \int_0^{2\pi} d^3_{0,m}(\theta_P, t)\log H(\theta_P, t, \theta, x_3)\,\mathrm{d}\theta_P = -b^3_{0,m}(t). \tag{B 14}$$

The last equality holds because the function ψ^M_t is harmonic in $\mathbb{R}^2\setminus C(t)$, of constant value $-b^3_{0,m}(t)$ over this path $C(t)$ (see equation (4.14)) and thereafter equal to $-b^3_{0,m}(t)$ everywhere in the closed set $\mathrm{Cs}(t)$. Inspection of (4.14), (B 9) and (B 13), (B 14) finally ensures that $T_{1,m}(x_3) = 0$. By introducing the functions v_l, w_l and $\boldsymbol{r}^{0,m} = (r^{0,m}_1; r^{0,m}_2)$ (see (4.45) and (4.43)) one deduces from (B 4) the forms (4.41), (4.42) for $t^i_{1,m}$ and after some algebra the results (4.42), (4.43) and (4.46)–(4.49).

If $t^i_{1,-1} := 0$ then, for $m \geqslant -1$,

$$D^l_{j,m}(x_3) = -\delta_{j,l}\partial_{x_3}\{K^t[t^3_{1,m+1}] - W_t\{K^v[t^3_{1,m}]\} - \tfrac{1}{2}K^t[t^3_{1,m}]\}, \tag{B 15}$$

$$2D^3_{j,m}(x_3) = -\partial_{x_3}\{K^t[t^j_{1,m+1}] - V_t\{K^v[t^j_{1,m}]\}\}. \tag{B 16}$$

The results (B 5), (B 15), (B 16) and also $K^t[t^3_{1,0}] = 0$ immediately yield

$$T_{2,-1}(x_3) = S(x_3)[D^1_{1,-1}(x_3) + D^2_{2,-1}(x_3)] = 0$$

and, for $m \geqslant 0$ (see the definition (B 14), of $\psi^M_t(u)$)

$$\begin{aligned} T_{2,m}(x_3) = {} & 2S(x_3)D^1_{1,m}(x_3) - S(x_3)\partial_{x_3}\{K^t[t^3_{1,m}]\} - 2\int_{\mathrm{Cs}(x_3)} \partial_{x_3}\{\psi^M_t(d^3_{1,m})\}\,\mathrm{d}S_M \\ & + \tfrac{1}{2}\delta_{m,0}\oint_{C(x_3)} [(a^{2,0}_1(x_3)x_1^2 + a^{1,1}_1(x_3)x_1x_2 + a^{0,2}_1(x_3)x_2^2)n^0_1 \\ & + (a^{2,0}_2(x_3)x_1^2 + a^{1,1}_2(x_3)x_1x_2 + a^{0,2}_2(x_3)x_2^2)n^0_2]\,\mathrm{d}l_M, \end{aligned} \tag{B 17}$$

with $t^i_{0,m} = 0$ since $(a^{0,0}_i(x_3)) = (0)$ and consequently (see (4.17)), for $m \geqslant 0$,

$$L^{\theta,x_3}[t^3_{1,m}] = K^{x_3}[t^3_{1,m+1}] - W_{x_3}\{K^t[t^3_{1,m}]\} + \tfrac{1}{4}\delta_{m,0}(a^{1,0}_3(x_3)x_1 + a^{0,1}_3(x_3)x_2) = e_m. \tag{B 18}$$

Since $e_m(\theta, x_3)$ is a sum of a function of x_3 and a linear function of (x_1, x_2) it is clear that

$$\psi^M_t(d^3_{1,m}) = -e_m(\theta, x_3)$$

for M inside $\mathrm{Cs}(x_3)$. This remark and property (3.7) applied to $(p,q) = (1,0)$ and $(p,q) = (0,1)$ show that $T_{2,m}(x_3) = 0$. The solution $t^i_{2,m}$ is deduced after introducing the functions u_3, u_4, u_5 by (4.59) and

$$v^1_i, \qquad w^1_i, \qquad \boldsymbol{r}^{1,m} = (r^{1,m}_1; r^{1,m}_2)$$

(for $m \geqslant 0$) by the well-posed integral equations

$$2L^{\theta,x_3}[v_i^1] = -L^{\theta,x_3}\left[\sum_{j=1}^{2}(x_j^P - x_j)\tau_j^i\right], \quad 2L^{\theta,x_3}[w_i^1] = L_1^{\theta,x_3}\left[\sum_{j=1}^{2}(x_j^P - x_j)\tau_j^i\right], \tag{B 19}$$

$$S_j^{\theta,x_3}[r_1^{1,m}; r_2^{1,m}] = \partial_{x_3}\left\{\left[\frac{1}{2} - \frac{1}{c_0(t)}\right]K^t[t_{1,m}^3]x_j\right\} + L_1^{\theta,x_3}[(x_j^P - x_j)t_{1,m}^3], \quad m \geqslant 1, \tag{B 20}$$

$$2S_j^{\theta,x_3}[r_1^{1,0}; r_2^{1,0}] = \sum_{k+l=2} a_j^{k,l}(x_3)x_k x_l - 2\partial_{x_3}\{K^t[t_{1,1}^3]\} + L_1\left[\sum_{j=1}^{2}(x_j^P - x_j)t_{1,0}^j\right]. \tag{B 21}$$

Thus $t_{2,-1}^1 = t_{2,-1}^2 = 0$ and since $t_{1,-1}^1 = t_{1,-1}^2 = 0$ then, for $j \in \{1,2\}$ and $m \geqslant 0$, the quantities $t_{2,m}^j$ obey

$$t_{2,m}^j(\theta, x_3) = \sum_{l=1}^{2} G_m^l(x_3)t_j^l(\theta, x_3) + (1 - \delta_{m,0})r_j^{1,m}(\theta, x_3) + \delta_{m,0}r_j^{1,0}(\theta, x_3), \tag{B 22}$$

$$\begin{aligned} K^{x_3}[t_{2,m+1}^j] = G_m^j(x_3) + V_{x_3}\{K^t[t_{2,m}^j]\} \\ - \partial_{x_3}\{K^t[x_j^P t_{1,m+1}^3] - (1 - \delta_{m,-1})V_t[K^v[x_j^P t_{1,m}^3]]\}, \end{aligned} \tag{B 23}$$

$$(G_m^1(x_3); G_m^2(x_3)) = \boldsymbol{A}_{x_3}(K^t[t_{2,m}^1] - \gamma_m^1(t); K^t[t_{2,m}^2] - \gamma_m^2(t)), \tag{B 24}$$

$$\gamma_m^j(t) = (1 - \delta_{m,0})K^t[r_j^{1,m}] + \delta_{m,0}K^t[r_j^{1,0}], \tag{B 25}$$

and for $m \geqslant -1$ then $K^{x_3}[t_{2,-1}^3] = 0$ and

$$\begin{aligned} t_{2,m}^3 = G_m^3(x_3)u_0 + \sum_{l=1}^{2} D_{l,m}^3(x_3)u_l + \tfrac{1}{4}\delta_{0,m}[a_3^{2,0}(x_3)u_3 + a_3^{1,1}(x_3)u_4 + a_3^{0,2}(x_3)u_5] \\ + (1 - \delta_{m,-1})\sum_{l=1}^{2}\{v_l\partial_{x_3}F_m^l + F_m^l w_l\} + \delta_{m,0}[v_1^1\partial_{x_3}C_{2,0}^1 + C_{2,0}^1(x_3)w_1^1 \\ + v_2^1\partial_{x_3}C_{1,0}^2 + C_{1,0}^2(x_3)w_2^1 + v_3^1\partial_{x_3}C_{1,0}^1 + C_{1,0}^1(x_3)w_3^1], \end{aligned} \tag{B 26}$$

$$\begin{aligned} & c_0(x_3)G_m^3(x_3) \\ & \quad = K^{x_3}[t_{2,m}^3] - \sum_{l=1}^{2} D_{l,m}^3(x_3)K^{x_3}[u_l] \\ & \qquad - \tfrac{1}{4}\delta_{0,m}\{a_3^{2,0}(x_3)K^{x_3}[u_3] + a_3^{1,1}(x_3)K^{x_3}[u_4] + a_3^{0,2}(x_3)K^{x_3}[u_5]\} \\ & \qquad - (1 - \delta_{m,-1})\sum_{l=1}^{2}\left\{\partial_{x_3}F_m^l K^{x_3}[v_l] + F_m^l K^{x_3}[w_l]\right\} \\ & \qquad + \delta_{m,0}(K^{x_3}[v_1^1]\partial_{x_3}C_{2,0}^1 + C_{2,0}^1 K^{x_3}[w_1^1] \\ & \qquad + K^{x_3}[v_2^1]\partial_{x_3}C_{1,0}^2 + C_{1,0}^2 K^{x_3}[w_2^1] + K^{x_3}[v_3^1]\partial_{x_3}C_{1,0}^1 + C_{1,0}^1 K^{x_3}[w_3^1]), \end{aligned} \tag{B 27}$$

$$
\begin{aligned}
K^{x_3}[t^3_{2,m+1}] \\
&= T_{x_3}\{K^t[t^3_{2,m}]\} - \sum_{l=1}^{2} D^3_{l,m}(x_3)K^{x_3}[u_l]/c_0(x_3) \\
&\quad - \partial_{x_3}\{K^t[x_1^P t^1_{1,m+1} + x_2^P t^2_{1,m+1}] - V_t[K^v[x_1^P t^1_{1,m} + x_2^P t^2_{1,m}]]\}/[2c_0(x_3)] \\
&\quad - \delta_{m,0}(a_3^{2,0}(x_3)K^{x_3}[u_3] + a_3^{1,1}(x_3)K^{x_3}[u_4] + a_3^{0,2}(x_3)K^{x_3}[u_5])/[4c_0(x_3)] \\
&\quad + (\delta_{m,-1} - 1)\sum_{l=1}^{2}\left\{\partial_{x_3}F^l_m \frac{K^{x_3}[v_l]}{c_0(x_3)} + F^l_m \frac{K^{x_3}[w_l]}{c_0(x_3)}\right\} \\
&\quad - \delta_{m,0}\left\{\frac{K^{x_3}[v_1^1]}{c_0(x_3)}\partial_{x_3}C^1_{2,0} + C^1_{2,0}(x_3)\frac{K^{x_3}[w_1^1]}{c_0(x_3)} + \frac{K^{x_3}[v_2^1]}{c_0(x_3)}\partial_{x_3}C^2_{1,0} \right. \\
&\quad \left. + C^2_{1,0}(x_3)\frac{K^{x_3}[w_2^1]}{c_0(x_3)} + \frac{K^{x_3}[v_3^1]}{c_0(x_3)}\partial_{x_3}C^1_{1,0} + C^1_{1,0}(x_3)\frac{K^{x_3}[w_3^1]}{c_0(x_3)}\right\}. \qquad \text{(B 28)}
\end{aligned}
$$

Appendix C. The exact solution for an ellipsoidal particle

This appendix gives the solution $\boldsymbol{d}^e$ (see (3.11)) for an ellipsoidal particle embedded in a simple external Stokes flow $\boldsymbol{u}_\infty$. More precisely, the ellipsoid and the external flow are, respectively, defined by

$$
\frac{X_1^2}{a_1^2} + \frac{X_2^2}{a_2^2} + \frac{X_3^2}{a_3^2} = 1, \quad e = a_1, \quad a_2 = \eta a_1, \quad L = 2a_3, \qquad \text{(C 1)}
$$

$$
\begin{aligned}
\boldsymbol{u}^\infty(M) &= \left[U_1 + \frac{w_1 X_2 - w X_1}{2a_3}\right]\boldsymbol{e}_1 \\
&\quad + \left[U_2 + \tfrac{1}{2}w_2 + \frac{w_2 X_3}{2a_3}\right]\boldsymbol{e}_2 + \left[U_3 + \frac{w_3 X_1 + w X_3}{2a_3}\right]\boldsymbol{e}_3 \\
&= [U_1 - \epsilon w x_1 + \epsilon w_1 x_2]\boldsymbol{e}_1 + [U_2 + w_2 x_3]\boldsymbol{e}_2 + [U_3 + w(x_3 - \tfrac{1}{2}) + \epsilon w_3 x_1]\boldsymbol{e}_3. \qquad \text{(C 2)}
\end{aligned}
$$

Under our notation, the following links hold:

$$
\epsilon = \frac{e}{L} = \frac{a_1}{2a_3}, \quad X_1 = x_1' = a_1 x_1, \quad X_2 = x_2' = a_1 x_2, \quad X_3 = x_3' - a_3 = a_3(2x_3 - 1). \qquad \text{(C 3)}
$$

The above external flow $\boldsymbol{u}_\infty$ satisfies the Stokes equations and will permit us to recover as special cases the uniform or pure shear flows. The form (C 2) suggests that to obtain the surface force $\boldsymbol{f}^e = \sigma \cdot \boldsymbol{n}$ only for the uniform flow $\boldsymbol{u}_1^\infty := U_1\boldsymbol{e}_1$, the pure shear flow $\boldsymbol{u}_2^\infty := VX_2\boldsymbol{e}_1$ and the extensional-type flow

$$
\boldsymbol{u}_3^\infty := W(X_3\boldsymbol{e}_1 - X_1\boldsymbol{e}_3).
$$

We denote by $\boldsymbol{f}^k$ the solution pertaining to the flow $\boldsymbol{u}_k^\infty$. The solutions $\boldsymbol{f}^2$ and $\boldsymbol{f}^3$ are given by Jeffery (1922). Note that the results (26) of this latter paper suffer from misprints (γ_0 to be replaced by γ_0' in the numerators of H and H' and similar remarks for above coefficients F, F', G and G'). The solution $\boldsymbol{f}^1$ is available in Oberbeck

(1876) (see also Lamb 1932). If we discard the constant pressure term, the solutions $\boldsymbol{f}^k = f_i^j \boldsymbol{e}_i$ become

$$f_i^1 = \frac{4\mu U_1}{a_1 a_2 a_3[\chi_0 + a_1^2\alpha_0]}\frac{\delta_{i1}}{\sqrt{N}}, \quad N(X_1, X_2, X_3) = \frac{X_1^2}{a_1^4} + \frac{X_2^2}{a_2^4} + \frac{X_3^2}{a_3^4}, \tag{C 4}$$

$$f_1^2 = \frac{2\mu V X_2[\alpha_0 + a_2^2\gamma_0']}{a_1 a_2^3 a_3\sqrt{N}\gamma_0'[a_1^2\alpha_0 + a_2^2\beta_0]}, \quad f_2^2 = \frac{2\mu V X_1[\beta_0 - a_1^2\gamma_0']}{a_1^3 a_2 a_3\sqrt{N}\gamma_0'[a_1^2\alpha_0 + a_2^2\beta_0]}, \quad f_3^2 = 0, \tag{C 5}$$

$$f_1^3 = \frac{4\mu W X_1[(2-\alpha_0)A - \beta_0 B - \gamma_0 C]}{a_1^3 a_2 a_3[\beta_0''\gamma_0'' + \alpha_0''\gamma_0'' + \alpha_0''\beta_0'']\sqrt{N}}, \quad 6A = -2\alpha_0'' - \gamma_0'', \tag{C 6}$$

$$f_2^3 = \frac{4\mu W X_2[-\alpha_0 A + (2-\beta_0)B - \gamma_0 C]}{a_1 a_2^3 a_3[\beta_0''\gamma_0'' + \alpha_0''\gamma_0'' + \alpha_0''\beta_0'']\sqrt{N}}, \quad 6B = \alpha_0'' - \gamma_0'', \tag{C 7}$$

$$f_3^3 = \frac{4\mu W X_3[-\alpha_0 A - \beta_0 B + (2-\gamma_0)C]}{a_1 a_2 a_3^3[\beta_0''\gamma_0'' + \alpha_0''\gamma_0'' + \alpha_0''\beta_0'']\sqrt{N}}, \quad C = -A - B, \tag{C 8}$$

with, for

$$\Delta(t) = \{(a_1^2 + t)(a_2^2 + t)(a_2^2 + t)\}^{1/2}, \qquad \chi_0 := \int_0^\infty \mathrm{d}t/\Delta(t)$$

and

$$\gamma_0 := \int_0^\infty \frac{\mathrm{d}t}{(a_3^2 + t)\Delta(t)}, \quad \gamma_0' := \int_0^\infty \frac{(a_1^2 + t)^{-1}\,\mathrm{d}t}{(a_2^2 + t)\Delta(t)}, \quad \gamma_0'' := \int_0^\infty \frac{(a_1^2 + t)^{-1} t\,\mathrm{d}t}{(a_2^2 + t)\Delta(t)}, \tag{C 9}$$

with symmetrical integrals for $\alpha_0, \alpha_0', \alpha_0'', \beta_0, \beta_0'$ and β_0''. For the ellipsoid we get

$$f(\theta, x_3) = h(x_3)g_\eta(\theta), \quad h^2(x_3) = 4x_3(1 - x_3), \quad g_\eta^2(\theta) = \frac{1 + \tan^2\theta}{1 + \tan^2\theta/\eta^2}, \tag{C 10}$$

$$\frac{f^2 s_\epsilon^2}{4g_\eta^4(\theta)} = x_3(1 - x_3)\frac{1 + \tan^2\theta/\eta^4}{1 + \tan^2\theta/\eta^2} + \epsilon^2(1 - 2x_3)^2 = \tfrac{1}{4}a_1^2 N(X_1, X_2, X_3). \tag{C 11}$$

Accordingly, definition (3.11) becomes $8\pi\mu d_i = a_1^2 F_i g_\eta^2(\theta)$ where $F_i = \sqrt{N} f_i$. By operating as many times as necessary cyclical changes of indices, we deduce the density $\boldsymbol{d}^e = d_i^e \boldsymbol{e}_i$ associated to the external flow $\boldsymbol{u}^\infty$. Curtailing the details, we thus obtain, up to a constant pressure $p_0(\epsilon)$,

$$\begin{aligned} d_1^e = \frac{g_\eta^2(\theta)}{4\pi\eta}\bigg\{ & \frac{2U_1}{I_0^\epsilon + I_1^\epsilon} + \epsilon w_1 x_2\left[\frac{1 + I_1^\epsilon/I_4^\epsilon}{I_1^\epsilon + I_2^\epsilon}\right] + w_3(x_3 - \tfrac{1}{2})\left[\frac{-1 + I_1^\epsilon/I_5^\epsilon}{I_1^\epsilon + I_3^\epsilon}\right] - \frac{p_0(\epsilon)a_1}{2\mu}\eta x_1 \\ & - \frac{\epsilon w\eta x_1}{E(\epsilon)}\left[\tfrac{1}{3}I_3^\epsilon(2I_9^\epsilon + \epsilon^2 I_7^\epsilon) + \frac{1}{6\epsilon^2}\left[\left(\frac{1}{\eta} - \tfrac{1}{2}I_1^\epsilon\right)(I_9^\epsilon + 2\epsilon^2 I_7^\epsilon) - \frac{I_2^\epsilon}{2\eta^2}(I_9^\epsilon - \epsilon^2 I_7^\epsilon)\right]\right]\bigg\}, \end{aligned} \tag{C 12}$$

$$\begin{aligned} d_2^e = \frac{g_\eta^2(\theta)}{4\pi\eta}\bigg\{ & \frac{2U_2 + w_2}{I_0^\epsilon + I_2^\epsilon} + \epsilon w_1 x_1\left[\frac{-1 + I_2^\epsilon/I_4^\epsilon}{I_1^\epsilon + I_2^\epsilon}\right] + w_2(x_3 - \tfrac{1}{2})\left[\frac{1 + I_2^\epsilon/I_6^\epsilon}{I_2^\epsilon + I_3^\epsilon}\right] - \frac{p_0(\epsilon)a_1}{2\mu}\frac{x_2}{\eta} \\ & - \frac{\epsilon w x_2}{\eta E(\epsilon)}\left[\tfrac{1}{3}I_3^\epsilon(2I_9^\epsilon + \epsilon^2 I_7^\epsilon) + \frac{1}{6\epsilon^2}\left[\left(\frac{1}{\eta} - \frac{I_1^\epsilon}{2\eta^2}\right)(I_9^\epsilon - \epsilon^2 I_7^\epsilon)\tfrac{1}{2}I_2^\epsilon(I_9^\epsilon + \epsilon^2 I_7^\epsilon)\right]\right]\bigg\}, \end{aligned} \tag{C 13}$$

$$d_3^\epsilon = \frac{g_\eta^2(\theta)}{4\pi\eta}\left\{\frac{2U_3}{I_0^\epsilon + I_3^\epsilon} + \epsilon w_3 x_1\left[\frac{1+I_3^\epsilon/I_5^\epsilon}{I_1^\epsilon + I_3^\epsilon}\right] + \epsilon w_2 x_2\left[\frac{-1+I_3^\epsilon/I_6^\epsilon}{I_2^\epsilon + I_3^\epsilon}\right] - \frac{2\epsilon p_0(\epsilon)a_1}{\mu}\eta(x_2 - \tfrac{1}{2}) \right.$$
$$\left. + \frac{w(x_3-\frac{1}{2})}{E(\epsilon)}\left[\tfrac{1}{3}I_3^\epsilon(I_9^\epsilon + 2\epsilon^2 I_7^\epsilon) + \frac{I_2^\epsilon}{3\eta^2}(I_9^\epsilon - \epsilon^2 I_7^\epsilon) + \frac{2}{3}\left(\frac{1}{\eta} - 2\epsilon^2 I_3^\epsilon(2I_9^\epsilon + \epsilon^2 I_7^\epsilon)\right)\right]\right\}, \tag{C 14}$$

with $E(\epsilon) = I_9^\epsilon(I_7^\epsilon + I_8^\epsilon) + \epsilon^2 I_7^\epsilon I_8^\epsilon$, quantities I_i^ϵ depending upon (ϵ, η) and such that

$$I_0^\epsilon = \int_0^\infty \frac{(t+4\eta^2\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+1)(t+4\epsilon^2)\}^{1/2}}, \qquad I_1^\epsilon = \int_0^\infty \frac{4\epsilon^2(t+4\epsilon^2)^{-3/2}\,\mathrm{d}t}{\{(t+1)(t+4\eta^2\epsilon^2)\}^{1/2}}, \tag{C 15}$$

$$I_2^\epsilon = \int_0^\infty \frac{4\eta^2\epsilon^2(t+4\eta^2\epsilon^2)^{-3/2}\,\mathrm{d}t}{\{(t+1)(t+4\epsilon^2)\}^{1/2}}, \qquad I_3^\epsilon = \int_0^\infty \frac{(t+1)^{-3/2}\,\mathrm{d}t}{\{(t+4\epsilon^2)(t+4\eta^2\epsilon^2)\}^{1/2}}, \tag{C 16}$$

$$I_4^\epsilon = \int_0^\infty \frac{16\eta^2\epsilon^4(t+1)^{-1/2}\,\mathrm{d}t}{\{(t+4\eta^2\epsilon^2)(t+4\epsilon^2)\}^{3/2}}, \quad I_5^\epsilon = \int_0^\infty \frac{4\epsilon^2(t+4\eta^2\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+1)(t+4\epsilon^2)\}^{3/2}}, \tag{C 17}$$

$$I_6^\epsilon = \int_0^\infty \frac{4\eta^2\epsilon^2(t+4\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+4\eta^2\epsilon^2)(t+1)\}^{3/2}}, \qquad I_7^\epsilon = \int_0^\infty \frac{t(t+4\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+1)(t+4\eta^2\epsilon^2)\}^{3/2}}, \tag{C 18}$$

$$I_8^\epsilon = \int_0^\infty \frac{(t+4\eta^2\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+4\epsilon^2)(t+1)\}^{3/2}}, \qquad I_9^\epsilon = \int_0^\infty \frac{\epsilon^2 t(t+1)^{-1/2}\,\mathrm{d}t}{\{(t+4\epsilon^2)(t+4\eta^2\epsilon^2)\}^{3/2}}. \tag{C 19}$$

We deduce the asymptotic behaviour of $\boldsymbol{d}^\epsilon$ by expanding the integrals I_i^ϵ. By invoking Sellier (1994) we get, for $0 < \eta \leqslant 1$ and as ϵ goes to zero,

$$I_0^\epsilon(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2(1+\eta^2)\epsilon^2\log\epsilon + O(\epsilon^2), \tag{C 20}$$

$$I_1^\epsilon(\epsilon) = \frac{2}{\eta+1} + 4\epsilon^2\log\epsilon + O(\epsilon^2), \quad I_2^\epsilon(\epsilon) = \frac{2\eta}{\eta+1} + 4\eta^2\epsilon^2\log\epsilon + O(\epsilon^2), \tag{C 21}$$

$$I_3^\epsilon(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2 - 6(1+\eta^2)\epsilon^2\log\epsilon + O(\epsilon^2), \tag{C 22}$$

$$I_4^\epsilon(\epsilon) = \frac{2\eta}{(\eta+1)^2} + O(\epsilon^2), \quad I_5^\epsilon(\epsilon) = \frac{2}{\eta+1} + 12\epsilon^2\log\epsilon + O(\epsilon^2), \tag{C 23}$$

$$I_6^\epsilon(\epsilon) = \frac{2\eta}{\eta+1} + 12\eta^2\epsilon^2\log\epsilon + O(\epsilon^2), \quad I_9^\epsilon(\epsilon) = \frac{1}{2(1+\eta^2)} + \epsilon^2\log\epsilon + O(\epsilon^2), \tag{C 24}$$

$$I_7^\epsilon(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2 - \frac{2\eta}{1+\eta} + O(\epsilon^2), \tag{C 25}$$

$$I_8^\epsilon(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2 - \frac{2}{1+\eta} + O(\epsilon^2). \tag{C 26}$$

Appendix D.

This appendix derives the solutions $u_0, u_1, u_2, \boldsymbol{t}^1, \boldsymbol{t}^2$ for the elliptical cross-section whose boundary $\mathcal{E}(x_3)$ obeys (5.1). Each point $P(\theta_P, x_3)$ of $\mathcal{E}(x_3)$ is identified by its elliptical angle ψ_P such that

$$x_P = h(x_3)\cos\psi_P, \quad y_P = \eta h(x_3)\sin\psi_P, \quad \tan\theta_P = \eta\tan\psi_P. \tag{D 1}$$

This choice allows us to derive the basic relations

$$\mathrm{d}l_P = \eta h^2(x_3)\delta(P)\,\mathrm{d}\psi_P, \quad [fs_0](P) = h^2(x_3)g_\eta^2(\theta)\delta(P), \quad h^4(x_3)\delta^2(M) = x_1^2 + \frac{x_2^2}{\eta^4}, \tag{D 2}$$

$$L^{\theta,x_3}[u] = -\eta h^2(x_3)\int_0^{2\pi} s(\psi_P)\log[PM]\,\mathrm{d}\psi_P, \quad K^{x_3}[u] = \eta h^2(x_3)\int_0^{2\pi} s(\psi_P)\,\mathrm{d}\psi_P, \tag{D 3}$$

$$2\log[PM] = \log\{h^2(x_3)[1-\cos(\psi_P-\psi)][1+\eta^2+(\eta^2-1)\cos(\psi_P+\psi)]\}, \tag{D 4}$$

with $s(\psi_P) = u(P)\delta(P)$. Accordingly, it is straightforward to obtain

$$u_0(\theta,x_3) = \frac{c_0(x_3)}{2\pi\eta h^2(x_3)\delta(M)}, \quad c_0(x_3) = K^{x_3}[u_0] = -\left\{\log[\tfrac{1}{2}(\eta+1)h(x_3)]\right\}^{-1}, \tag{D 5}$$

$$t_2^1 = 0, \quad t_1^1(\theta,x_3) = \frac{K^{x_3}[t_1^1]}{2\pi\eta h^2(x_3)\delta(M)}, \quad K^{x_3}[t_1^1] = -\left\{\log[\tfrac{1}{2}(\eta+1)h(x_3)] - \frac{1}{\eta+1}\right\}^{-1}, \tag{D 6}$$

$$t_1^2 = 0, \quad t_2^2(\theta,x_3) = \frac{K^{x_3}[t_2^2]}{2\pi\eta h^2(x_3)\delta(M)}, \quad K^{x_3}[t_2^2] = -\left\{\log[\tfrac{1}{2}(\eta+1)h(x_3)] - \frac{\eta}{\eta+1}\right\}^{-1}, \tag{D 7}$$

$$u_j(\theta,x_3) = \frac{(\eta+1)x_j}{2\pi\eta^j h^2(x_3)\delta(M)}, \quad K^{x_3}[u_j] = 0 \quad \text{for } j\in\{1,2\}, \tag{D 8}$$

$$\tau_1^1(\theta,x_3) = \frac{(1+2\eta)x_2}{4\pi\eta^2 h^2(x_3)\delta(M)}, \quad \tau_2^1(\theta,x_3) = \frac{x_1}{4\pi h^2(x_3)\delta(M)}, \quad K^{x_3}[\tau_j^1] = 0, \tag{D 9}$$

$$\tau_1^2(\theta,x_3) = \frac{x_2}{4\pi\eta^2 h^2(x_3)\delta(M)}, \quad \tau_2^2(\theta,x_3) = \frac{(\eta+2)x_1}{4\pi\eta h^2(x_3)\delta(M)}, \quad K^{x_3}[\tau_j^2] = 0, \tag{D 10}$$

$$\tau_1^3(\theta,x_3) = \frac{(1+\eta)^2 x_2}{2\pi\eta h^2(x_3)\delta(M)}, \quad \tau_2^3(\theta,x_3) = -\frac{(1+\eta)^2 x_2}{2\pi\eta^3 h^2(x_3)\delta(M)}, \quad K^{x_3}[\tau_j^3] = 0. \tag{D 11}$$

Those solutions $(\tau_1^i,;\tau_2^i)$ have been sought under the form

$$\tau_j^i(M) = \tau_j^i(\psi,x_3) = [A_j^i\cos\psi + B_j^i\sin\psi]/\delta(M), \quad j\in\{1,2\}. \tag{D 12}$$

For $\boldsymbol{\tau}^3$, we are led (see Gradshteyn & Ryzhik 1965) to

$$A_1^3 - \eta B_2^3 = (1+\eta)^2/[2\pi\eta h(x_3)], \quad B_1^3 = A_2^3 = 0, \tag{D 13}$$

and the solution (D 11) is selected by imposing the solution independent of the choice of directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ (replace $(\boldsymbol{e}_1,\boldsymbol{e}_2)$ by $(\boldsymbol{e}_2,\boldsymbol{e}_1)$).

Our assumptions imply that $0 < h(x_3) \leqslant 1$ for $x_3 \in]0,1[$ and $0 < \eta \leqslant 1$. This justifies the existence of $K^{x_3}[u_0]$, $K^{x_3}[t_1^1]$ and $K^{x_3}[t_2^2]$ for $x_3 \in]0,1[$. By using the link (combine (5.2) and (D 2))

$$\delta(M) = \frac{[fs_0](M)}{h^2(x_3)g_\eta^2(\theta)} = \frac{1}{h(x_3)}\left\{\frac{1+\tan^2\theta/\eta^4}{1+\tan^2\theta/\eta^2}\right\}^{1/2}. \tag{D 14}$$

We may detail the dependence upon (θ,x_3) of the solutions given by (D 5)–(D 10).

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