IMA Journal of Applied Mathematics (1999) **62**, 167–193

Asymptotic solution for the electrostatic field around a slender conducting body

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[Received 25 November 1997 and in revised form 20 May 1998]

This paper deals with the electrostatic field around a slender, conducting body, not necessarily of revolution, embedded in an applied potential. In contrast to previous works devoted to a body of revolution we do not place sources on a segment inside the body. Instead we spread a source density on the boundary of the body in order to obtain a well-posed problem. More precisely, the source strength satisfies a well-known Fredholm integral equation of the first kind. This latter is asymptotically inverted with respect to the slenderness ratio by invoking a systematic formula which provides, to any order, the asymptotic estimate of certain integrals. Several comparisons with the behaviour of exact solutions are also proposed.

1. Introduction

Handelsman & Keller (1967b) obtained the electrostatic field around a slender and axisymmetric conducting body by placing inside the body an unknown source density along a part of the axis of revolution. The source's strength and its extent are gauged by solving asymptotically with respect to a slenderness parameter the integral equation imposed by the boundary condition. In view of its advantages this method has been further employed for slender bodies (see Handelsman & Keller (1967a), Moran (1963) or Geer (1974, 1975) for the potential flow around a two-dimensional or three-dimensional slender body and Barshinger and Geer (1987) for the electrostatic field around a slender dielectric body) and also extended to the case of a thin oblate body of revolution by distributing singularities along a disk inside the body (see (Barshinger & Geer 1983) for the potential flow, (Barshinger & Geer 1981, Homentcovschi 1982) for the electrostatic field, (Homentcovschi 1983) for the scattering of a scalar wave and (Barshinger & Geer 1984) for a low-Reynolds-number flow about the thin oblate body). Except for the two-dimensional case (see Geer (1974)), these works only consider a body of revolution. Moreover, Cade (1994) recently highlighted that the associated integral equation may be sometimes ill-posed (the answer also depends on the applied potential and the global analyticity of the surface, as assumed for instance by Moran (1963), is not a sufficient condition for this integral equation to have a solution). In order to avoid these drawbacks and to deal with a slender body which is not necessarily of revolution we spread a source distribution on the boundary of the body. This point of view not only leads to a well-known Fredholm integral equation of the first kind but also to a well-posed problem. The price to pay seems to be the derivation of the asymptotic estimate (up to high orders) of a specific kind of integrals. This task is indeed not at all trivial but this key step is treated by applying a powerful and systematic formula derived in Sellier (1996).

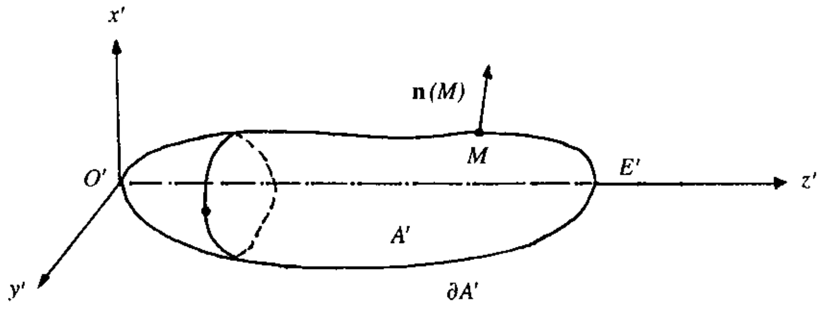

FIG. 1. A slender conducting body with an indication of the coordinate sytem

This paper is organized as follows. The notation and the boundary integral equation are proposed in Section 2. The asymptotic expansion of this integral equation is established in Section 3 by using a general result proved in Sellier (1996) and briefly reported in Appendix A. The asymptotic solution is thereafter derived in two different cases: a body kept at a fixed potential (see Section 4) and a body isolated with a given total charge (see Section 5). In order to discuss the validity of the proposed method the whole of Section 6 is devoted to several comparisons with the behaviour of the exact solution for a slender ellipsoid (in this case Appendix B and Appendix C exhibit the exact solution).

2. The governing boundary integral equation

Throughout this paper we consider a slender body $\mathcal{A}'$ which is an open, simply connected and bounded subset of $\mathbb{R}^3$. The boundary $\partial\mathcal{A}'$ of $\mathcal{A}'$ is assumed to be smooth enough except possibly at its two end points O' and E' (see Fig. 1).

For convenience the set (O', x', y', z') of Cartesian coordinates such that $\mathbf{e}_z := \mathbf{O'E'}/O'E'$ and the set of cylindrical coordinates (r', θ, z') are introduced. By setting $L := O'E'$ and $e := \mathrm{Max}(r')$ for $M \in \partial\mathcal{A}'$, which is the maximum 'radius' of the body, the slenderness ratio ϵ is written $\epsilon = e/L \ll 1$. The slender body is 'straight' in the sense that it collapses, as the slenderness ratio goes to zero, to the straight segment $O'E'$.

By now $\mathcal{A}'$ is a perfectly conducting slender body embedded in a given and applied electrostatic potential ϕ_0 which is induced by electrostatic sources lying outside $\mathcal{A}' \cup \partial\mathcal{A}'$ (ϕ_0 is actually supposed to be harmonic in a neighbourhood of $\mathcal{A}' \cup \partial\mathcal{A}'$). The introduction of $\mathcal{A}'$ disturbs the applied potential ϕ_0 and gives rise to a new potential $\phi := \phi_0 + \phi_{\partial\mathcal{A}'}$ such that, since perfectly conducting, the body $\mathcal{A}'$ becomes a domain where the total electrostatic field vanishes, that is, a domain of constant potential ϕ. Hence, $\Delta\phi = \Delta\phi_0 = \Delta\phi_{\partial\mathcal{A}'} = 0$ in $\mathcal{A}' \cup \partial\mathcal{A}'$ and the potential function $\phi_{\partial\mathcal{A}'}$ is indeed only due to the occurrence of free electrostatic sources on the boundary $\partial\mathcal{A}'$ (see Jackson (1975)). Consequently if ϵ_0 and q respectively designate the free space permittivity and the free surface-charge density arising on $\partial\mathcal{A}'$, the potential function ϕ and the electrostatic field $\mathbf{E} = -\mathrm{grad}[\phi]$ are written for $M \in \mathbb{R}^3 \setminus \partial\mathcal{A}'$

$$\phi(M) = \phi_0(M) + \iint_{\partial\mathcal{A}'} \frac{q(P)}{4\pi\epsilon_0 PM}\,\mathrm{d}S'_P \quad \mathbf{E}(M) = \mathbf{E}_0(M) + \iint_{\partial\mathcal{A}'} \frac{q(P)\mathbf{PM}}{4\pi\epsilon_0 PM^3} dS'_P, \tag{2.1}$$

where $\mathbf{E}_0 := -\mathrm{grad}[\phi_0]$. Observe that whereas ϕ_0 does not necessarily satisfy $\Delta\phi_0 = 0$ everywhere in $\mathbb{R}^3$ (there may exist outside $\mathcal{A}' \cup \partial\mathcal{A}'$ electrostatic sources creating ϕ_0) the potential function $\phi_q := \phi_{\partial\mathcal{A}'}$ is smooth in $\mathbb{R}^3 \setminus \partial\mathcal{A}'$ with $\Delta\phi_q = -q/\epsilon_0\delta_{\partial\mathcal{A}'}$ (if $\delta_{\partial\mathcal{A}'}$ designates the Dirac distribution on boundary $\partial\mathcal{A}'$) and vanishes together with $\mathbf{E}_q := -\mathrm{grad}[\phi_q]$ far from $\mathcal{A}'$. In view of (2.1) the electrostatic problem reduces to the determination of the unknown density q. Once this is achieved one can derive ϕ and $\mathbf{E}$ in $\mathbb{R}^3$ but also get without too much algebra the total charge Q on the body and the electrostatic field on $\partial\mathcal{A}'$ (outside $\mathcal{A}'$). An application of the well-known Gauss theorem (see Jackson (1975) or Sommerfeld (1952)) indeed provides the link between $q(M)$ and $\mathbf{E}(M)$ which is normal to $\partial\mathcal{A}'$ for $M \in \partial\mathcal{A}'\setminus\{O', E'\}$. More precisely, if $\mathbf{n}(M)$ stands for the unit, outward normal vector at $M \in \partial\mathcal{A}' \setminus \{O', E'\}$ (see Fig. 1) then

$$Q = \iint_{\partial\mathcal{A}'} q(P)dS'(P); \qquad \mathbf{E}(M) = \frac{q(M)}{\epsilon_0}\mathbf{n}(M), \quad M \in \partial\mathcal{A}' \setminus \{O', E'\}. \tag{2.2}$$

Recall that $\mathbf{E} = \mathbf{0}$ inside $\mathcal{A}'$ and in (2.2) $\mathbf{E}(M)$ is the limit of the field $\mathbf{E}(P)$ as $P \in \mathbb{R}^3 \setminus (\mathcal{A}' \cup \partial\mathcal{A}')$ goes to M. Of course if $\mathbf{n}(M)$ admits a sense at O' or E' the previous link between q and $\mathbf{E}$ remains valid there.

In this paper we address two different circumstances for the electrostatic potential. In case 1, the constant value of the function ϕ in $\mathcal{A}'$ and denoted by d is known. Thereafter the source density q obeys the following boundary integral equation:

$$d - \phi_0(M) = \mathcal{L}[q](M) := \iint_{\partial\mathcal{A}'} \frac{q(P)}{4\pi\epsilon_0 PM}\,\mathrm{d}S'_P, \qquad M \in \partial\mathcal{A}'. \tag{2.3}$$

Thus, q is the solution to a Fredholm integral equation of the first kind. According to the usual results (see for instance Dautray & Lions (1988a)) such a problem may also be seen as the determination of a harmonic function $\phi_q \in \mathbb{R}^3 \setminus \partial\mathcal{A}'$ solution to outer and inner Dirichlet boundary-value problems, vanishing at infinity and admitting a simple-layer representation (of density q). If $H^{s/2}(\partial\mathcal{A}')$ and $H^{-s/2}(\partial\mathcal{A}')$ denote for real s the usual dual Sobolev spaces, this point of view ensures that (consult Dautray & Lions (1988b, Chapter XI) if $\phi_{0|\partial\mathcal{A}'}$, the restriction of ϕ_0 to the boundary $\partial\mathcal{A}'$, belongs to $H^{1/2}(\partial\mathcal{A}')$ then the integral equation (2.3) admits a unique solution, denoted by $\mathcal{L}^{-1}[d - \phi_0]$, in the space $H^{-1/2}(\partial\mathcal{A}')$. Consequently, and under the assumption that $\phi_{0|\partial\mathcal{A}'} \in H^{1/2}(\partial\mathcal{A}')$, the problem of finding the unknown source density q satisfying (2.3) is a well-posed problem.

In case 2, the perfectly-conducting body is isolated with a given and not necessarily zero total charge Q. Equation (2.3) holds once again but the value of the constant d is determined by enforcing the condition for Q. This leads also to a well-posed problem and for $\phi_{0|\partial\mathcal{A}'} \in H^{1/2}(\partial\mathcal{A}')$ the condition bearing on the total charge Q takes, thanks to (2.2), the form

$$d\iint_{\partial\mathcal{A}'} \mathcal{L}^{-1}[1](P)\,\mathrm{d}S'_P = Q + \iint_{\partial\mathcal{A}'} \mathcal{L}^{-1}[\phi_0](P)\,\mathrm{d}S'_P. \tag{2.4}$$

Thus, once d has been provided by using (2.4), the treatment of case 2 reduces to that of case 1. As seen later the constant d will depend upon ϵ for the slender body. Hence, we write d_ϵ in equation (2.3). For instance case 2 will permit us (see Sections 4 and 5) to give the polarizability of an uncharged ($Q = 0$) slender body $\mathcal{A}'$ embedded in a uniform applied electrostatic field $\mathbf{E}_0$.

To conclude this section we remark that the outer Dirichlet boundary-value problem (with ϕ_f vanishing at infinity)

$$\Delta\phi_f = 0 \quad \text{in } \mathbb{R}^3 \setminus (\mathcal{A}' \cup \partial\mathcal{A}'); \qquad \phi_f(M) = d - \phi_0(M), \quad M \in \partial\mathcal{A}' \tag{2.5}$$

may be tackled by using a source distribution of unknown density f in $\mathcal{A}'$. For a body of revolution and an axisymmetric applied flow or potential ϕ_0 such a method has actually been employed by many authors (see (Barshinger & Geer 1987; Geer 1974, 1975; Handelsman & Keller 1967a,b) and also (Moran 1963)) by spreading sources only on a part of the axis of the body. Unfortunately, this approach presents several drawbacks. First the new density f does not possess a physical sense and subsequent efforts are required in obtaining the physical free source density q through the relation $q = -\epsilon_0[\![\partial\phi_f/\partial n]\!]$, where $[\![a]\!] := a^+ - a^-$ if a^+ and a^- respectively designate the limit values of function a obtained on $\partial\mathcal{A}'$ from outside $\mathcal{A}'$ and from inside. Moreover, Cade (1994) has shown that the new associated integral equation of the first kind imposed on the density f does not in general admit solutions. In contrast to these objections the method here proposed consists this time in asymptotically expanding and inverting the well-posed integral equation (2.3) with respect to the small slenderness parameter ϵ. This method is valid for a body not necessarily of revolution and without any restrictions regarding the applied potential ϕ_0.

3. Asymptotic expansion of the integral equation

In this key section we establish the asymptotic behaviour of the Fredholm integral equation of the first kind (2.3), with respect to the small slenderness parameter ϵ of the problem. The first step consists in rewriting (2.3) in terms of ϵ through the choice of a set of non-dimensional coordinates. If the left-hand side, $d_\epsilon - \phi_0(M)$, of (2.3) is thereafter easily expanded, the asymptotic behaviour of the term $\mathcal{L}[q](M)$ comes, under specific assumptions on the shape of the body especially at its end points O' and E', from the application of a systematic method valid for a large class of integrals depending upon a small parameter.

Let us define the new non-dimensional coordinates (x, y, z, r) and the positive and single-valued body-shape function $f(\theta, z)$ by $x'/x = y'/y = e, z'/z = r'/r = L$ and also if $M \in \partial\mathcal{A}'$ then $M(r', \theta, z') = M(\epsilon f(\theta, z), \theta, z) := M(\theta, z)$ with $f(\theta, z) = O(1)$ for $\theta \in [0, 2\pi]$ and $0 \leqslant z \leqslant 1$ and also $f(\theta, 0) = f(\theta, 1) = 0$. For convenience we also set $f_v^k := \partial^k f/\partial v^k$ for $k \in \mathbb{N}$ and $v \in \{\theta, z\}$. Moreover, we note that $\phi_0(x', y', z') = \phi_0(x, y, z) = \phi_0(r, \theta, z)$ and for $M \in \partial\mathcal{A}'$ we have $\phi_0(M) = \phi_0[\epsilon f(\theta, z), \theta, z]$. The property $0 \leqslant \epsilon f(\theta, z) = O(\epsilon) \ll 1$ for $M \in \partial\mathcal{A}'$ makes it possible to expand the left-hand side of (2.3), denoted by $a(M) = a(\theta, z)$, under the form

$$a(M) = a(\theta, z) = d_\epsilon - a_0(z) - \sum_{n=1}^{\infty} a_n(\theta, z)\epsilon^n \qquad \text{for } M(\theta, z) \in \partial\mathcal{A}', \tag{3.1}$$

where the new functions a_n obey for $n \geqslant 0$ and $r := r'/L$ the definitions

$$a_0(z) = \phi_0(r, \theta, z)_{r=0}, \quad a_n(\theta, z) = \frac{f^n(\theta, z)}{n!}\left(\frac{\partial^n \phi_0}{\partial r^n}\right)(0, \theta, z) \quad \text{for } n \geqslant 1. \tag{3.2}$$

According to Geer (1975), being harmonic in a neighbourhood of the body, the applied potential $\phi_0(r, \theta, z)$ indeed presents for small values of the new non-dimensional variable r the following behaviour:

$$\phi_0(r, \theta, z) = A_0(r^2, z) + \sum_{k=1}^{\infty}\left\{r^k A_k(r^2, z)\cos k\theta + r^k B_k(r^2, z)\sin k\theta\right\}, \tag{3.3}$$

where A_k and B_k are regular functions of r^2 and z near $r = 0$ for $0 \leqslant z \leqslant 1$. Because $\phi_0(r, \theta, z)_{r=0} = A_0(0, z)$, a_0 now depends indeed on the angular variable θ.

As regards the right-hand side of (2.3) if one introduces the new source density λ by $4\pi\epsilon_0\lambda(P) = eq(P)$ and the functions s_ϵ and $H(\theta_P, z_P, \theta, z)$ such that

$$s_\epsilon := \left\{1 + (f^{-1}f_\theta^1)^2 + (\epsilon f_z^1)^2\right\}^{1/2}, \quad \mathrm{d}S'_P = eL[fs_\epsilon](\theta_P, z_P)\,\mathrm{d}\theta_P\,\mathrm{d}z_P, \tag{3.4}$$

$$H(\theta_P, z_P, \theta, z) := \left\{f^2(\theta_P, z_P) + f^2(\theta, z) - 2\cos(\theta_P - \theta)f(\theta, z)f(\theta_P, z_P)\right\}^{1/2}, \tag{3.5}$$

then the linear operator $\mathcal{L}[q]$ becomes, for $M(\theta, z) \in \partial\mathcal{A}'$,

$$\mathcal{L}[q](M) = \mathcal{L}[\lambda f s_\epsilon](\theta, z) = \int_0^{2\pi}\left[\int_0^1 \frac{[\lambda f s_\epsilon](\theta_P, z_P)\,\mathrm{d}z_P}{[(z_P - z)^2 + \epsilon^2 H^2(\theta_P, z_P, \theta, z)]^{1/2}}\right]\mathrm{d}\theta_P. \tag{3.6}$$

In view of (3.6) the term $\mathcal{L}[q](M)$ depends upon the small parameter ϵ via the product ϵH, the solution λ whose asymptotic estimate with respect to ϵ is sought and also the function s_ϵ depending on the angular and axial variations of the body-shape function f. At this stage one may be tempted to expand s_ϵ first. If the body has pointed ends O' and E' with $f_z^1 = O(1)$ everywhere on $\partial\mathcal{A}'$ it is indeed possible to build a uniform expansion of $s_\epsilon(\theta_P, z_P)$ on $[0, 1]$. Unfortunately, this procedure breaks down for rounded ends. In these circumstances and for almost each θ of $[0, 2\pi]$ we demand the function $f^2(\theta, .)$ to be analytic in $[0, 1]$ with the following behaviours respectively near zero on the right and near one on the left:

$$f^2(\theta, z) = \sum_{n\geqslant 1} c_n(\theta)z^n, \quad 2[ff_z^1](\theta, z) = \sum_{n\geqslant 1} nc_n(\theta)z^{n-1} \quad \text{as } z \to 0^+, \tag{3.7}$$

$$f^2(\theta, z) = \sum_{n\geqslant 1} b_n(\theta)(1 - z)^n, \quad 2[ff_z^1](\theta, z) = -\sum_{n\geqslant 1} nb_n(\theta)(1 - z)^{n-1} \quad \text{as } z \to 1^-, \tag{3.8}$$

with $0 < c_1(\theta) = O(1)$ and $0 < b_1(\theta) = O(1)$. Such requirements actually extend the assumptions proposed for a body of revolution in other works (see (Barshing & Geer 1974; Geer 1974, 1975; Handelsman & Keller 1967a,b; Moran 1963)) and makes analytic in $[0, 1]$ the area $A(z)$ of the non-dimensional cross-section $CS(z)$ such that

$$A(z) := \int_0^{2\pi}\left[\int_0^{f(\theta,z)} r\,\mathrm{d}r\right]d\theta = \int_0^{2\pi} [f^2(\theta, z)]/2\,\mathrm{d}\theta. \tag{3.9}$$

For instance for strictly positive and small enough values of z one obtains $(\epsilon f_z^1)^2 \sim \epsilon^2 c_1(\theta)/[4z]$. Accordingly $1 + (f^{-1} f_\theta^1)^2 \gg (\epsilon f_z^1)^2$ is not true any more for $0 < z \leqslant \mu = O(\epsilon^2)$. However, for $z = O(\epsilon^2)$, $f s_\epsilon \sim \epsilon c_1(\theta)/2$ and these remarks clearly suggest we keep together the terms f and s_ϵ. We therefore introduce the unknown function $v_\epsilon(\theta_P, z_P) := [\lambda f s_\epsilon](\theta_P, z_P)$ and such a procedure remains adequate both for pointed and for rounded ends. Observe that, when rewritten in terms of the non-dimensional variables (r, θ, z), the formulae (2.1), (2.2) naturally involve this new unknown function v_ϵ. If the function h and the linear operator $I_{\epsilon h}^z$ are defined for $z \in]0, 1[$ as

$$h(u) := H(\theta_P, z + u, \theta, z), \qquad I_{\epsilon,h}^z[g] := \int_{-z}^{1-z} \frac{g(u+z)\,\mathrm{d}u}{[u^2 + \epsilon^2 h^2(u)]^{1/2}} \tag{3.10}$$

then, by using for $\mathcal{L}[v_\epsilon]$ the change of variable $u := z_P - z$, one immediately finds

$$\mathcal{L}[v_\epsilon](\theta, z) = \int_0^{2\pi} I_{\epsilon,h}^z[v_\epsilon]\,\mathrm{d}\theta_P \quad \text{for} \quad M(\theta, z) \in \partial\mathcal{A}'. \tag{3.11}$$

Note that the definition (3.5) of the positive function $H(\theta_P, z_P, \theta, z)$ shows that H vanishes if and only if $\theta_P = \theta$ and $z_P = z$. Hence, for $\theta_P \in [0, 2\pi] \setminus \{\theta\}$ the above form of $\mathcal{L}[v_\epsilon](\theta, z)$ involves for almost each θ_P a strictly positive function $h(u)$ (depending also on (θ_P, θ, z)). This justifies why such an integral $I_{\epsilon,h}^z[v_\epsilon]$ is usual (in the Lebesgue sense). The next step consists in building for smooth-enough functions h and g the asymptotic expansion of $I_{\epsilon,h}^z[g]$ with respect to the small parameter ϵ. For ϵ set to zero the integral $I_{0,h}^z[g]$ turns out to be singular (except possibly if $g(z) = 0$). This feature is typical of the existence of a singular expansion of $I_{\epsilon,h}^z[g]$ with respect to ϵ. Among usual methods available in such a case one may think about the method of matched asymptotic expansions (see (Van Dyke 1975)) which leads to tedious algebra. Other classical procedures (see (Bleistein & Handelsman 1975, Estrada & Kanwal 1994, Van Dyke 1975)) unfortunately fail. In their approach Handelsman & Keller (1967) were led to consider the asymptotic estimate of

$$J_{\epsilon'}^z[g] = \int_0^1 \frac{g(v)\,\mathrm{d}v}{[(v-z)^2 + \epsilon' S(z)]^{1/2}} = \int_{-z}^{1-z} \frac{g(u+z)\,\mathrm{d}u}{[u^2 + \epsilon' S(z)]^{1/2}} \tag{3.12}$$

and they provided in this case the asymptotic behaviour of $I_{\epsilon'}^z[g]$, formally up to any order, by using a large amount of algebra, with intricate formulae for the coefficients of this estimate. Note that $J_{\epsilon'}^z[g]$ is a particular case of $I_{\epsilon,h}^z[g]$ with $\epsilon' = \epsilon^2$ and this time $h^2 = S(z)$ with no dependence on variable u. For our more general case the treatment proposed by Handelsman and Keller could perhaps work but undoubtedly with much algebra. Instead of employing this method we treat $I_{\epsilon,h}^z[g]$ as a particular case of a larger class of possibly hypersingular integrals handled in Sellier (1996). Not only does this approach based on the use of the concept of integration in the finite-part sense of Hadamard (see (Hadamard 1932, Schwartz 1966, Sellier 1994)) authorize us to deal with many cases encountered but it provides the sought expansion up to any order and in a simple form. For the present integral $I_{\epsilon,h}^z[g]$ if $g_m := \mathrm{Max}_{[0,1]}|g|$ and functions h and g are smooth enough near zero the following behaviour holds for small enough ϵ (see Sellier (1994, 1996) and Appendix A for further explanations)

$$I_{\epsilon,h}^z[g] = I_0^z[g]\log\epsilon + I_1^z[g] + I_2^z[g]\epsilon^2\log\epsilon + I_3^z[g]\epsilon^2 + O(g_m\epsilon^4\log\epsilon), \tag{3.13}$$

where the new linear operators I_0^z and I_n^z, for $n \in \{1,2,3\}$, admit the definitions

$$I_0^z[g] := -2g(z), \quad I_2^z[g] := \frac{d^2}{du^2}\left[g(z+u)\frac{[h^2(u)]}{2}\right]_{u=0}, \tag{3.14}$$

$$I_1^z[g] := 2\{\log 2 - \log h(0)\}g(z) + fp\int_0^1 \frac{g(t)}{|t-z|}\,dt, \tag{3.15}$$

$$I_3^z[g] := -fp\int_0^1 \frac{g(t)h^2(t-z)}{2|t-z|^3}\,dt + \frac{d^2}{du^2}\left[\frac{g(z+u)h^2(u)}{2}\left(\tfrac{1}{2} + \log[\tfrac{1}{2}h(u)]\right)\right]_{u=0}, \tag{3.16}$$

with the symbol fp indicating an integration to hold in the finite-part sense of Hadamard. Such a symbol allows us to propose a pleasant, since synthetic, form for the expansion of $I_{\epsilon,h}^z[g]$. Observe that $I_0^z[g]$ is a two-dimensional term in the sense that it only involves $g(z)$ whereas $I_2^z[g]$ is a weakly three-dimensional correction (functions g and H are needed in a neighbourhood of z). Since they take into account the values of the function g on the whole set $[0,1]$ the remaining terms $I_1^z[g]$ and $I_3^z[g]$ are strongly three-dimensional terms. Results (3.13) to (3.16) agree with those of Handelsman & Keller (1967b) in case $h^2(u) = S(z)$. Although we restrict for this work our attention to the first orders for the behaviour (3.13), the proposed method permits us to deal with higher orders. By combining the definition of the function $h(u)$, results (3.6) and (3.13) to (3.16) it is straightforward to express the integral equation (2.3) as

$$d_\epsilon - a_0(z) - \sum_{n=1}^{\infty} a_n(\theta,z)\epsilon^k = \mathcal{L}_0^z[v_\epsilon]\log\epsilon + \mathcal{L}_1^{\theta,z}[v_\epsilon] + \mathcal{L}_2^{\theta,z}[v_\epsilon]\epsilon^2\log\epsilon$$
$$+\mathcal{L}_3^{\theta,z}[v_\epsilon]\epsilon^2 + O(v_{\epsilon m}\epsilon^4\log\epsilon) \quad \text{for } M(\theta,z) \in \partial\mathcal{A}', \tag{3.17}$$

where the linear operators $\mathcal{L}_0^z$ and $\mathcal{L}_i^{\theta,z}$ follow from (3.14) to (3.16). More precisely, if the linear operator T_z obeys

$$T_z[\alpha(t)] := \alpha(z)\log 2 + fp\int_0^1 \frac{\alpha(t)}{2|t-z|}\,dt, \tag{3.18}$$

then one gets the basic definitions

$$\mathcal{L}_0^z[g] = -2\int_0^{2\pi} g(\theta_P,z)\,d\theta_P, \quad \mathcal{L}_2^{\theta,z}[g] = \frac{d^2}{dz_P^2}\left[\int_0^{2\pi}\frac{H^2(\theta_P,z_P,\theta,z)}{2g^{-1}(\theta_P,z_P)}\,d\theta_P\right]_{z_P=z}, \tag{3.19}$$

$$\mathcal{L}_1^{\theta,z}[g] = -2\int_0^{2\pi} g(\theta_P,z)\log[H(\theta_P,z,\theta,z)]\,d\theta_P - T_z \circ \{\mathcal{L}_0^z[g]\}, \tag{3.20}$$

$$\mathcal{L}_3^{\theta,z}[g] = \frac{d^2}{dz_P^2}\left[\int_0^{2\pi} g(\theta_P,z_P)\frac{H^2(\theta_P,z_P,\theta,z)}{2}\left(\tfrac{1}{2} + \log[\tfrac{1}{2}H(\theta_P,z_P,\theta,z)]\right)d\theta_P\right]_{z_P=z}$$
$$-fp\int_0^1\left\{\frac{\int_0^{2\pi} g(\theta_P,t)H^2(\theta_P,t,\theta,z)\,d\theta_P}{2|t-z|^3}\right\}dt. \tag{3.21}$$

In order to sort the different terms arising in the asymptotic expansion of the integral equation (3.18) it is assumed throughout this paper that for each $n \geqslant 0$ we have

$\partial^n\phi/\partial r^n = O(1), a_n = O(1)$. By superposition one will be able to build the asymptotic solution as soon as each coefficient a_n admits a decomposition in terms of a finite sequence (ϵ^p), $p \in \mathbb{N}$.

4. The asymptotic solution for case 1

In this section we build for case 1 the asymptotic behaviour of the unknown function v_ϵ governed by the boundary equation (3.17). In such circumstances $d_\epsilon = d$ does not depend upon ϵ and we set $d_{0,0} = d$. Inspection of equality (3.17), especially of its remainder compared to the leading term $\mathcal{L}_0^z[v_\epsilon]\log\epsilon$, suggests we seek, for a small slenderness parameter ϵ and $(\theta, z) \in [0, 2\pi]\times]0, 1[$, the solution $v_\epsilon(\theta, z)$ in the form

$$v_\epsilon(\theta, z) = \sum_{i\geqslant 0}^{I} \mu_i(\epsilon)v_i(\theta, z) + O\left(\frac{\epsilon^4}{\log\epsilon}\right), \quad \mu_0(\epsilon) \gg \dots \gg \mu_I(\epsilon) \gg \mu_0(\epsilon)\epsilon^4\log\epsilon. \tag{4.1}$$

When reintroducing such a formal behaviour into (3.17) one has to consider, for given functions $a(z)$ and $b(\theta, z)$, the basic problem

$$\mathcal{L}_0^z[g] = a(z), \quad \mathcal{L}_1^{\theta,z}[g] = b(\theta, z) \quad \text{for } (\theta, z) \in [0, 2\pi]\times]0, 1[. \tag{4.2}$$

Finding the function g for given (a, b) turns out to be of the utmost importance for this work. In view of the definitions satisfied by the operators $\mathcal{L}_0^z$ and $\mathcal{L}_1^{\theta,z}$ it is convenient to set $g = ufs_0$ (see (3.4) for the link between s_0 and f). Thus, if the closed path $C(z)$ designates the boundary of the non-dimensional cross-section $CS(z)$, after noting that $[fs_0](P)\,d\theta_P = dl_P$ and using the definition of the function H, one immediately obtains for $(\theta, z) \in [0, 2\pi]\times]0, 1[$

$$K^z[u] := \oint_{C(z)} u(P)\,dl_P = a'(z) := -a(z)/2, \tag{4.3}$$

$$L^{\theta,z}[u] := -\oint_{C(z)} u(P)\log[PM]\,dl_P = b'(\theta, z) := \{b(\theta, z) + T_z[a(t)]\}/2. \tag{4.4}$$

For data (a', b') and known cross-section $CS(z)$, $z \in]0, 1[$, inverting the problem (4.3), (4.4) reduces it to a classical question. For given z the integral equation of the first kind (4.4) bearing on the density u is indeed associated with interior and exterior Dirichlet problems for the Laplace equation in the plane. For $b' = 0$ it may present non-trivial solutions (see for instance (Hsiao & MacCamy 1973; Hssiao & Wendland 1977, 1981; Hsaio 1986)). Nevertheless, if this occurs for at least one value of z in $]0, 1[$ it always remains possible, through an adequate choice of the radial length scale e (by choosing $e > 2\text{Max}(r')$ instead of $e := \text{Max}(r')$) to ensure that $0 < \delta(z) < 1$ for $z \in]0, 1[$ if $\delta(z) := \text{Max}\{PM;$ for $P, M \in C(z)\}$ designates the diameter of the non-dimensional cross-section $CS(z)$. Under such an assumption (see (Hsiao & Wendland 1981)), the operator $L^{\theta,z}[.]$ is a continuous, bijective mapping from $H^s(C(z))$ onto $H^{s+1}(C(z))$ for all $s \in \mathbb{R}$ and the equation (4.4) admits a unique solution $u_{b'} := \{L^{\theta,z}\}^{-1}[b']$ in $H^s(C(z))$. Consequently, if for instance $b' \in H^{1/2}(\mathcal{C}(z))$ then the set (4.3), (4.4) possesses a unique solution in $H^{-1/2}(\mathcal{C}(z))$ if and only if (a', b') satisfies the basic and linear compatibility condition

$$K^z\big(\{L^{\theta,z}\}^{-1}[b']\big) = a'(z). \tag{4.5}$$

Of course when $a' = b' = 0$, equation (4.5) is true and, since unique, the solution is $u = 0$. At this stage we also outline that the compatibility relation (4.5) never holds (whatever cross-section $\mathcal{C}(z)$ is) if $(a', b') = (0, 1)$, that is, $K^z(\{L^{\theta,z}\}^{-1}[1]) \neq 0$ (see for instance Giroire (1987)). Taking into account these properties of the basic system (4.3), (4.4) it is thus possible to deduce inductively the sequence $\mu_i(\epsilon)$ together with the links between the associated functions v_i. One actually finds that $v_\epsilon(\theta, z)$ becomes, for $(\theta, z) \in [0, 2\pi]\times]0, 1[$,

$$v_\epsilon(\theta, z) = \sum_{n=0}^{3}\sum_{m=-1}^{\infty} v_{n,m}(\theta, z)\epsilon^n[\log\epsilon]^{-m} + O(\epsilon^4\log^2\epsilon), \tag{4.6}$$

with $v_{0,-1} = v_{1,-1} = 0$ and the following set of induction relations for $n \in \{0, 1\}$:

$$\mathcal{L}_0^z[v_{n,0}] = d_{n,-1}, \quad \mathcal{L}_1^{\theta,z}[v_{n,0}] = e_n(\theta, z) - \mathcal{L}_0^z[v_{n,1}], \tag{4.7}$$

$$\mathcal{L}_1^{\theta,z}[v_{n,m}] = d_{n,m} - \mathcal{L}_0^z[v_{n,m+1}] \quad \text{for} \quad m \geqslant 1, \tag{4.8}$$

where $e_0(\theta, z) = e_0(z) = d_{0,0} - a_0(z)$; $e_1(\theta, z) = d_{1,0} - a_1(\theta, z)$ and, for $n \in \{2, 3\}$,

$$\mathcal{L}_0^z[v_{n,-1}] = 0, \quad \mathcal{L}_1^{\theta,z}[v_{n,-1}] = d_{n,-1} - \mathcal{L}_2^{\theta,z}[v_{n-2,0}] - \mathcal{L}_0^z[v_{n,0}], \tag{4.9}$$

$$\begin{aligned}\mathcal{L}_1^{\theta,z}[v_{n,m}] &= d_{n,m} - \delta_{m,0}a_n(\theta, z)\\ &\quad -\mathcal{L}_2^{\theta,z}[v_{n-2,m+1}] - \mathcal{L}_3^{\theta,z}[v_{n-2,m}] - \mathcal{L}_0^z[v_{n,m+1}], \quad \text{for} \quad m \geqslant 0\end{aligned} \tag{4.10}$$

with $\delta_{x,y} := 0$ unless $x = y$ when $\delta_{x,x} := 1$ and for case 1, $d_{n,m} := 0$ except that $d_{0,0} = d$. Since it will be useful when handling case 2, the real data $d_{n,m}$ have been introduced in the above equalities (4.7) to (4.10). Observe that this set of equations is triangular with at each approximation order an adequate right-hand side only depending on the previous orders. Therefore this system can be progressively solved from top to bottom. This is actually worked out by inverting for given functions a, e and unknown functions u, w the equations (for $(\theta, z) \in [0, 2\pi]\times]0, z[)$)

$$\mathcal{L}_0^z[ufs_0] = a(z) = -2K^z[u] \; \forall z \in]0, 1[, \tag{4.11}$$

$$\mathcal{L}_1^{\theta,z}[ufs_0] = e(\theta, z) - \mathcal{L}_0^z[wfs_0] = e(\theta, z) + 2K^z[w]. \tag{4.12}$$

The unique solution u exists if and only if the function $K^z[w]$ is such that the compatibility condition (4.5) holds. By introducing $u_1(\theta, z) := \{L^{\theta,z}\}^{-1}[1]$ and $c_1(z) := K^z[u_1] \neq 0$, one easily gets the useful relations

$$K^z[w] = \big(T_z + \frac{\delta_{z,t}}{c_1(z)}\big)\{K^t[u]\} - \frac{K^z(\{L^{\theta,z}\}^{-1}[e])}{2c_1(z)}, \tag{4.13}$$

$$L^{\theta,z}[u] = -\frac{a(z)}{2c_1(z)} - \frac{K^z(\{L^{\theta,z}\}^{-1}[e])}{2c_1(z)} + \frac{e(\theta, z)}{2}. \tag{4.14}$$

Note that results (4.12), (4.13) take a simple form as soon as $e = e(z)$. In such a case (4.14) reduces to $L^{\theta,z}[u] = -a(z)/[2c_1(z)]$. For convenience we thus introduce the unknown functions $t_{n,m} := v_{n,m}/[fs_0]$ for $n \in \{0, 1, 2, 3\}$ and $m \geqslant -1$ and also, for each $z \in]0, 1[$, the linear operator $\mathcal{S}_z$ by

$$\mathcal{S}_z[\alpha(t)] = \left(T_z + \frac{\delta_{z,t}}{c_1(z)}\right)[\alpha(t)] = \left\{\log 2 + \frac{1}{c_1(z)}\right\}\alpha(z) + fp\int_0^1 \frac{\alpha(t)\mathrm{d}t}{2|t-z|}. \quad (4.15)$$

We shall set $\mathcal{S}_z^0[\alpha(t)] = \alpha(z)$ and $\mathcal{S}_z^m := \mathcal{S}_z \circ \mathcal{S}_z^{m-1}$ for $m \geqslant 1$. By successively choosing $u = t_{n,m}$ and $w = t_{n,m+1}$ these results (4.13), (4.14) authorize us to solve equations (4.7) to (4.10) by induction with $d_{n,m} = 0$ except possibly for $d_{0,0} = d$. If $u_1 := \{L^{\theta,z}\}^{-1}[1]$ and $u_{a_1} := \{L^{\theta,z}\}^{-1}[a_1]$ the reader may check that equations (4.7), (4.8) yield

$$v_{0,0} = 0;\quad v_{1,0}(\theta, z) = \frac{1}{2}\left(\frac{K^z[u_{a_1}]}{c_1(z)}u_1(\theta, z) - u_{a_1}(\theta, z)\right)[fs_0](\theta, z), \quad (4.16)$$

and also, for $m \geqslant 1$,

$$K^z[t_{0,m}] = \mathcal{S}_z^{m-1}\left\{\frac{a_0(z) - d}{2}\right\},\quad v_{0,m}(\theta, z) = \frac{\mathcal{S}_z^{m-1}\{a_0(z) - d\}}{2c_1(z)}[u_1 fs_0](\theta, z), \quad (4.17)$$

$$K^z[t_{1,m}] = \mathcal{S}_z^{m-1}\left(\frac{K^t[u_{a_1}]}{2c_1(t)}\right),\quad v_{1,m}(\theta, z) = \frac{\mathcal{S}_z^{m-1}\{K^t[u_{a_1}]/c_1(t)\}}{2c_1(z)}[u_1 fs_0](\theta, z). \quad (4.18)$$

Regarding the remaining equations (4.9), (4.10) it is now possible to calculate, for $n \in \{2, 3\}$ and $m \geqslant 0$, the functions

$$e_{n,-1} = -\mathcal{L}_2^{\theta,z}[v_{n-2,0}],\ e_{n,m} = -\delta_{m,0}a_n(\theta, z) - \mathcal{L}_2^{\theta,z}[v_{n-2,m+1}] - \mathcal{L}_3^{\theta,z}[v_{n-2,m}]. \quad (4.19)$$

Accordingly, one obtains the induction relation, for $n \in \{2, 3\}$,

$$K^z[t_{n,-1}] = 0,\quad K^z[t_{n,m}] = \mathcal{S}_z\{K^z[t_{n,m-1}]\} - \frac{K^z[\{L^{\theta,z}\}^{-1}[e_{n,m-1}]]}{2c_1(z)},\quad m \geqslant 0 \quad (4.20)$$

and also the solutions for $n \in \{2, 3\}$ and $m \geqslant -1$

$$v_{n,m}(\theta, z) = \{L^{\theta,z}\}^{-1}\left[\frac{K^z[t_{n,m}]}{c_1(z)} + \frac{e_{n,m}}{2} - \frac{K^z[\{L^{\theta,t}\}^{-1}[e_{n,m}]]}{2c_1(z)}\right][fs_0](\theta, z). \quad (4.21)$$

Result (4.21) together with definition (4.19) of $e_{n,m}$ shows that only $v_{n,0}$ depends upon $a_n(\theta, z)$. This dependence vanishes as soon as $\partial a_n/\partial\theta = 0$. It often happens that the applied potential function $\phi_0(r, \theta, z)$ obeys $(\partial\phi_0/\partial r)_{r=0} = 0$. In such circumstances $a_1 = 0$, that is, according to (4.18), $v_{1,m} = 0$ for $m \geqslant -1$, $e_{3,m} = -\delta_{m,0}a_3$ and if $u_{a_3} := \{L^{\theta,z}\}^{-1}[a_3]$ the results (4.20), (4.21) also yield

$$v_{3,-1} = 0,\quad v_{3,0}(\theta, z) = \{L^{\theta,z}\}^{-1}\left[\frac{K^z[u_{a_3}]}{2c_1(z)} - \frac{a_3}{2}\right][fs_0](\theta, z), \quad (4.22)$$

and if $m \geqslant 1$

$$K^z[t_{3,m}] = \mathcal{S}_z^{m-1}\left\{\frac{K^t[u_{a3}]}{c_1(t)}\right\}, \quad v_{3,m}(\theta,z) = \{L^{\theta,z}\}^{-1}\left[\frac{K^z[v_{3,m}]}{c_1(z)}\right][fs_0](\theta,z). \quad (4.23)$$

Once the asymptotic solution v_ϵ is determined we can calculate global quantities such as the total charge Q (see (2.2)) or the total moment (for instance with respect to the point O') $\mathbf{M}(O') = \iint_{\mathcal{A}'} q(P)\mathbf{O'P}\,\mathrm{d}S'_P = \mathcal{M}_x(O')\mathbf{e}_x + \mathcal{M}_y(O')\mathbf{e}_y + \mathcal{M}_z(O')\mathbf{e}_z$ of the slender body $\mathcal{A}'$. In terms of the functions $t_{n,m}$ one gets (with $v_{n,m} = t_{n,m}fs_0$)

$$\frac{Q}{4\pi\epsilon_0 L} = \sum_{n=0}^{3}\sum_{m=-1}^{\infty}\left\{\int_0^1 K^z[t_{n,m}]\,\mathrm{d}z\right\}\frac{\epsilon^n}{[\log\epsilon]^m} + O(\epsilon^4\log^2\epsilon), \quad (4.24)$$

$$\frac{\mathcal{M}_x(O')}{4\pi\epsilon_0 eL} = \sum_{n=0}^{3}\sum_{m=-1}^{\infty}\left\{\int_0^1\left(\int_0^{2\pi}\cos\theta[t_{n,m}f^2s_0](\theta,z)\,\mathrm{d}\theta\right)\mathrm{d}z\right\}\frac{\epsilon^n}{[\log\epsilon]^m} + O(\epsilon^4\log^2\epsilon), \quad (4.25)$$

$$\frac{\mathcal{M}_y(O')}{4\pi\epsilon_0 eL} = \sum_{n=0}^{3}\sum_{m=-1}^{\infty}\left\{\int_0^1\left(\int_0^{2\pi}\sin\theta[t_{n,m}f^2s_0](\theta,z)\,\mathrm{d}\theta\right)\mathrm{d}z\right\}\frac{\epsilon^n}{[\log\epsilon]^m} + O(\epsilon^4\log^2\epsilon), \quad (4.26)$$

$$\frac{\mathcal{M}_z(O')}{4\pi\epsilon_0 L^2} = \sum_{n=0}^{3}\sum_{m=-1}^{\infty}\left\{\int_0^1 zK^z[t_{n,m}]\,\mathrm{d}z\right\}\frac{\epsilon^n}{[\log\epsilon]^m} + O(\epsilon^4\log^2\epsilon). \quad (4.27)$$

Thus, relations (4.17), (4.18) and (4.20) for $K^z[t_{n,m}]$ easily provide the asymptotic estimate of Q and $P_z(O')$. In usual case of a slender body in a free space without any applied potential ($\phi_0 = 0$) the solution v_ϵ is proportional to $d = d_{0,0}$ and thereafter $Q = C_{\mathcal{A}'}d$, where $C_{\mathcal{A}'}$ denotes the slender-body capacity. For $d_{0,0} = 1$ and $a_1 = a_3 = 0$ we have $t_{1,m} = t_{3,m} = 0$, $v_{0,0} = 0$, $v_{0,m} = -\mathcal{S}_z^{m-1}[1]u_1 fs_0/2$ for $m \geqslant 1$ and (4.24) yields, with $K^z[t_{2,0}] = 0$,

$$\frac{C_{\mathcal{A}'}}{2\pi\epsilon_0 L} = -\left(\sum_{m=1}^{\infty}\left\{\int_0^1 \mathcal{S}_z^{m-1}[1]\,\mathrm{d}z - 2\epsilon^2\int_0^1 K^z[t_{2,m}]\,\mathrm{d}z\right\}[\log\epsilon]^{-m} + O(\epsilon^4\log^2\epsilon)\right). \quad (4.28)$$

In view of the equalities (4.17), (4.18) the term $K^z[t_{0,1}]$ only depends on the applied potential ϕ_0 via $a_0(z)$ and the potential d_ϵ on the body. According to (4.24) and (4.27) this property ensures that the terms associated with ϵ^0 arising in the behaviour of the total charge Q and $\mathbf{M}_z(O')$ are the same whatever the detailed body shape is for a given slenderness parameter. Since $\mathcal{S}_z^0[1] = 1$, the first term arising in (4.28) becomes $-2\pi\epsilon_0 L/\log\epsilon$ and as previously announced only depends on the slenderness ratio.

5. The asymptotic solution for case 2

Now we turn to the case 2, that is, when the total charge Q is given and the constant potential d_ϵ on the body must be determined thanks to condition (2.4). Observing that

$C_{\mathcal{A}'} = \iint_{\mathcal{A}'} \mathcal{L}^{-1}[1](P)\,\mathrm{d}S'_P$ and $\iint_{\mathcal{A}'} \mathcal{L}^{-1}[\phi_0](P)\,\mathrm{d}S'_P$ admit, according to the previous section, behaviours respectively given by (4.28) and (4.24) it is thus straightforward to derive for d_ϵ the following estimate:

$$d_\epsilon = \sum_{n=0}^{3} \sum_{m=-1}^{\infty} d_{n,m}\epsilon^n [\log\epsilon]^{-m} + O(\epsilon^4 \log^2 \epsilon) \tag{5.1}$$

and also to check that v_ϵ satisfies (4.6) once again with $v_{0,-1} = v_{1,-1} = 0$ and the inductive relations (4.7) to (4.10). By solving this pyramidal system as explained in Section 4 one can formally express each $t_{n,m}$ in terms of the family $(d_{n,m})$. This latter is therafter determined by enforcing, in view of equality (4.24), the conditions

$$4\pi\epsilon_0 \int_0^1 K^z[t_{n,m}]\,\mathrm{d}z = \delta_{n,0}\delta_{m,0}Q/L. \tag{5.2}$$

The combination of (4.7) and (5.2) readily shows that $d_{0,-1} = -Q/[2\pi\epsilon_0 L]$ and $d_{1,-1} = 0$. Moreover for the new operator $\mathcal{I}_z$ such that $\mathcal{I}_z^0[\alpha(t)] = \alpha(z)$ and

$$\mathcal{I}_z[\alpha(t)] = \mathcal{S}_z[\alpha(t)] - \int_0^1 \mathcal{S}_z[\alpha(t)]\,\mathrm{d}z, \tag{5.3}$$

with $\mathcal{I}_z^m := \mathcal{I}_z \circ \mathcal{I}_z^{m-1}$ for $m \geqslant 1$, the reader may successively check that

$$d_{0,0} = \int_0^1 \{a_0(t) - d_{0,-1}\mathcal{S}_t[1]\}\,\mathrm{d}t, \quad d_{1,0} = \int_0^1 \frac{K^z[u_{a_1}]}{c_1(z)}\,\mathrm{d}z, \tag{5.4}$$

$$K^z[t_{0,1}] = \frac{a_0(z)}{2} - \int_0^1 \frac{a_0(t)}{2}\,\mathrm{d}t - \frac{d_{0,-1}}{2}\mathcal{I}_z[1], \quad K^z[t_{1,1}] = \frac{K^z[u_{a_1}]}{2c_1(z)} - \int_0^1 \frac{K^t[u_{a_1}]}{2c_1(t)}\,\mathrm{d}t, \tag{5.5}$$

with $v_{1,0}$ still given by (4.16), $2c_1(z)v_{0,0}(\theta,z) = -d_{0,-1}[u_1 f s_0](\theta,z)$ and for $n \in \{0,1\}$ and $m \geqslant 1$ the relations

$$d_{n,m} = 2\int_0^1 \mathcal{S}_z \circ \mathcal{I}_z^{m-1}\{K^t[t_{n,1}]\}\,\mathrm{d}z, \quad K^z[t_{n,m}] = \mathcal{I}_z^{m-1}\{K^t[t_{n,1}]\}, \tag{5.6}$$

$$v_{n,m}(\theta,z) = \frac{\mathcal{I}_z^{m-1}\{K^t[t_{n,1}]\}}{c_1(z)}[u_1 f s_0](\theta,z). \tag{5.7}$$

For $n \in \{2,3\}$ if the functions $e_{n,m}$ remain defined by (4.19) one also finds that (4.20) is replaced by $K^z[t_{n,-1}] = 0$ and for $m \geqslant 0$ the inductive relations

$$K^z[t_{n,m}] = \mathcal{I}_z\{K^t[t_{n,m-1}]\} + \int_0^1 \frac{K^z[\{L^{\theta,z}\}^{-1}[e_{n,m-1}]]}{2c_1(z)}\,\mathrm{d}z - \frac{K^z[\{L^{\theta,z}\}^{-1}[e_{n,m-1}]]}{2c_1(z)} \tag{5.8}$$

whereas (4.21) still holds and $d_{n,m}$ is given for $m \geqslant -1$ by

$$d_{n,m} = 2\int_0^1 \left(\mathcal{S}_z\{K^t[t_{n,m}]\} - \frac{K^z[\{L^{\theta,z}\}^{-1}[e_{n,m}]]}{2c_1(z)}\right)\mathrm{d}z. \tag{5.9}$$

If Q is zero then $d_{0,-1}$ and $v_{0,0}$ vanish. In such circumstances $\mathbf{M} = \mathbf{M}(O')$ does not depend on O' and for a constant applied electrostatic field $\mathbf{E}_0$ the polarizability matrix $[P_{\mathcal{A}'}]$ of the body, such that $\mathbf{M} = [P_{\mathcal{A}'}].\mathbf{E}_0$, may be approximated by employing the above forms of functions $t_{n,m}$ and estimates (4.25) to (4.28). For instance, if $\mathbf{E}_0 = E_0\mathbf{e}_z$ then $\phi_0(r,\theta,z) = -E_0Lz = a_0(z)$ with $a_n = 0$ for $n \geqslant 1$ and $v_{1,m} = v_{3,m} = 0$ for $m \geqslant -1$. Hence, the derived behaviour of the vector $\mathbf{M}.\mathbf{e}_z$ only involves ϵ^2. Noting that $K^z[t_{0,1}] = -E_0L(2z-1)/4$ the first term arising in the expansion of $\mathbf{e}_z.[P_{\mathcal{A}'}].\mathbf{e}_z$ is equal to $-\pi\epsilon_0L^3/[6\log\epsilon]$ whatever the detailed shape of the body is.

Before closing this section we give some remarks valid for both cases 1 and 2. At this order of estimation one needs coefficients $a_n(\theta,z)$ for $n \in \{0,1,2,3\}$. It is worth expressing these quantities in terms of the body-shape function f and of the applied potential ϕ_0. If the function C is smooth enough and $C^{(i)}(x) := \partial^i C/[\partial x^i]$ for $i \in \mathbb{N}$ one obtains

$$g_n^k := \Big(\frac{\partial^n[r^kC(r^2)]}{\partial r^n}\Big)_{r=0} = \frac{n!C^{(m)}(0)}{m!} \text{ if } n = k+2m, m \in \mathbb{N} \quad \text{otherwise } g_n^k = 0. \tag{5.10}$$

For given ϕ_0 and $n \in \{0,1,2,3\}$ the derivatives $h_n(\theta,z) := [\partial^n\phi_0/\partial r^n](0,\theta,z)$ may be related to the functions A_k and B_k by combining the behaviour (3.3) with (5.10). Accordingly, for $n \in \{1,2,3\}$ this ensures the links

$$A_0(0,z) = \lim_{r\to 0}\phi_0(r,\theta,z), \quad A_0^{(1)}(0,z) = \int_0^{2\pi}\frac{h_2(\theta,z)}{2!\pi}\,\mathrm{d}\theta, \tag{5.11}$$

$$A_n(0,z) = \int_0^{2\pi}\frac{h_n(\theta,z)\cos n\theta}{n!\pi}\,\mathrm{d}\theta, \quad B_n(0,z) = \int_0^{2\pi}\frac{h_n(\theta,z)\sin n\theta}{n!\pi}\,\mathrm{d}\theta, \tag{5.12}$$

$$A_1^{(1)}(0,z) = \int_0^{2\pi}\frac{h_3(\theta,z)\cos\theta}{3!\pi}\,\mathrm{d}\theta, \quad B_1^{(1)}(0,z) = \int_0^{2\pi}\frac{h_3(\theta,z)\sin\theta}{3!\pi}\,\mathrm{d}\theta, \tag{5.13}$$

and also the following useful forms for the coefficients $a_n(\theta,z)$:

$$a_0(z) = A_0(0,z); \quad a_1(\theta,z) = f(\theta,z)\{A_1(0,z)\cos\theta + B_1(0,z)\sin\theta\}, \tag{5.14}$$

$$a_2(\theta,z) = f^2(\theta,z)\{A_0^{(1)}(0,z) + A_2(0,z)\cos 2\theta + B_2(0,z)\sin 2\theta\}, \tag{5.15}$$

$$a_3(\theta,z) = f^3(\theta,z)\{A_1^{(1)}(0,z)\cos\theta + B_1^{(1)}(0,z)\sin\theta + A_3(0,z)\cos 3\theta + B_3(0,z)\sin 3\theta\}, \tag{5.16}$$

where $C^{(1)}(0,z) := [\partial C/\partial r](0,z)$. Hence, $v_{1,m} = 0$ for $m \geqslant -1$ as soon as $h_1(\theta,z)$ admits no component of the first azimuthal mode. If in addition the modal analysis of $h_3(\theta,z)$ reveals no contribution to modes $\mathrm{e}^{\mathrm{i}\theta}$ or $\mathrm{e}^{3\mathrm{i}\theta}$ then $v_{3,m} = 0$ for $m \geqslant -1$.

6. Applications and comparisons

In view of the results derived in Sections 4 and 5 one basic step consists in inverting the two integral equations of the first kind

$$L^{\theta,z}[u_1] = 1, \qquad L^{\theta,z}[u_{a_1}] = a_1(\theta,z) \tag{6.1}$$

for $z \in]0,1[$ and also the more general problem $L^{\theta,z}[u] = a(\theta,z)$ in getting $v_{2,m}$ or $v_{3,m}$. For this purpose a numerical treatment seems quite adequate and the reader is for instance referred to (Hsiao & MacCamy 1973; Hsiao & Wedland 1977, 1981) for detailed explanations and also to Hsiao (1986) for the numerical stability. One can thereafter build the asymptotic estimates (4.6), (4.24) to (4.27) or (5.1) by computing the proposed formulae. However, for a small slenderness ratio ϵ, it remains somewhat difficult to sort the different terms $T_{nm}\epsilon^n/[\log\epsilon]^m$ arising in the previous results for given n but varying m. Thus it appears quite desirable to discuss comparisons when analytical results are available both for asymptotic and exact solutions. Hence this section is restricted to such an example. 5
More precisely, we consider a body entirely described by the real value $\eta > 0$ and a positive function h with $h(0) = h(1) = 0$ and such that each non-dimensionnal cross-section $CS(z)$ admits for boundary $C(z) = \mathcal{E}(z)$ an ellipse with axis of symmetry $\mathbf{e}_x$ and $\mathbf{e}_y$ and of equation (in dimensionless variables x, y and z)

$$x^2 + \frac{y^2}{\eta^2} = h^2(z). \tag{6.2}$$

We report below the solutions obtained (additional details are available in Appendix C). In these circumstances $f(\theta,z) = h(z)g_\eta(\theta)$ with

$$g_\eta(\theta) = \left\{\frac{1+\tan^2\theta}{1+\frac{\tan^2\theta}{\eta^2}}\right\}^{1/2}, \quad [fs_0](\theta,z) = h(z)(1+\tan^2\theta)\frac{\left\{1+\frac{\tan^2\theta}{\eta^4}\right\}^{1/2}}{\left\{1+\frac{\tan^2\theta}{\eta^2}\right\}^{3/2}}. \tag{6.3}$$

The function u_1 solution to $L^{\theta,z}[u_1] = 1$ is easy to guess. More precisely, one finds

$$u_1(\theta,z) = \frac{c_1(z)}{2\pi\eta h(z)}\left\{\frac{1+\frac{\tan^2\theta}{\eta^2}}{1+\frac{\tan^2\theta}{\eta^4}}\right\}^{1/2}, \quad c_1(z) = K^z[u_1] = -\frac{1}{\log[\frac{1}{2}(\eta+1)h(z)]}. \tag{6.4}$$

By a suitable choice of directions $\mathbf{e}_x$ and $\mathbf{e}_y$ it remains always possible to restrict the study to the case $\mathrm{Max}_{z\in[0,1]}h(z) = 1$ and $\eta \leqslant 1$. Hence, the relations (6.4) admit a sense for almost each z in $[0,1]$. The solution u_{a_1} such that $L^{\theta,z}[u_{a_1}] = a_1(\theta,z)$ with a_1 defined by (5.14) becomes (see Appendix C)

$$u_{a_1}(\theta,z) = \frac{\eta+1}{2\pi\eta}\left\{\frac{1+\tan^2\theta}{1+\frac{\tan^2\theta}{\eta^4}}\right\}^{1/2}\left[A_1(0,z)\cos\theta + B_1(0,z)\frac{\sin\theta}{\eta}\right], \quad K^z[u_{a_1}] = 0. \tag{6.5}$$

6.1 *Case* 1

Here we obtain $v_{0,-1} = v_{1,-1} = 0$ and also, since $K^z[u_{a_1}] = 0$, the results (4.17), (4.18) become $K^z[t_{1,m}] = 0$ for $m \geqslant 0$ and

$$v_{0,m}(\theta, z) = \frac{F_m(z)}{2\pi\eta}\frac{1+\tan^2\theta}{1+\dfrac{\tan^2\theta}{\eta^2}} \quad \text{with} \quad K^z[t_{0,m}] = F_m(z),\ m \geqslant 0, \tag{6.6}$$

$$v_{1,m}(\theta, z) = -\delta_{m,0}\frac{(\eta+1)h(z)g_\eta^3(\theta)}{4\pi\eta}\left[A_1(0,z)\cos\theta + B_1(0,z)\frac{\sin\theta}{\eta}\right] \text{ if } m \geqslant 0 \tag{6.7}$$

with, for this case, functions F_m obeying

$$F_0 = 0, \quad F_m(z) = S_z^{m-1}\left[\frac{a_0(z)-d}{2}\right] \quad \text{if} \quad m \geqslant 1. \tag{6.8}$$

For $m \geqslant -1$ and $n \in \{2, 3\}$ the definitions of $e_{n,m}$ proposed by (4.18) reduce to

$$e_{2,-1} = -\mathcal{L}_2^{\theta,z}[v_{0,0}], \quad e_{2,0} = -a_2 - \mathcal{L}_2^{\theta,z}[v_{0,1}] - \mathcal{L}_3^{\theta,z}[v_{0,0}], \tag{6.9}$$

$$e_{2,m} = -\mathcal{L}_2^{\theta,z}[v_{0,m+1}] - \mathcal{L}_3^{\theta,z}[v_{0,m}] \text{ for } m \geqslant 1, \tag{6.10}$$

$$e_{3,-1} = -\mathcal{L}_2^{\theta,z}[v_{1,0}], \quad e_{3,0} = -a_3 - \mathcal{L}_3^{\theta,z}[v_{1,0}], \quad e_{3,m} = 0 \quad \text{for} \quad m \geqslant 1. \tag{6.11}$$

Thanks to material provided by Appendix C the reader may check that $v_{2,-1} = 0$ and, for $m \geqslant 0$,

$$\begin{aligned} v_{2,m}(\theta,z) = \frac{(\eta+1)^2 g_\eta^2(\theta)}{2\pi\eta(\eta^2+1)}\Bigg\{ & \frac{\eta^2+1}{(\eta+1)^2}K^z[t_{2,m}] + \frac{1+\eta^2}{2}\left(O_2[F_m] + \frac{h^2F_{m+1}^{(2)}}{2}\right)(z) \\ & + \frac{1-\eta^2}{2}O_3[F_m](z) - \left(O_2[F_m] + \frac{h^2F_{m+1}^{(2)}}{2}\right)(z)g_\eta^2(\theta) \\ & - O_3[F_m](z)g_\eta^2(\theta)\cos 2\theta \\ & + \delta_{m,0}\bigg[\frac{1+\eta^2}{2}h^2(z)A_0^{(1)}(0,z) + \frac{1-\eta^2}{2}h^2(z)A_2(0,z) - h^2(z)A_0^{(1)}(0,z)g_\eta^2(\theta) \\ & - h^2A_2(0,z)g_\eta^2(\theta)\cos 2\theta - \frac{(\eta+1)^2}{\eta}h^2(z)B_2(0,z)g_\eta^2(\theta)\sin 2\theta\bigg]\Bigg\}, \end{aligned} \tag{6.12}$$

where O_2, O_3 (and also O_1 for the incoming relation (6.13)) designate operators detailed in Appendix C whereas (4.20) provides $K^z[t_{2,0}] = 0$ together with, for $m \geqslant 1$, the induction relation

$$\begin{aligned} K^z[t_{2,m}] = {} & \mathcal{S}_z\{K^z[t_{2,m-1}]\} + \tfrac{1}{8}(\eta^2+1)\big\{(h^2F_m)^{(2)} + h^2F_m^{(2)}\big\}(z) \\ & + \tfrac{1}{4}\delta_{m,1}\big\{(1+\eta^2)h^2(z)A_0^{(1)}(0,z) + (1-\eta^2)h^2(z)A_2(0,z)\big\} \\ & + \tfrac{1}{4}(1-\delta_{m,1})\big\{2O_1[F_{m-1}](z) + (1+\eta^2)O_2[F_{m-1}](z) + (1-\eta^2)O_3[F_{m-1}](z)\big\}. \end{aligned} \tag{6.13}$$

Finally, (6.11) easily leads to

$$v_{3,-1}(\theta,z) = -\frac{(\eta+1)^2 h(z) g_\eta^3(\theta)}{16\pi\eta}\left[(h^2 A_1(0,z))^{(2)}\cos\theta + (h^2 B_1(0,z))^{(2)}\sin\theta\right] \tag{6.14}$$

and for $v_{3,0}(\theta,z)$ detailed in Appendix C

$$K^z[t_{3,m}] = 0 \quad \text{for } m \geqslant -1, \quad v_{3,m}(\theta,z) = \delta_{m,0} v_{3,0}(\theta,z) \quad \text{for } m \geqslant 0. \tag{6.15}$$

For instance assume that $\phi_0 = 0$. If the new funtions W and S satisfy

$$W(z) := \log\left[\frac{16z(1-z)}{(\eta+1)^2 h^2(z)}\right] = 2\mathcal{S}_z[1], \quad S(z) := h^2(z) \quad \text{for } z \in [0,1] \tag{6.16}$$

one easily gets the general results

$$v_{0,-1} = v_{0,0} = v_{2,-1} = v_{2,0} = v_{1,m} = v_{3,m} = 0 \text{ for } m \geqslant -1, \tag{6.17}$$

$$v_{0,1}(\theta,z) = -d\frac{g_\eta^2(\theta)}{4\pi\eta}, \quad v_{0,2}(\theta,z) = -d\frac{g_\eta^2(\theta)W(z)}{8\pi\eta}, \tag{6.18}$$

$$v_{0,3}(\theta,z) = -d\frac{g_\eta^2(\theta)}{16\pi\eta}\left\{W^2(z) + \int_{-z}^{1-z}\log\left[\frac{S(z)(z+u)(1-z-u)}{S(z+u)z(1-z)}\right]\frac{du}{|u|}\right\}, \tag{6.19}$$

$$\begin{aligned} v_{2,1}(\theta,z) = &-d\frac{(\eta+1)^2 g_\eta^2(\theta)}{4\pi\eta(\eta^2+1)}\left\{\frac{(\eta^2+1)^2 S^{(2)}(z)}{8(\eta+1)^2}\right.\\ &+\frac{\eta-1}{\eta+1}\left[\frac{1-\eta^2}{2} - g_\eta^2(\theta)\cos 2\theta\right]hh^{(2)}(z) + \left[\frac{1+\eta^2}{2} - g_\eta^2(\theta)\right]\left[\frac{[hh^{(2)}](z)}{2}\right.\\ &\left.\left.+\frac{1-2z+2z^2}{4z^2(1-z)^2}S(z) - \frac{(\eta^2+1)S^{(2)}(z)}{4(1+\eta)^2} - \frac{\eta[h^{(1)}]^2(z)}{(1+\eta)^2} + \frac{[SW^{(2)}](z)}{4}\right]\right\}, \end{aligned} \tag{6.20}$$

$$\begin{aligned} \frac{Q}{4\pi\epsilon_0 L} = &-\frac{d}{\log\epsilon}\left[\frac{1}{2} + \frac{1}{4\log\epsilon}\int_0^1 W(z)\,dz + \frac{1}{8(\log\epsilon)^2}\int_0^1\left(W^2(z)\right.\right.\\ &\left.\left.+\int_{-z}^{1-z}\log\left[\frac{S(z)(z+u)(1-z-u)}{S(z+u)z(1-z)}\right]\frac{du}{|u|}\right)dz + O\left(\frac{1}{(\log\epsilon)^3}\right)\right]\\ &-\frac{d(\eta^2+1)\epsilon^2}{16\log\epsilon}\left[S^{(1)}(1) - S^{(0)}(0) + \frac{1}{\log\epsilon}\left\{\frac{S^{(1)}(0)}{2}\log\left[\frac{(1+\eta)^2 S^{(1)}(0)}{16}\right]\right.\right.\\ &-\frac{S^{(1)}(1)}{2}\log\left[-\frac{(1+\eta)^2 S^{(1)}(1)}{16}\right] + \int_0^1\left(\frac{[SW^{(2)}](z)}{2} + \frac{1}{2}\left[\frac{S(z)-zS^{(1)}(0)}{z^2}\right.\right.\\ &\left.\left.\left.+\frac{S(z)+(1-z)S^{(1)}(1)}{(1-z)^2}\right] + (1-\eta^2)\left[hh^{(2)}(z) + \frac{[h^{(1)}]^2(z)}{1+\eta^2}\right]\right)dz\right\}\\ &\left.+O\left(\frac{1}{(\log\epsilon)^2}\right)\right]. \end{aligned} \tag{6.21}$$

For $\eta = 1$ the body is one of revolution with $g_\eta = 1$, and several terms arising on the right-hand side of equalities (6.20), (6.21) vanish. The asymptotic estimate of the body capacity $C_{\mathcal{A}'} = Q/d$ exactly agrees with the result of Handelsman & Keller (1967b). Since the source density σ is placed on the exact boundary (not on a part of the axis of revolution) the terms $v_{2,0}$ and $v_{2,1}$ differ from the second approximation given in this latter paper. If the body is a slender ellipsoid with $0 < \eta \leqslant 1$ we have $S(z) = h^2(z) = 4z(1-z)$. Thus, (6.18) to (6.20) immediately lead to

$$v(\theta, z) = -d\frac{g_\eta^2(\theta)}{4\pi\eta}\left\{\frac{1}{\log\epsilon}\left[1 + \frac{\log[2/(\eta+1)]}{\log\epsilon} + \left(\frac{\log[2/(\eta+1)]}{\log\epsilon}\right)^2 + O\left(\frac{1}{(\log\epsilon)^3}\right)\right]\right.$$
$$\left. -\frac{\epsilon^2}{\log\epsilon}\left[1 + \eta^2 + O\left(\frac{1}{\log\epsilon}\right)\right]\right\}. \tag{6.22}$$

This result agrees perfectly with the asymptotic expansion of the exact solution proposed in Appendix B (with $(\alpha, \beta, \gamma) = (0, 0, 0)$).

6.2 *Case* 2

In this case we still obtain $v_{0,-1} = v_{1,-1} = v_{2,-1} = 0$, results (6.6), (6.7), (6.9) to (6.12) and (6.14), (6.15) remain valid with definitions (6.8) here replaced by

$$F_0 = -\frac{d_{0,-1}}{2}; \quad F_m = \mathcal{I}_z^{m-1}\left\{\frac{a_0(t)}{2} - \int_0^1 \frac{a_0(t)}{2}\,dt - \frac{d_{0,-1}}{2}\mathcal{I}_z[1]\right\} \quad \text{for } m \geqslant 1. \tag{6.23}$$

This time $K^z[t_{2,m}]$ is given by

$$K^z[t_{2,0}] = d_{0,-1}\frac{\eta^2+1}{16}\left\{\int_0^1 [h^2]^{(2)}(z)\,dz - [h^2]^{(2)}(z)\right\} \tag{6.24}$$

and the induction relation (5.8), expressed by using the link

$$-\frac{K^z[\{L^{\theta,z}\}^{-1}[e_{n,m}]]}{c_1(z)} = \frac{1+\eta^2}{4}\left\{2O_2[F_m] + (h^2F_{m+1})^{(2)} + h^2F_{m+1}^{(2)}\right\}(z)$$
$$+O_1[F_m](z) + \frac{1-\eta^2}{2}O_3[F_m](z) + \delta_{0,m}\left\{\frac{1-\eta^2}{2}h^2A_2(0,z) + h^2A_0^{(1)}(0,z)\right\}. \tag{6.25}$$

Finally $d_{1,m} = d_{3,m} = 0$, $d_{0,0}$ is given by (5.4), $d_{0,m} = 2\int_0^1 S_z \circ F_m\,dz$ if $m \geqslant 1$,

$$d_{2,-1} = -\frac{\eta^2+1}{8}d_{0,-1}\int_0^1 [h^2]^{(2)}(z)\,dz \tag{6.26}$$

and $d_{2,m}$ is obtained for $m \geqslant 0$, by combining (5.9), (6.25), (6.26).

We detail the solution in three different cases.

Case 2.1. Here we choose a uniform and axial external field with $\phi_0 = z$ and a body of total charge Q. Hence $a_0(z) = A_0(0, z) = z$, $A_0^{(1)}(0, z) = 0$ and the results take the following form:

$$v_{0,-1} = v_{0,-0} = v_{2,-1} = 0, \quad v_{1,m} = \delta_{m,0} v_{1,0}, \quad v_{3,m} = \delta_{m,-1} v_{3,-1} \quad \text{for } m \geqslant -1, \tag{6.27}$$

$$d_{0,-1} = -\frac{Q}{2\pi\epsilon_0 L}, \quad d_{0,0} = \frac{\gamma}{2} + \frac{Q}{4\pi\epsilon_0 L}\int_0^1 W(z)dz, \tag{6.28}$$

$$v_{0,0}(\theta, z) = \frac{Q}{4\pi\epsilon_0 L}\frac{g_\theta^2(\theta)}{2\pi\eta}, \quad d_{2,-1} = \frac{(\eta^2+1)Q}{16\pi\epsilon_0 L}[S^{(1)}(1) - S^{(1)}(0)], \tag{6.29}$$

$$v_{0,1}(\theta, z) = \frac{g_\theta^2(\theta)}{4\pi\eta}\left\{\gamma(z-\tfrac{1}{2}) + \frac{Q}{2\pi\epsilon_0 L}\left[W(z) - \int_0^1 W(z)\,\mathrm{d}z\right]\right\}, \tag{6.30}$$

$$\begin{aligned} d_{0,1} = {} & \gamma\int_0^1 \frac{W(z)}{4}(z-\tfrac{1}{2})\,\mathrm{d}z + \frac{Q}{8\pi\epsilon_0 L}\left\{\int_0^1 W^2(z)\,\mathrm{d}z - \left(\int_0^1 W(z)\,\mathrm{d}z\right)^2\right. \\ & \left. + \int_0^1\left(\int_{-z}^{1-z}\log\left[\frac{S(z)(z+u)(1-z-u)}{S(z+u)z(1-z)}\right]\frac{\mathrm{d}u}{|u|}\right)\mathrm{d}z\right\}, \end{aligned} \tag{6.31}$$

$$\begin{aligned} v_{0,2}(\theta, z) = {} & \frac{g_\theta^2(\theta)}{8\pi\eta}\left\{\gamma(z-\tfrac{1}{2})[W(z)-2] + \int_0^1 \gamma W(z)(z-\tfrac{1}{2})\,\mathrm{d}z\right. \\ & + \frac{Q}{4\pi\epsilon_0 L}\left[\int_{-z}^{1-z}\log\left[\frac{S(z)(z+u)(1-z-u)}{S(z+u)z(1-z)}\right]\frac{\mathrm{d}u}{|u|} + \left(\int_0^1 W(z)\,\mathrm{d}z\right)^2\right. \\ & - \int_0^1 W^2(z)\,\mathrm{d}z \\ & - W(z)\int_0^1 W(z)\,\mathrm{d}z + W^2(z) - \int_0^1\left(\int_{-z}^{1-z}\log\left[\frac{S(z)(z+u)(1-z-u)}{S(z+u)z(1-z)}\right]\right. \\ & \left.\left.\left.\frac{\mathrm{d}u}{|u|}\right)\mathrm{d}z\right]\right\}, \end{aligned} \tag{6.32}$$

$$\begin{aligned} v_{2,0}(\theta, z) = {} & \frac{Q}{4\pi\epsilon_0 L}\frac{(\eta+1)^2 g_\eta^2(\theta)}{4\pi\eta(\eta^2+1)}\left\{\frac{(\eta^2+1)^2}{8(\eta+1)^2}[S^{(2)}(z) + S^{(1)}(0) - S^{(1)}(1)]\right. \\ & + \frac{\eta-1}{\eta+1}\left[\frac{1-\eta^2}{2} - g_\eta^2(\theta)\cos 2\theta\right]hh^{(2)}(z) + \left[\frac{1+\eta^2}{2} - g_\eta^2(\theta)\right]\left[\frac{[hh^{(2)}](z)}{2}\right. \\ & \left.\left. + \frac{1-2z+2z^2}{4z^2(1-z)^2}S(z) - \frac{(\eta^2+1)S^{(2)}(z)}{4(1+\eta)^2} - \frac{\eta[h^{(1)}]^2(z)}{(1+\eta)^2} + \frac{[SW^{(2)}](z)}{4}\right]\right\}, \end{aligned} \tag{6.33}$$

with $\gamma = 1$ and $v_{1,0} = v_{3,-1} = 0$.

For $Q = 0$ and $\eta = 1$ these results agree with Handelsman & Keller (1967b). It is also possible to express the total moment $\mathbf{M}(O')$ depending for non-zero Q upon the point O' (see formulae (4.25) to (4.27)). For a slender ellipsoid with $0 < \eta \leqslant 1$, $S(z) = 4z(1-z)$

many terms vanish for above results. One obtains $v_{2,0} = 0$ and

$$v(\theta, z) = \frac{g_\eta^2(\theta)}{2\pi\eta}\left\{\frac{Q}{4\pi\epsilon_0 L} + \frac{z - 1/2}{2\log\epsilon} \times\left[1 + \frac{1}{\log\epsilon}\left(\log\left[\frac{2}{\eta+1}\right] - 1\right) + O\left(\frac{1}{(\log\epsilon)^2}\right)\right]\right\}, \tag{6.34}$$

$$d_\gamma(\epsilon) = -\frac{Q}{2\pi\epsilon_0 L}\log\epsilon + \frac{\gamma}{2} + \frac{Q}{2\pi\epsilon_0 L}\log[\frac{2}{\eta+1}] + O(\frac{1}{(\log\epsilon)^2}) -\epsilon^2\log\epsilon\left[\frac{(\eta^2+1)Q}{2\pi\epsilon_0 L} + O(\frac{1}{\log\epsilon})\right] \quad \text{with } \gamma = 1. \tag{6.35}$$

These results agree perfectly with the asymptotic expansion of the exact solution (see Appendix B with $(\alpha, \beta, \gamma) = (0, 0, 1)$).

Case 2.2. This is the case of a uniform and transverse external field with $\phi_0 = x'/L = \epsilon x$. In this circumstance $A_1(r^2, z) = 1$ is the unique non-zero function of (r^2, z) arising on the right-hand side of (3.3). For given total charge Q on the body our results (6.27) to (6.33) remain valid with $\gamma = 0$ and this time

$$v_{1,0}(\theta, z) = -\frac{(\eta+1)h(z)g_\eta^3(\theta)\cos\theta}{4\pi\eta} = -\frac{g_\eta^2(\theta)}{2\pi\eta}\frac{(\eta+1)x}{2}, \tag{6.36}$$

$$v_{3,-1}(\theta, z) = -\frac{(\eta+1)^2 S^{(2)}(z)h(z)g_\eta^3(\theta)\cos\theta}{16\pi\eta} = -\frac{g_\eta^2(\theta)}{2\pi\eta}\frac{(\eta+1)^2 S^{(2)}(z)x}{8}. \tag{6.37}$$

Thus, for a slender ellipsoid one gets $d(\epsilon) = d_0(\epsilon)$ (see (6.35)) and also

$$v(\theta, z) = \frac{g_\eta^2(\theta)}{2\pi\eta}\left\{\frac{Q}{4\pi\epsilon_0 L} - \frac{\eta+1}{2}\epsilon x + (\eta+1)^2 x\epsilon^3\log\epsilon + o(\epsilon^3\log\epsilon)\right\}. \tag{6.38}$$

Equality (6.38) quite agrees with the behaviour of the exact solution (see Appendix B with $(\alpha, \beta, \gamma) = (\epsilon, 0, 0)$).

Case 2.3. We finally address the case of a uniform and transverse external field with $\phi_0 = y'/L = \epsilon y$. Here $B_1(r^2, z) = 1$ is non-zero. Formulae (6.27) to (6.33) are still valid with $\gamma = 0$ and this time

$$v_{1,0}(\theta, z) = -\frac{g_\eta^2(\theta)}{2\pi\eta}\frac{(\eta+1)y}{\eta}, \quad v_{3,-1}(\theta, z) = -\frac{g_\eta^2(\theta)}{2\pi\eta}\frac{(\eta+1)^2 S^{(2)}(z)y}{8}, \tag{6.39}$$

$$v(\theta, z) = \frac{g_\eta^2(\theta)}{2\pi\eta}\left\{\frac{Q}{4\pi\epsilon_0 L} - \frac{\eta+1}{2\eta}\epsilon y + (\eta+1)^2 y\epsilon^3\log\epsilon + o(\epsilon^3\log\epsilon)\right\}, \tag{6.40}$$

as Appendix B confirms (this time use $(\alpha, \beta, \gamma) = (0, \epsilon, 0)$).

7. Conclusion

By resorting to a well-posed boundary integral formulation we asymptotically built the electrostatic source density q arising on the conductor surface. The derived results hold not only for an arbitrary imposed potential but also for a slender conducting body of general cross-section. Accordingly, this work encompasses previous studies devoted to a body of revolution. Term-by-term comparisons with the asymptotic behaviour of the exact solutions clearly show the validity of the proposed approach. Finally, it is worth noting that the estimate of the density q would also easily provide asymptotic approximations of the electrostatic potential and field, uniformly valid outside the body (invoke the relations (2.1)).

Acknowledgment

The author is indebted to Professor J. C. Nédélec for useful comments.

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Appendix A

In this Appendix we give the systematic formula here employed for the derivation of the asymptotic estimate of $I^z_{\epsilon,h}[g]$ (see (3.10)). For detailed explanations the reader is referred to Sellier (1996, Theorem 3). Accordingly one obtains for a positive integer N

$$\int_{-z}^{1-z} \frac{g(u+z)\,\mathrm{d}u}{[u^2+\epsilon^2 h^2(u)]^{1/2}} = \sum_{n=0}^{N} \frac{\partial^n K(1,0)}{n!}\left[fp\int_{-z}^{1-z} \frac{\mathrm{sgn}(u) g(u+z)}{u^{n+1}}[h(u)]^n\,\mathrm{d}u\right]\epsilon^n$$
$$+\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{i=0}^{n-l} \frac{g^{(l)}(z) a^i_{n-l-i}}{l!i!}\left[fp\int_{-\infty}^{\infty} \partial^i K[t,h(0)] t^n\,\mathrm{d}t\right]\epsilon^n$$
$$-2\sum_{n=0}^{N}\sum_{l=0}^{n}\sum_{i=0}^{n-l}\sum_{j=0}^{l} \frac{[h(0)]^l g^{(j)}(z) a^i_{l-j}}{l!i!j!}\partial^n K(1,0)\epsilon^n \log\epsilon + O(\epsilon^{N+1}\log\epsilon); \qquad (A.1)$$

if $K(x,y) := [x^2+y^2]^{-1/2}$ for $n \in \mathbb{N}$ then $\partial^n K(x,y) := [\partial^n K/\partial y^n](x,y)$, $g^{(n)}$ is the derivative of order n and for $(i,p) \in \mathbb{N}\times\mathbb{N}$ the coefficients a^i_p obey $a^0_0 = 1$, $a^0_p = 0$ for $p \geqslant 1$ and for $i \geqslant 1$

$$\left[\frac{h(u)-h(0)}{u}\right]^i = \sum_p a^i_p u^p \quad \text{as} \quad u \to 0. \qquad (A.2)$$

If $N = 2$ a careful calculation of the integrals in the finite-part sense of Hadamard arising on the right-hand side of (A.1) (see for instance Schwartz (1966) and Sellier (1994)) yields result (3.14). Since $\partial^3 K(1,0) = 0$ and for $i \in \{0,1,2,3\}$ each integral $fp \int_{-\infty}^{\infty} \partial^i K[t, h(0)] t^n \, dt$ vanishes for symmetry reasons the remainder indeed becomes $O(\epsilon^4 \log \epsilon)$.

Appendix B

By invoking the material available in Lamb (1945, pp.149–153) we give the exact solution $v(\theta, z)$ for the equipotential ellipsoid

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1; \quad a = e; \quad b = \eta a; \quad L = 2c; \quad \epsilon = \frac{a}{2c}, \tag{B.1}$$

when the applied potential ϕ_0 reads

$$\phi_0(X, Y, Z) = \frac{\alpha}{a} X + \frac{\beta}{a} Y + \frac{\gamma}{2c} Z + \frac{\gamma}{2}. \tag{B.2}$$

We thus seek the total potential function ϕ outside the ellipsoid in the form

$$\begin{aligned}\phi(X, Y, Z) = \phi_0(X, Y, Z) + C_0 \int_\lambda^\infty \frac{du}{\Delta(u)} + C_1 X \int_\lambda^\infty \frac{du}{(a^2+u)\Delta(u)} \\ + C_2 Y \int_\lambda^\infty \frac{du}{(b^2+u)\Delta(u)} + C_3 Z \int_\lambda^\infty \frac{du}{(c^2+u)\Delta(u)},\end{aligned} \tag{B.3}$$

where $\Delta(u) := \{(a^2+u)(b^2+u)(c^2+u)\}^{1/2}$. The ellipsoid (B.1) is associated with $\lambda = 0$ and if d and Q respectively denote its potential and total charge one obtains

$$d = \frac{\gamma}{2} + C_0 \int_0^\infty \frac{du}{\Delta(u)}, \quad Q = 8\pi\epsilon_0 C_0, \quad \frac{\alpha}{a} + C_1 \int_0^\infty \frac{du}{(a^2+u)\Delta(u)} = 0, \tag{B.4}$$

$$\frac{\beta}{a} + C_2 \int_0^\infty \frac{du}{(b^2+u)\Delta(u)} = \frac{\gamma}{2c} + C_3 \int_0^\infty \frac{du}{(c^2+u)\Delta(u)} = 0, \tag{B.5}$$

and the charge surface density σ on $\lambda = 0$ takes the following form:

$$\sigma(X, Y, Z) = \frac{2\epsilon_0}{abc\left\{\dfrac{X^2}{a^4} + \dfrac{Y^2}{b^4} + \dfrac{Z^2}{c^4}\right\}^{1/2}} \left[C_0 + \frac{C_1 X}{a^2} + \frac{C_2 Y}{b^2} + \frac{C_3 Z}{c^2}\right]. \tag{B.6}$$

According to our notation $X = x' = ax$, $Y = y' = ay$, $Z + c = z' = 2cz$ and also

$$v(\theta, z) = \frac{a\sigma f s_\epsilon}{4\pi\epsilon_0}, \quad h^2(z) = 4z(1-z), \quad g_\eta^2(\theta) = \frac{1 + \tan^2\theta}{1 + \tan^2\theta/\eta^2}. \tag{B.7}$$

It follows that

$$f^2 s_\epsilon^2 = 4 g_\eta^4(\theta)\left[z(1-z)\frac{1 + \tan^2\theta/\eta^4}{1 + \tan^2\theta/\eta^2} + \epsilon^2(1-2z)^2\right] = a^2 g_\eta^4 \left[\frac{X^2}{a^4} + \frac{Y^2}{b^4} + \frac{Z^2}{c^4}\right]. \tag{B.8}$$

Case 1

The potential value d is known. Previous equalities immediately lead to

$$v(\theta, z) = \frac{g_\eta^2(\theta)}{2\pi\eta}\left\{\frac{d-\gamma/2}{I_0(\epsilon)} - \frac{\alpha x}{I_1(\epsilon)} - \frac{\beta y}{I_2(\epsilon)} - \frac{\gamma(z-1/2)}{I_3(\epsilon)}\right\}, \quad \frac{Q}{4\pi\epsilon_0 L} = \frac{d-\gamma/2}{I_0(\epsilon)}, \tag{B.9}$$

if the quantities $I_i(\epsilon)$ depending upon (ϵ, η) obey

$$I_0(\epsilon) = \int_0^\infty \frac{(t+4\eta^2\epsilon^2)^{-1/2}\,\mathrm{d}t}{\{(t+1)(t+4\epsilon^2)\}^{1/2}}, \quad I_1(\epsilon) = \int_0^\infty \frac{4\epsilon^2(t+4\epsilon^2)^{-3/2}\,\mathrm{d}t}{\{(t+1)(t+4\eta^2\epsilon^2)\}^{1/2}}, \tag{B.10}$$

$$I_2(\epsilon) = \int_0^\infty \frac{4\eta^2\epsilon^2(t+4\eta^2\epsilon^2)^{-3/2}\,\mathrm{d}t}{\{(t+1)(t+4\epsilon^2)\}^{1/2}}, \quad I_3(\epsilon) = \int_0^\infty \frac{(t+1)^{-3/2}\,\mathrm{d}t}{\{(t+4\epsilon^2)(t+4\eta^2\epsilon^2)\}^{1/2}}. \tag{B.11}$$

Case 2

In this case Q is given and d is unknown. The results are

$$v(\theta, z) = \frac{g_\eta^2(\theta)}{2\pi\eta}\left\{\frac{Q}{4\pi\epsilon_0 L} - \frac{\alpha x}{I_1(\epsilon)} - \frac{\beta y}{I_2(\epsilon)} - \frac{\gamma(z-1/2)}{I_3(\epsilon)}\right\}, \quad d = \frac{\gamma}{2} + \frac{Q I_0(\epsilon)}{4\pi\epsilon_0 L} \tag{B.12}$$

if $I_i(\epsilon)$ remains defined by (B.10), (B.11).

Previous results (B.9) or (B.12) are exact solutions. When ϵ goes to zero the asymptotic estimate of $v(\theta, z)$ comes from the asymptotic expansion of each integral $I_i(\epsilon)$. Such behaviour comes from applying Sellier (1994, Theorem 3). Hence one may check that, as $\epsilon \to 0$,

$$I_0(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2(1+\eta^2)\epsilon^2\log\epsilon + O(\epsilon^2), \tag{B.13}$$

$$I_1(\epsilon) = \frac{2}{\eta+1} + 4\epsilon^2\log\epsilon + O(\epsilon^2), \quad I_2(\epsilon) = \frac{2\eta}{\eta+1} + 4\eta^2\epsilon^2\log\epsilon + O(\epsilon^2), \tag{B.14}$$

$$I_3(\epsilon) = -2\log\epsilon + 2\log[2/(\eta+1)] - 2 - 6(1+\eta^2)\epsilon^2\log\epsilon + O(\epsilon^2). \tag{B.15}$$

Appendix C

For given $z \in]0, 1[$ we introduce for each point P of $\mathcal{E}(z)$ its elliptical angle $\psi_P \in [0, 2\pi]$ such that

$$x_P = h(z)\cos\psi_P; \qquad y_P = \eta h(z)\sin\psi_P; \qquad \tan\theta_P = \eta\tan\psi_P, \tag{C.1}$$

where the last equality outlines the link between the usual polar angle θ_P and this convenient elliptical angle ψ_P associated with each point P of $\mathcal{E}(z)$. Hence $f^2(\theta_P, z) = x_P^2 + y_P^2 = h^2(z)g_\eta^2(\theta_P)$ where the function g_η obeys (6.3) and

$$\frac{\mathrm{d}l_P}{\sqrt{\dfrac{x_P^2}{h^4(z)} + \dfrac{y_P^2}{\eta^4 h^4(z)}}} = \eta h^2(z)\,\mathrm{d}\psi_P; \quad \frac{[f s_0](P)}{\sqrt{\dfrac{x_P^2}{h^4(z)} + \dfrac{y_P^2}{\eta^4 h^4(z)}}} = h^2(z)\frac{1+\tan^2\theta}{1+\dfrac{\tan^2\theta}{\eta^2}}. \tag{C.2}$$

For $M \in \mathcal{E}(z)$ with polar angle θ and elliptical angle ψ the following equality holds:

$$2\log[PM] = \log\{h^2(z)[1-\cos(\psi_P-\psi)][1+\eta^2+(\eta^2-1)\cos(\psi_P+\psi)]\}. \quad \text{(C.3)}$$

When inverting the integral equation $L^{\theta,z}[u] = a(\theta)$ the idea consists in writing the operators $L^{\theta,z}[u]$, $K^z[u]$ and data $a(\theta)$ in terms of elliptical angles ψ_P and ψ by invoking relations (C.1) to (C.3). For $s(\psi_P) = u(P)\{x_P^2/h^4(z) + y_P^2/\eta^4 h^4(z)\}^{1/2}$ we have

$$L^{\theta,z}[u] = -\eta h^2(z)\int_0^{2\pi} s(\psi_P)\log[PM]\,\mathrm{d}\psi_P;\quad K^z[u] = \eta h^2(z)\int_0^{2\pi} s(\psi_P)\,\mathrm{d}\psi_P. \quad \text{(C.4)}$$

This remark (C.4) makes it possible to find the solution u for specific functions $a(\theta)$ by setting $s(\psi_P) = s(\theta_P)$. As a result we present below several useful couples (s, a) with the value of $K^z[u]$ only when it is non-zero. More precisely the reader may check (use for instance Gradshteyn & Ryzhik (1965)) that

$$a = 1;\ s = s_1 = \frac{c_1(z)}{2\pi\eta h^2(z)};\ K^z[u] = c_1(z) = -\left\{\log[\frac{\eta+1}{2}h(z)]\right\}^{-1}, \quad \text{(C.5)}$$

$$a = g_\eta(\theta)\cos\theta,\ s = \frac{(\eta+1)g_\eta(\theta)\cos\theta}{2\pi\eta h^2(z)};\ a = g_\eta(\theta)\sin\theta,\ s = \frac{(\eta+1)g_\eta(\theta)\sin\theta}{2\pi\eta^2 h^2(z)}, \quad \text{(C.6)}$$

$$a = g_\eta^2(\theta);\ s = \frac{b_0 g_\eta^2(\theta) + (1+\eta^2)b_1}{2\pi\eta h^2(z)};\ K^z[u] = \frac{1+\eta^2}{2}c_1(z), \quad \text{(C.7)}$$

$$a = g_\eta^2(\theta)\cos 2\theta;\ s = \frac{b_0 g_\eta^2(\theta)\cos 2\theta + (1-\eta^2)b_1}{2\pi\eta h^2(z)};\ K^z[u] = \frac{1-\eta^2}{2}c_1(z), \quad \text{(C.8)}$$

$$a = g_\eta^2(\theta)\sin 2\theta;\ s = \frac{(\eta+1)^2 g_\eta^2(\theta)\sin 2\theta}{2\pi\eta^2 h^2(z)} = \frac{(\eta+1)^2 g_\eta^2(\theta)\sin 2\psi}{2\pi\eta h^2(z)}, \quad \text{(C.9)}$$

$$a = g_\eta^3(\theta)\cos\theta;\ s = \frac{1}{2\pi\eta h^2(z)}\left\{b_2 g_\eta^3(\theta)\cos\theta + b_3 g_\eta(\theta)\cos\theta\right\}, \quad \text{(C.10)}$$

$$a = g_\eta^3(\theta)\sin\theta;\ s = \frac{1}{2\pi\eta h^2(z)}\left\{b_4 g_\eta^3(\theta)\sin\theta + b_5 g_\eta(\theta)\sin\theta\right\}, \quad \text{(C.11)}$$

$$a = g_\eta^3(\theta)\cos 3\theta;\ s = \frac{1}{2\pi\eta h^2(z)}\left\{b_2 g_\eta^3(\theta)\cos 3\theta + b_6 g_\eta(\theta)\cos\theta\right\}, \quad \text{(C.12)}$$

$$a = g_\eta^3(\theta)\sin 3\theta;\ s = \frac{1}{2\pi\eta h^2(z)}\left\{b_4 g_\eta^3(\theta)\sin 3\theta + b_7 g_\eta(\theta)\sin\theta\right\}, \quad \text{(C.13)}$$

where the quantities b_i depend upon η in the following way:

$$b_0 = \frac{2(\eta+1)^2}{(\eta^2+1)};\ 2b_1 = c_1(z) - b_0;\ b_2 = \frac{3(1+\eta)^3}{1+3\eta^2};\ b_4 = \frac{3(1+\eta)^3}{\eta(\eta^2+3)}, \quad \text{(C.14)}$$

$$b_3 = -\frac{(\eta+1)(1+3\eta)(3+\eta^2)}{2(1+3\eta^2)};\ b_5 = -\frac{(\eta+1)(1+3\eta^2)(3+\eta)}{2(\eta^2+3)}, \quad \text{(C.15)}$$

$$b_6 = \frac{3(\eta-1)(\eta+1)^2(1+3\eta)}{2(1+3\eta^2)};\ b_7 = \frac{3(\eta-1)(\eta+1)^2(3+\eta)}{2(\eta^2+3)}. \quad \text{(C.16)}$$

Accordingly, one derives (6.4) to (6.7). Moreover by introducing for $i \in \{0, 1\}$ the new operators

$$E_0^i[w] = \int_0^{2\pi} w(\theta_P) \log^i[PM]\, \mathrm{d}\theta_P, \quad E_1^i[w] = E_0^i[\cos(\alpha) g_\eta(\alpha) w(\alpha)], \tag{C.17}$$

$$E_2^i[w] = E_0^i[\sin(\alpha) g_\eta(\alpha) w(\alpha)], \quad E_3^i[w] = E_0^i[g_\eta^2 w], \tag{C.18}$$

one notes that $E_k^0[w]$ does not depend on (θ, z) and may easily deduce the following forms for $\mathcal{L}_k^{\theta,z}[t(z_P)w(\theta_P)]$, $k \in \{2, 3\}$:

$$\begin{aligned}
2\mathcal{L}_2^{\theta,z}[t(z_P)w(\theta_P)] &= [h^2 t]^{(2)}(z) E_3^0[w] + [h^2 t^{(2)}](z) E_0^0[w] g_\eta^2(\theta) \\
&\quad -2h(z)[ht]^{(2)}(z) g_\eta(\theta)\{\cos\theta E_1^0[w] + \sin\theta E_2^0[w]\}, \qquad (C.19)\\
\mathcal{L}_3^{\theta,z}[t(z_P)w(\theta_P)] &= (\tfrac{1}{2} - \log 2)\mathcal{L}_2^{\theta,z}[t(z_P)w(\theta_P)] - h^2(z) E_0^0[w] N_z[t] g_\eta^2(\theta) \\
&\quad -E_3^0[w] N_z[h^2 t] + 2h(z) N_z[ht] g_\eta(\theta)\{\cos\theta E_1^0[w] + \sin\theta E_2^0[w]\} \\
&\quad +\tfrac{1}{2} d_1[t](z) E_3^0[w] - \tfrac{1}{2} d_2[t](z) g_\eta(\theta)\{\cos\theta E_1^0[w] + \sin\theta E_2^0[w]\} \\
&\quad +\tfrac{1}{2} d_3[t](z) E_3^1[w] - d_4[t](z) g_\eta(\theta)\{\cos\theta E_1^1[w] + \sin\theta E_2^1[w]\} \\
&\quad +\tfrac{1}{2}[t^{(2)} h^2](z) g_\eta^2(\theta) E_0^1[w] + t(z)[h^{(1)}]^2(z) T[w], \qquad (C.20)
\end{aligned}$$

if the new operators d_i for $i \in \{1, 2, 3, 4\}$, N_z or T satisfy the definitions $N_z[u] = fp \int_0^1 u(x)dx/[2|x - z|^3]$ and

$$d_1[t] = 2hh^{(1)}t^{(1)} + t(hh^{(1)})^{(1)}, \quad d_4[t] = t^{(2)}h^2 + 2hh^{(1)}t^{(1)} + thh^{(2)}, \tag{C.21}$$

$$d_2[t] = hh^{(1)}t^{(1)} + thh^{(2)}, \quad d_3[t] = t^{(2)}h^2 + 4hh^{(1)}t^{(1)} + 2t(hh^{(1)})^{(1)}, \tag{C.22}$$

$$\begin{aligned}
T[w] &= \int_0^{2\pi} \frac{g_\eta^2(\theta_P)\{g_\eta(\theta_P) - \cos(\theta_P - \theta) g_\eta(\theta)\}^2 w(\theta_P)\, \mathrm{d}\theta_P}{g_\eta^2(\theta_P) + g_\eta^2(\theta) - 2\cos(\theta_P - \theta) g_\eta(\theta_P) g_\eta(\theta)} \\
&= E_3^0[w] - \eta^3 \int_0^{2\pi} \frac{\sin^2(\psi_P - \psi) w(\theta_P)[1 + \tan^2\psi_P][1 + \eta^2 \tan^2\psi_P]^{-1}\, \mathrm{d}\psi_P}{[1 - \cos(\psi_P - \psi)][1 + \eta^2 + (\eta^2 - 1)\cos(\psi_P + \psi)]},
\end{aligned} \tag{C.23}$$

where the last form comes from the link (C.1) between $\mathrm{d}\theta_P$ and $\mathrm{d}\psi_P$. Inspection of results (6.6), (6.7) suggests we use three couples (t, w) when calculating the terms $\mathcal{L}_2^{\theta,z}[v_{0,m}]$, $\mathcal{L}_3^{\theta,z}[v_{0,m}]$, $\mathcal{L}_2^{\theta,z}[v_{1,0}]$ and $\mathcal{L}_3^{\theta,z}[v_{1,0}]$. More precisely, we choose

$$t_1(z) = F_m(z) := K^z[t_{0,m}]; \; t_2(z) = h(z) A_1(0, z); \; t_3(z) = h(z) B_1(0, z), \tag{C.24}$$

$$w_1(\theta) = \frac{g_\eta^2(\theta)}{2\pi\eta}; \; w_2(\theta) = -\frac{(\eta + 1) g_\eta^3(\theta)\cos\theta}{4\pi\eta}; \; w_3(\theta) = -\frac{(\eta + 1) g_\eta^3(\theta)\sin\theta}{4\pi\eta^2}. \tag{C.25}$$

Accordingly, definitions (C.17), (C.18) immediately yield $E_j^i[w_k] = 0$ except for the following cases:

$$E_0^0[w_1] = 1; \quad E_3^0[w_1] = \tfrac{1}{2}(1 + \eta^2); \quad E_1^0[w_2] = -\tfrac{1}{4}(\eta + 1); \quad E_2^0[w_3] = -\tfrac{1}{4}\eta(\eta + 1). \tag{C.26}$$

Observe that for $d := x_P^2/h^4(z) + y_P^2/\eta^4 h^4(z)$, for any function $\gamma(P) = \gamma(\psi_P)$,

$$E_0^1[\gamma w_n] = \int_0^{2\pi} [\gamma w_n](P) \log[PM]\, d\theta_P = -L^{\theta,z}[\gamma w_n/(g_\eta^2(\theta) h^2(z)\sqrt{d})]. \quad \text{(C.27)}$$

This yields the following relations:

$$E_0^1[w_2] = \frac{g_\eta(\theta)\cos\theta}{2};\ E_1^1[w_2] = \frac{\eta+1}{4}\left\{\frac{\cos 2\psi}{b_0} + \frac{1}{c_1(z)}\right\};\ E_2^1[w_2] = \frac{\eta^2 \sin 2\psi}{4(\eta+1)}, \quad \text{(C.28)}$$

$$E_0^1[w_3] = \frac{g_\eta(\theta)\sin\theta}{2};\ E_1^1[w_3] = \frac{E_2^1[w_2]}{\eta};\ E_2^1[w_3] = \frac{\eta(\eta+1)}{4}\left\{\frac{1}{c_1(z)} - \frac{\cos 2\psi}{b_0}\right\}, \quad \text{(C.29)}$$

$$E_3^1[w_1] = -\frac{\eta^2+1}{2}\left\{\frac{1-\eta}{2(1+\eta)}\cos 2\psi + \frac{1}{c_1(z)}\right\};\ E_0^1[w_1] = -\frac{1}{c_1(z)}, \quad \text{(C.30)}$$

$$E_3^1[w_2] = \frac{\eta+1}{2b_2} g_\eta^3(\theta)\cos\theta - \frac{b_3}{2b_2} g_\eta(\theta)\cos\theta;\ E_1^1[w_1] = -\frac{g_\eta(\theta)\cos\theta}{\eta+1}, \quad \text{(C.31)}$$

$$E_3^1[w_3] = \frac{\eta+1}{2\eta b_4} g_\eta^3(\theta)\sin\theta - \frac{b_5}{2b_4} g_\eta(\theta)\sin\theta;\ E_2^1[w_1] = -\frac{\eta g_\eta(\theta)\sin\theta}{\eta+1}. \quad \text{(C.32)}$$

Hence, for both cases 1 and 2 we deduce that

$$\mathcal{L}_2^{\theta,z}[v_{0,m}] = \tfrac{1}{4}(\eta^2+1)[h^2 F_m]^{(2)}(z) + [h^2 F_m^{(2)}](z)\tfrac{1}{2} g_\eta^2(\theta) \quad \text{for } m \geqslant 0, \quad \text{(C.33)}$$

$$\mathcal{L}_3^{\theta,z}[v_{0,m}] = O_1[F_m] + O_2[F_m] g_\eta^2(\theta) + O_3[F_m] g_\eta^2(\theta)\cos 2\theta \quad \text{for } m \geqslant 0, \quad \text{(C.34)}$$

$$\mathcal{L}_2^{\theta,z}[v_{1,0}] = \tfrac{1}{4}(\eta+1) h(z) g_\eta(\theta)\{[ht_2]^{(2)}(z)\cos\theta + \eta[ht_3]^{(2)}(z)\sin\theta\}, \quad \text{(C.35)}$$

$$\begin{aligned}\mathcal{L}_3^{\theta,z}[v_{1,0}] = {} & C_1[t_2] g_\eta(\theta)\cos\theta + S_1[t_3] g_\eta(\theta)\sin\theta + C_2[t_2] g_\eta^3(\theta)\cos\theta \\ & + S_2[t_3] g_\eta^3(\theta)\sin\theta + C_3[t_2] g_\eta^3(\theta)\cos 3\theta + S_3[t_3] g_\eta^3(\theta)\sin 3\theta, \quad \text{(C.36)}\end{aligned}$$

if the new operators O_i, C_i and S_i are defined as

$$\begin{aligned}O_1[u] = {} & \tfrac{1}{4}(\eta^2+1)\Bigg\{(\tfrac{1}{2} - \log 2)(h^2 u)^{(2)} - 2N_z[h^2 u] + d_1[u] \\ & + \left[\frac{\eta^2+1}{2(\eta+1)^2} + \log\left(\frac{\eta+1}{2} h\right)\right] d_3[u] \\ & + \left[2 - \frac{4\eta^2}{(1+\eta^2)(1+\eta)^2}\right][h^{(1)}]^2 u\Bigg\}, \quad \text{(C.37)}\end{aligned}$$

$$\begin{aligned}O_2[u] = {} & (\tfrac{1}{2} - \log 2)\frac{h^2 u^{(2)}}{2} - h^2 N_z[u] - \frac{\eta^2+1}{4(1+\eta)^2} d_3[u] + \frac{d_4[u]}{2} \\ & + \log\left(\frac{\eta+1}{2} h\right)\frac{h^2 u^{(2)}}{2} - \frac{\eta}{(1+\eta)^2}[h^{(1)}]^2 u;\quad O_3[u] = -\frac{\eta-1}{2(1+\eta)} d_4[u], \quad \text{(C.38)}\end{aligned}$$

$$C_1[u] = \frac{\eta+1}{4}\left\{(\tfrac{1}{2} - \log 2)h[hu]^{(2)} - 2N_z[hu] + \frac{d_2[u]}{2}\right\} + \frac{(1+3\eta)(3+\eta^2)}{24(1+\eta)^2}d_3[u]$$
$$+\left[\frac{\eta+1}{4}\log\left(\frac{\eta+1}{2}h\right) + \frac{\eta^2+1-4\eta^3}{8(\eta+1)}\right]d_4[u] + \frac{\eta^2(3-\eta)}{2(1+\eta)}[h^{(1)}]^2u, \quad \text{(C.39)}$$

$$\frac{S_1[u]}{\eta} = \frac{\eta+1}{4}\left\{(\tfrac{1}{2} - \log 2)h[hu]^{(2)} - 2N_z[hu] + \frac{d_2[u]}{2\eta}\right\} + \frac{(1+3\eta^2)(3+\eta)}{24(1+\eta)^2}d_3[u]$$
$$+\left[\frac{\eta+1}{4}\log\left(\frac{\eta+1}{2}h\right) + \frac{\eta^3+\eta-4}{8\eta(\eta+1)}\right]d_4[u] + \frac{3\eta-1}{2\eta(1+\eta)}[h^{(1)}]^2u, \quad \text{(C.40)}$$

$$C_2[u] = \frac{1+3\eta^2}{12(1+\eta)^2}d_3[u] + \frac{3(\eta-1)(1+\eta+2\eta^2)}{16(\eta+1)}d_4[u]$$
$$+\frac{h^2u^{(2)}}{4} + \frac{3\eta(1-6\eta+\eta^2)}{8(1+\eta)}[h^{(1)}]^2u, \quad \text{(C.41)}$$

$$S_2[u] = \frac{\eta^2+3}{12(1+\eta)^2}d_3[u] - \frac{3(\eta-1)(2+\eta+2\eta^2)}{16(\eta+1)}d_4[u]$$
$$+\frac{h^2u^{(2)}}{4} + \frac{3(1-6\eta+\eta^2)}{8\eta^2(\eta+1)}[h^{(1)}]^2u, \quad \text{(C.42)}$$

$$C_3[u] = \frac{(\eta-1)(1+\eta+2\eta^2)}{16(\eta+1)}d_4[u] + \frac{\eta(1+\eta^2-6\eta)}{8(\eta+1)}[h^{(1)}]^2u, \quad \text{(C.43)}$$

$$S_3[u] = \frac{(\eta-1)(2+\eta+\eta^2)}{16\eta^2(1+\eta)}d_4[u] - \frac{1+\eta^2-6\eta}{8\eta(\eta+1)}[h^{(1)}]^2u. \quad \text{(C.44)}$$

In view of these results both (4.20) and (5.8) ensure that $K^z[t_{3,0}] = 0$. Accordingly, (6.10) and (4.21) yield (with $t_2(z) = h(z)A_1(0,z)$, $t_3(z) = h(z)B_1(0,z)$),

$$v_{3,0}(\theta,z) = -\frac{g_\eta^2(\theta)}{4\pi\eta}\{b_2(h^3A_3 + C_3[t_2])g_\eta^3(\theta)\cos 3\theta + (h^3B_3 + S_3[t_2])g_\eta^3(\theta)\sin 3\theta$$
$$+b_2(h^3A_1^{(1)} + C_2[t_2])g_\eta^3(\theta)\cos\theta + (h^3B_1^{(1)} + S_2[t_3])g_\eta^3(\theta)\sin\theta$$
$$+\big[(\eta+1)C_1[t_2] + b_3(h^3A_1^{(1)} + C_2[t_2]) + b_6(h^3A_3 + C_3[t_2])\big]g_\eta(\theta)\cos\theta$$
$$+\big[(\eta+1)S_1[t_3] + b_5(h^3B_1^{(1)} + S_2[t_3]) + b_7(h^3B_3 + S_3[t_3])\big]g_\eta(\theta)\sin\theta\}. \quad \text{(C.45)}$$