A general and formal slender-body theory in the non-lifting case

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In this paper, an alternative and integral method is proposed in order to build a formal slender-body theory valid up to high orders with respect to the slenderness ratio ϵ and for a non-lifting body which is not necessarily of circular cross-section. The method consists of asymptotically expanding and inverting the Fredholm integral equation of the second kind bearing on the unknown source density, which may be spread on the boundary of the body. For such a treatment, the concept of integration in the finite-part sense of Hadamard is powerful. The source density is then given up to order $o(\epsilon^3)$ and the pressure coefficient is provided on the body up to order $O(\epsilon^4 \log \epsilon)$. Throughout this paper, special attention is also paid to the main considered case of the axially symmetric slender body.

1. Introduction

A few decades ago, slender-body theory was one of the most popular theories in aerodynamics. Such a success was, in part, explained by the poor abilities of computers and also by the lack of analytical and exact solutions (except, possibly, for peculiar shapes of the body). Nowadays, the constant improvement of numerical tools allows us to compute the flow around bodies of arbitrary form. Nevertheless, it remains highly desirable to be able both to predict, and to explain, at a reasonable time cost, the main features of a flow around a body; clearly the numerical way does not provide the whole answer. Hence, in the case of a slender body, a formal and asymptotic theory remains quite useful.

For such a slender body, it is obviously possible to exhibit a small slenderness ratio ϵ (see § 2) and thereafter seek an asymptotic expansion of the flow with respect to this small parameter. At leading order, it seems quite reasonable to claim that the flow past any cross-section is independent of that past any other. Such an assumption was applied by Munk (1924) to the lateral flow past elongated bodies of revolution. Actually, for further orders or non-lateral flows, two different approaches have been proposed. The first one consists of asymptotically solving the local and differential problem of singular perturbation associated with the potential function. This is achieved by applying the famous method of matched asymptotic expansions (see Ashley & Landhal 1965; Cole & Kevorkian 1981; Van Dyke 1975). The solution up to second order was proposed by Van Dyke (1959) for an axisymmetric flow and by Euvrard (1983) for a non-lifting slender body, not necessarily of circular cross-section. The second point of view is to spread unknown singularities on a subset of the axis of the body. In the case of axisymmetric flow, one actually spreads a source density whose adequate strength (per unit length) and support are obtained

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by solving the integral equation associated with the flow-tangency condition on the boundary of the body. Such an approach has been pionereed by Landweber (1951, 1959) and also carried out by Moran (1963) and detailed by Handelsman & Keller (1967). Recently, Cade (1994) cast some doubts on the legitimacy of such a method. He actually proved that the integral equation successively handled by Moran (1963) and Handelsman & Keller (1967) does not, in general, possess solutions.

As far as the author knows, the integral approach has only been applied to deal with the axisymmetric flow. Moreover, at the end of his paper devoted to analytical aspects of slender body theory, Tuck (1992) writes that the method of matched asymptotic expansions seems to be 'the only sensible way to discuss slender bodies with a general cross-section'. The aim of this work is to present an alternative treatment valid in the non-lifting case and for a body not necessarily of circular cross-section and consisting in spreading this time a source density on the exact boundary. The flow-tangency condition leads to a Fredholm integral equation of the second kind for this unknown distribution. This equation is asymptotically expanded and solved by using the basic concept of integration in the finite-part sense of Hadamard. More precisely, this paper is organized as follows. After giving general assumptions, the next section presents the basic integral equation bearing on the source density to be spread on the boundary of the slender body. Such an integral equation is asymptotically expanded with respect to the slenderness ratio ϵ, which compares the thickness of the body to its length, in § 3. Such a step is achieved by invoking a general formula, as detailed in Appendix A. The asymptotic estimate of the solution is thereafter given in § 4, up to order $o(\epsilon^3)$, whereas § 5 exhibits the asymptotic expansion of pressure coefficient on the body, up to order $O(\epsilon^4 \log \epsilon)$.

2. The governing integral equation

(a) General assumptions and notations

Throughout this paper we consider (see figure 1) a rigid and slender body $\mathcal{A}'$ at rest, which is actually an open and bounded subset of $\mathbb{R}^3$, such that there exist a set of cylindrical coordinates (r', θ, z'), with usual associated unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ and a smooth and positive function $F(\theta, z')$, fulfilling the following properties.

(i) For each point $P = (r'_P, \theta_P, z'_P)$ belonging to the boundary $\partial\mathcal{A}'$ of $\mathcal{A}'$ then $0 \leqslant z'_P \leqslant L$, $0 \leqslant \theta_P \leqslant 2\pi$ and $r'_P = F(\theta_P, z'_P)$, with $F(\theta_P, 0) = F(\theta_P, L) = 0$. The body may admit pointed ends O' and E'. Moreover, if $z'_P \in [0, L] \setminus \{0, L\}$, then $F(\theta_P, z'_P) > 0$ and $F(0, z'_P) = F(2\pi, z'_P)$ for $0 \leqslant z'_P \leqslant L$.

(ii) If $e = \max[F(\theta, z')]$ for $(\theta, z') \in [0, 2\pi] \times [0, L]$, then the slenderness ratio $\epsilon = e/L$ obeys $0 < \epsilon \ll 1$.

(iii) If non-dimensional coordinates (r, z) are introduced by $M = (r', \theta, z') = (er, \theta, Lz)$, then the new positive shape function f defined by $F(\theta, z') = ef(\theta, z)$ is such that $|\partial^i_v f(\theta, z)| = O(1)$ for $(\theta, z) \in [0, 2\pi] \times [0, 1]$, $v \in \{\theta, z\}$, $i \in \mathbb{N}$ and $\partial^i_v f(\theta, z) := \partial^i f(\theta, z)/\partial v^i$. Finally, $\mathcal{A}$ and $\partial\mathcal{A}$, respectively, denote the new body obeying $r = f(\theta, z)$ and its boundary with associated end points O and E.

For further questions it is also worth intoducing the set of Cartesian coordinates (O', x', y', z'), with $M = (x', y', z') = (ex, ey, Lz)$ and unit vectors $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ directed as shown by figure 1.

For our non-lifting body $\mathcal{A}'$, the problem consists of studying the steady, incompressible and irrotational flow of an inviscid, constant density (ρ_∞) fluid presenting at infinity (when $z' \to -\infty$) given velocity $\boldsymbol{u}_\infty = u_\infty \cos(\alpha\epsilon)\boldsymbol{e}_z + u_\infty \sin(\alpha\epsilon)\boldsymbol{e}_x$ with

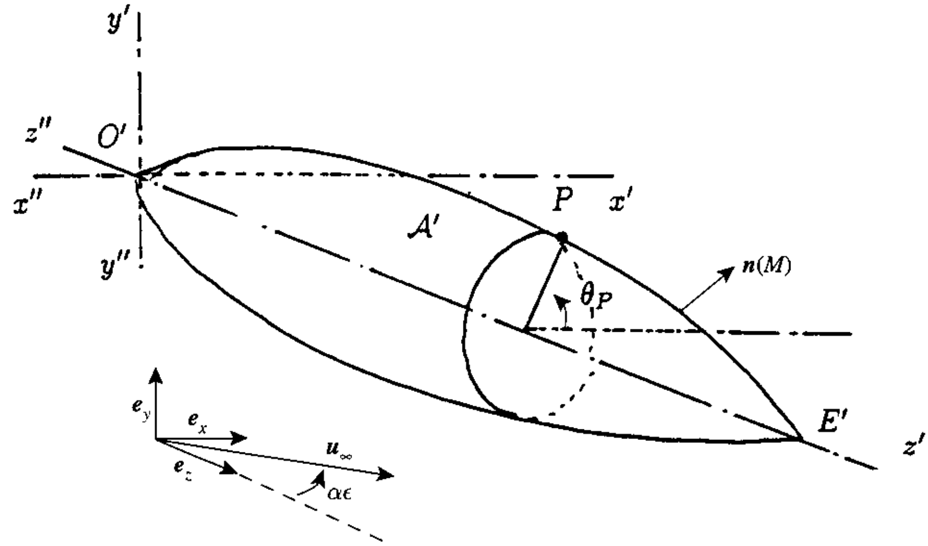

Figure 1.

$\alpha = O(1)$ and pressure p_∞. For this irrotational flow, the fluid velocity $\boldsymbol{u}(M)$ writes $\boldsymbol{u}(M) = \boldsymbol{u}_\infty + \mathrm{grad}_M[\phi]$, where potential function ϕ obeys the well-known differential problem, noted $(\mathcal{P})$,

$$\Delta\phi := \mathrm{div}[\mathrm{grad}(\phi)] = 0, \quad \text{in } \mathbb{R}^3 \setminus (\mathcal{A}' \cup \partial\mathcal{A}'), \tag{2.1}$$

$$\mathrm{grad}[\phi](M) \to \mathbf{0}, \quad \text{as } z' \to -\infty, \tag{2.2}$$

$$\mathrm{grad}[\phi](M) \cdot \boldsymbol{n}(M) = -\boldsymbol{u}_\infty \cdot \boldsymbol{n}(M), \quad \text{for } M \in \partial\mathcal{A}' \setminus \{O', E'\}, \tag{2.3}$$

where (see figure 1) $\boldsymbol{n}(M)$ designates the unit vector at point M of $\partial\mathcal{A}' \setminus \{O', E'\}$, which is directed outwards $\mathcal{A}'$, and the equation (2.3) is the flow-tangency condition $\boldsymbol{u}(M)\cdot\boldsymbol{n}(M) = 0$ on the body. This equation (2.3) is imposed for $M \in \partial\mathcal{A}'\setminus\{O', E'\}$, since the ends O' or E' may be pointed. Once the well-posed problem $(\mathcal{P})$ is solved, one is both eager and able to deduce quantities of physical interest such as pressure coefficient C_p everywhere in the flow, especially on the body itself. By invoking the usual Bernoulli's theorem, one indeed obtains the following convenient link between the pressure p and the field velocity $\boldsymbol{u}(M)$:

$$C_\mathrm{p}(M) := \frac{2[p(M) - p_\infty]}{\rho_\infty u_\infty^2} = 1 - \left[\frac{\boldsymbol{u}(M)}{u_\infty}\right]^2, \quad \text{for } M \in \mathbb{R}^3 \setminus \mathcal{A}'. \tag{2.4}$$

Note that in the special case of an axially symmetric flow (obtained for an axially symmetric body and $\alpha = 0$), one could also solve the problem bearing on the associated stream function Ψ (see Handelsman & Keller 1967).

(b) A Fredholm integral equation of the second kind

If the unique solution of the well-posed problem $(\mathcal{P})$ may be approximated, with not so much additional effort, by available numerical methods, it remains quite impossible to analytically express this solution for such an arbitrary body shape. As outlined in the introduction, the occurrence of the small slenderness ratio ϵ, allows us to build an asymptotic estimate of the unknown potential ϕ by resorting to two possible approaches: the method of matched asymptotic expansions; or the inversion of an integral equation bearing on a source density to spread on the axis of the body. Here we choose to spread singularities on the boundary $\partial A'$ itself. Thanks to the third Green's identity (see Kellogg 1953), it is indeed well known that any potential function may be obtained by spreading on $\partial\mathcal{A}'$ source and normal doublet distributions. For our non-lifting body, we restrict this choice to a source distribution on $\partial\mathcal{A}'$. Accordingly, the potential function ϕ writes, for $M \in \mathbb{R}^3 \setminus \mathcal{A}'$,

$$\phi(M) = -\frac{1}{4\pi}\iint_{\partial\mathcal{A}'} \frac{q(P)}{PM}\mathrm{d}S'_P, \tag{2.5}$$

where the source density $q(P)$ is unknown. Such a potential not only exists and is smooth in $\mathbb{R}^3 \setminus \mathcal{A}'$, but also fulfils conditions (2.1) and (2.2). Thus, the source density q is determined by imposing the flow-tangency condition (2.3). More precisely (see Kellogg 1953), the equality (2.3) leads, for the unknown function q, to the following Fredholm integral equation of the second kind:

$$\tfrac{1}{2}q(M) + \frac{1}{4\pi}\iint_{\partial\mathcal{A}'} q(P)\frac{\boldsymbol{PM}\cdot\boldsymbol{n}(M)}{PM^3}\mathrm{d}S'_P = -\boldsymbol{u}_\infty\cdot\boldsymbol{n}(M), \quad M \in \partial\mathcal{A}' \setminus \{O', E'\}. \tag{2.6}$$

If $u_\infty w(M)$ denotes the normal velocity induced at point M of $\partial\mathcal{A} \setminus \{O, E\}$ by the perturbation potential ϕ (see (2.5)), and $d(M) = -\boldsymbol{u}_\infty \cdot \boldsymbol{n}(M)/u_\infty$, then (2.6) writes $w(M) = d(M)$ for $M \in \partial\mathcal{A}' \setminus \{O', E'\}$.

3. Asymptotic expansion of the integral equation

(a) *An integral equation depending on a small parameter*

In order to exhibit the small parameter ϵ, we rewrite (2.6) in terms of the non-dimensional variables (r_P, z_P, r, z). First, the reader may check that

$$\boldsymbol{n}(M) = \frac{\boldsymbol{e}_r - (f^{-1}f^1_\theta)(\theta,z)\boldsymbol{e}_\theta - \epsilon f^1_z(\theta,z)\boldsymbol{e}_z}{s_\epsilon(M)}, \quad \frac{\mathrm{d}S'_P}{eL} = [fs_\epsilon](P)\,\mathrm{d}\theta_P\,\mathrm{d}z_P, \tag{3.1}$$

if the basic function s_ϵ obeys the following definition:

$$s_\epsilon(M) := \{1 + (f^{-1}f^1_\theta)^2(\theta,z) + [\epsilon f^1_z(\theta,z)]^2\}^{1/2}. \tag{3.2}$$

The integral arising on the left-hand side of (2.6) is regular. This feature justifies the application of a change of variables $(r'_P, \theta_P, z'_P) = (er_P, \theta_P, Lz_P)$ and of Fubini's theorem. Thus one obtains, after some algebra, and for $(\theta, z) \in [0, 2\pi]\times]0, 1[$,

$$\frac{q(M)s_\epsilon(M)}{2u_\infty} + \frac{\epsilon^2}{4\pi u_\infty}\int_0^{2\pi}\left[\int_0^1 \frac{A(\theta_P,z_P,\theta,z)[qfs_\epsilon](P)}{[(z_P-z)^2+\epsilon^2H^2(\theta_P,z_P,\theta,z)]^{3/2}}\mathrm{d}z_P\right]\mathrm{d}\theta_P$$
$$= \epsilon\cos(\alpha\epsilon)f^1_z(\theta,z) - \sin(\alpha\epsilon)[\cos(\theta) + \sin(\theta)(f^{-1}f^1_\theta)(\theta,z)], \tag{3.3}$$

where the new functions A and H depend on $(\theta_P, z_P, \theta, z)$ and obey

$$H = \{f^2(\theta_P,z_P) + f^2(\theta,z) - 2\cos(\theta_P-\theta)f(\theta,z)f(\theta_P,z_P)\}^{1/2},$$
$$A = (z_P-z)f^1_z(\theta,z) + f(\theta,z) - f(\theta_P,z_P)\big[\cos(\theta_P-\theta) - \sin(\theta_P-\theta)(f^{-1}f^1_\theta)(\theta,z)\big].$$

Observe that $H(\theta_P, z_P, \theta, z)$ is zero if and only if $\theta_P = \theta$ and $f(\theta_P, z_P) = f(\theta, z)$. Hence, the integral arising on the left-hand side of (3.3) rewrites $\int_0^{2\pi} B(\theta_P, \theta, z)\mathrm{d}\theta_P$, where B designates a regular integration for $\theta_P \in [0, 2\pi] \setminus \{\theta\}$. The Fredholm integral equation (3.3) clearly depends on the slenderness parameter ϵ and the aim of this study is to asymptotically invert this equation by taking into account not only the basic assumption $0 < \epsilon \ll 1$, but also additional conditions bearing on the boundary $\partial\mathcal{A}$ of the non-dimensional slender body $\mathcal{A}$. As will be discussed, and for an asymptotic expansion of the unknown density $q(\theta, z)$, with respect to ϵ, up to order $O(\epsilon^N)$, these new assumptions will deeply depend on order N but also on the subset of $]0, 1[$, where such an asymptotic solution holds. Clearly, inspection of equality (3.3) at least suggests to ensure condition $|f^1_{z_P}(\theta_P, z_P)| = O(1)$ and $|(f^{-1}f^1_\theta)(\theta, z)| = O(1)$ in order to sort the different terms for an asymptotic treatment. Usually, a slender-body theory is said to be formal (see Tuck 1992) as soon as these conspicuous restrictions are disregarded. Of course, such a formal theory may present a lack of accuracy near the points where restrictions break down, i.e. near the edges of the body. It remains of interest to give examples for which the formal theory exactly applies up to the chosen order of approximation since the needed improvements to formal theory require an extensive and tedious study (especially for high orders) of the flow in the vincinity of end points which is achieved by employing the matched asymptotic expansions technique (see Van Dyke 1954). Despite its potential drawbacks, a formal theory up to high orders for an arbitrary shape remains of interest and often allows us to deduce good approximations of integrated quantities (see Tuck 1992).

(*b*) *A systematic method for expanding the integral equation*

At a first stage, it is only assumed that $|f^1_{z_P}(\theta_P, z_P)| = O(1)$ for $(\theta_P, z_P) \in [0, 2\pi] \times]0, 1[$ and a formal asymptotic expansion of equation (3.3) is thereafter derived by disregarding the potential restrictions. Under this assumption, and if $|f^{-1}f^1_\theta(\theta, z)| = O(1)$ for the considered point $M = (f(\theta, z), \theta, z)$ of $\partial\mathcal{A}$, the introduction of the new unknown $\lambda = u_\infty^{-1} q$ and the use for function $s_\epsilon(P)$ of its Taylor expansion, allows us to rewrite (3.3) as

$$\begin{aligned}
&\tfrac{1}{2}\lambda(\theta, z)\sqrt{1 + (f^{-1}f^1_\theta)^2(\theta, z)} + \int_0^{2\pi} I^{\theta,z}_{\epsilon,\theta_P}[\lambda(\theta_P, z_P) g_0(\theta, z, \theta_P, z_P)]\,\mathrm{d}\theta_P \\
&\quad + \epsilon^2 \left\{ \frac{\lambda(\theta, z)[f^1_z(\theta, z)/2]^2}{\sqrt{1 + (f^{-1}f^1_\theta)^2(\theta, z)}} + \int_0^{2\pi} I^{\theta,z}_{\epsilon,\theta_P}[\lambda(\theta_P, z_P) g_2(\theta, z, \theta_P, z_P)[1 + O(\epsilon^2)]]\,\mathrm{d}\theta_P \right\} \\
&\quad = \epsilon\cos(\alpha\epsilon) f^1_z(\theta, z) - \sin(\alpha\epsilon)[\cos(\theta) + \sin(\theta)(f^{-1}f^1_\theta)(\theta, z)] + O[\lambda(\theta, z)\epsilon^4], \quad (3.4)
\end{aligned}$$

where the new functions g_0 and g_2 obey

$$g_0(\theta, z, \theta_P, z_P) = f(\theta_P, z_P)\sqrt{1 + (f^{-1}f^1_{\theta_P})^2(\theta_P, z_P)}\,A(\theta_P, z_P, \theta, z), \quad (3.5)$$

$$g_2(\theta, z, \theta_P, z_P) = \frac{f(\theta_P, z_P)[f^1_{z_P}(\theta_P, z_P)]^2 A(\theta_P, z_P, \theta, z)}{2\sqrt{1 + (f^{-1}f^1_{\theta_P})^2(\theta_P, z_P)}}, \quad (3.6)$$

and the important linear operator $I^{\theta,z}_{\epsilon,\theta_P}$ is defined for $\theta_P \in [0, 2\pi] \setminus \{\theta\}$ and satisfies

$$\begin{aligned}
I^{\theta,z}_{\epsilon,\theta_P}[v(z_P)] &= \frac{\epsilon^2}{4\pi}\int_0^1 \frac{v(z_P)\,\mathrm{d}z_P}{[(z_P - z)^2 + \epsilon^2 H^2(\theta_P, z_P, \theta, z)]^{3/2}} \\
&= \frac{\epsilon^2}{4\pi}\int_{-z}^{1-z} \frac{v(u + z)\,\mathrm{d}u}{[u^2 + \epsilon^2 h^2(u)]^{3/2}}.
\end{aligned}$$

In this latter equality, the change of variable $z_P = z + u$ has been applied to the regular integration over $[0, 1]$ (remember that H is non-zero for $\theta_P \neq \theta$) and clearly h obeys $h(u) := H(\theta_P, u + z, \theta, z)$.

The next key step consists in expanding the above quantity $I^{\theta,z}_{\epsilon,\theta_P}[v(z_P)]$ with respect to the small slenderness parameter ϵ. This task leads us to handle, for function $w(u)$, the new integral

$$J(\epsilon) = \int_{-z}^{1-z} \frac{w(u)\,\mathrm{d}u}{[u^2 + \epsilon^2 h^2(u)]^{3/2}}. \tag{3.7}$$

Since $z \in\,]0, 1[$, then $0 \in\,]-z, 1-z[$ and by setting, without any caution, $\epsilon = 0$ in (3.7), one gets a hypersingular integral. Such a property is typical of a singular expansion of $J(\epsilon)$ with respect to ϵ. Thereafter, it is not at all trivial to deal with $J(\epsilon)$ and one may be tempted to derive the associated expansion by applying the method of matched asymptotic expansions, i.e. by introducing two subsets of $]-z, 1-z[$; the inner one where $u = O(\epsilon)$ (since $h(0) > 0$), the outer one where $u = O(1)$, and thereafter matching the associated expansions via adequate matching rules (see Van Dyke 1975). Such a technique actually presents a substantial drawback since it requires algebra of monumental complexity as the order of approximation increases. Moreover, it is quite impossible to derive with this method a systematic formula, i.e. valid up to any order provided that both functions w and h fulfil adequate smoothness assumptions. It is worth noting that $J(\epsilon)$ is a special case of the next class of integrals

$$M_h(\epsilon) = \mathrm{fp} \int_{\mathcal{D}} w(u) K[u, \epsilon h(u)]\,\mathrm{d}u, \tag{3.8}$$

where $\mathcal{D}$ denotes an open subset of $\mathbb{R}$ containing zero and K is a kernel function which may be singular at $(0, 0)$ and satisfies, for the present work, a pseudo-homogeneous property: $K(\alpha u, \alpha v) = \mathrm{sgn}(\alpha)\alpha^Q K(u, v)$, for $\alpha \neq 0$, with $\mathrm{sgn}(\alpha) = \alpha/|\alpha|$ and Q an integer (positive or negative). Observe that the integration over $\mathcal{D}$ is to handle in the finite-part sense of Hadamard (this explains the occurrence of the symbol fp) and for detailed explanations regarding this basic concept the reader is referred to Hadamard (1932), Lavoine (1959), Schwartz (1966) and Sellier (1994). Such a notion joins the usual Lebesgue's integration as soon as the integrand is regular. For $\epsilon > 0$, the property of kernel K yields

$$M_h(\epsilon) = \epsilon^Q\, \mathrm{fp} \int_{\mathcal{D}} w(u) K[\epsilon^{-1} u, h(u)]\,\mathrm{d}u. \tag{3.9}$$

Hence, one recognizes a specific integral depending on large parameter ϵ^{-1}. This feature suggests the use of available methods such as integration by parts, Mellin's transform (see Bleistein & Handelsman 1975; Wong 1989), or the recent and powerful distributional approach developed by Estrada & Kanwal (1990, 1994). Unfortunately, none of these techniques apply to the present case. Thereafter, a systematic formula has been derived (see Sellier 1996) in order to expand $M_h(\epsilon)$. The result is obtained by extending an earlier work (Sellier 1994) presenting a new method for expanding a class of integrals by using the concept of integration in the finite-part sense of Hadamard. One important feature of this method is to express the expansion in terms of an asymptotic sequence $\epsilon^\gamma \log^m \epsilon$, whose associated coefficients may be integrals in the sense of Hadamard, even if the initial integral $M_h(\epsilon)$ turns out to be a regular integration. The general asymptotic formula, together with the conditions to be checked by functions w and h, are presented in Appendix A. By the way, this result provides, for $-\infty < a < x < b < +\infty$, the asymptotic behaviour of the classical integral

$$M_\epsilon^x[w] = \int_a^b \frac{w(\xi)\,\mathrm{d}\xi}{\sqrt{(\xi - x)^2 + \epsilon^2}} = \int_{a-x}^{b-x} \frac{w(u+x)\,\mathrm{d}u}{\sqrt{u^2+\epsilon^2}}, \tag{3.10}$$

which is obtained by choosing $h(u) = 1$, $K(u,v) = [u^2+v^2]^{-1/2}$. Integral $M_\epsilon^x[w]$ indeed plays a central role (see Tuck 1992) in a formal slender-body theory for a body of revolution. Usually, such an expansion is derived by taking a Fourier (see Thwaites 1960) or Laplace (see Ursell 1962; Tuck 1992) transform of $M_\epsilon^x[w]$ and inverting the associated transform term to term for the expansion of a transformed integral. Such a method is restrictive since it makes assumptions for the coefficients associated to the behaviour of the transform of $M_\epsilon^x[w]$.

For $\theta_P \in [0, 2\pi] \setminus \{\theta\}$, the following asymptotic behaviour holds (see Appendix A):

$$I_{\epsilon,\theta_P}^{\theta,z}[v(z_P)] = I_{0,\theta_P}^{\theta,z}[v(z_P)] + I_{1,\theta_P}^{\theta,z}[v(z_P)]\epsilon^2\log\epsilon + I_{2,\theta_P}^{\theta,z}[v(z_P)]\epsilon^2 + o(v_M\epsilon^2), \tag{3.11}$$

with $v_M = \max_{z\in\,]0,1[}|v(z)|$ and the definitions

$$I_{0,\theta_P}^{\theta,z}[v] = \frac{v(z)}{2\pi H^2(\theta_P, z, \theta, z)}, \quad I_{1,\theta_P}^{\theta,z}[v] = -\frac{1}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}z_P^2}[v(z_P)]_{z_P=z}, \tag{3.12}$$

$$4\pi I_{2,\theta_P}^{\theta,z}[v] = \mathrm{fp}\int_0^1 \frac{v(z_P)\,\mathrm{d}z_P}{|z_P - z|^3} - \frac{\mathrm{d}^2}{\mathrm{d}z_P^2}[\{1+\log[\tfrac{1}{2}H(\theta_P, z_P, \theta, z)]\}v(z_p)]_{z_P=z}. \tag{3.13}$$

Observe that the leading term $I_{0,\theta_P}^{\theta,z}[v(z_P)]$ is a local quantity since it only involves functions v and H at point z. It is actually a two-dimensional contribution of the whole section $z_P = z$. Remaining terms $I_{1,\theta_P}^{\theta,z}[v(z_P)]$ and $I_{2,\theta_P}^{\theta,z}[v(z_P)]$ are, respectively, weakly and strongly three-dimensional corrections because they require the knowledge of the functions v or H, respectively, in a neighbourhood of z and on $]0,1[$. Keeping in mind the assumption $|(f^{-1}f_\theta^1)(\theta,z)| = O(1)$, (3.11) yields, for the integral equation (3.4), the next asymptotic expansion

$$\begin{aligned}
&\tfrac{1}{2}\lambda(\theta,z)\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)} + \int_0^{2\pi} I_{0,\theta_P}^{\theta,z}[\lambda g_0]\,\mathrm{d}\theta_P + \left\{\int_0^{2\pi} I_{1,\theta_P}^{\theta,z}[\lambda g_0]\,\mathrm{d}\theta_P\right\}\epsilon^2\log\epsilon \\
&\quad + \left\{\frac{\lambda(\theta,z)[\tfrac{1}{2}f_z^1(\theta,z)]^2}{\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}} + \int_0^{2\pi}(I_{2,\theta_P}^{\theta,z}[\lambda g_0] + I_{0,\theta_P}^{\theta,z}[\lambda g_2])\,\mathrm{d}\theta_P\right\}\epsilon^2 + o[\lambda(\theta,z)\epsilon^2] \\
&\quad = \epsilon f_z^1(\theta,z)[1-\tfrac{1}{2}\alpha^2\epsilon^2] - [\cos(\theta)+\sin(\theta)(f^{-1}f_\theta^1)(\theta,z)]\left[\alpha\epsilon - \frac{\alpha^3}{3!}\epsilon^3\right] + o(\epsilon^3).
\end{aligned} \tag{3.14}$$

In view of (3.4), the left-hand side of (3.14) is actually the asymptotic expansion of $w(M)s_\epsilon(M)$, whereas the right-hand side provides the behaviour of $d(M)s_\epsilon(M)$. Hence, it is straightforward to derive the asymptotic expansion of relation $w(M) = d(M)$, for $M \in \partial\mathcal{A}' \setminus \{O, E\}$. One immediately obtains

$$\begin{aligned}
w(M) = w(\theta,z) &= L_0^{\theta,z}[\lambda] + L_1^{\theta,z}[\lambda]\epsilon^2\log\epsilon + L_2^{\theta,z}[\lambda]\epsilon^2 + o[\lambda(\theta,z)\epsilon^2] \\
&= d_1(\theta,z)\epsilon + d_3(\theta,z)\epsilon^3 + o(\epsilon^3),
\end{aligned} \tag{3.15}$$

if the functions $d_1(\theta,z)$ and $d_3(\theta,z)$ obey

$$d_1(\theta,z) = \frac{f_z^1(\theta,z) - \alpha b(\theta,z)}{[1+(f^{-1}f_\theta^1)^2(\theta,z)]^{1/2}}, \quad b(\theta,z) = \cos(\theta) + \sin(\theta)(f^{-1}f_\theta^1)(\theta,z), \tag{3.16}$$

$$d_3(\theta,z) = \frac{b(\theta,z)\alpha^3 - 3f_z^1(\theta,z)\alpha^2}{6[1+(f^{-1}f_\theta^1)^2(\theta,z)]^{1/2}} - \frac{[f_z^1(\theta,z)-\alpha b(\theta,z)][f_z^1(\theta,z]^2}{2[1+(f^{-1}f_\theta^1)^2(\theta,z)]^{3/2}}, \tag{3.17}$$

and, thanks to the above results (3.12)–(3.13), the new linear operators $L_i^{\theta,z}$ are defined for $i \in \{0,1,2\}$ as

$$L_0^{\theta,z}[\lambda] = \tfrac{1}{2}\lambda(\theta,z) + \frac{1}{2\pi\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}}\int_0^{2\pi}\frac{\lambda(\theta_P,z)g_0(\theta,z,\theta_P,z)}{H^2(\theta_P,z,\theta,z)}\mathrm{d}\theta_P, \tag{3.18}$$

$$L_1^{\theta,z}[\lambda] = -\frac{1}{4\pi\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}}\int_0^{2\pi}\frac{\mathrm{d}^2}{\mathrm{d}z_P^2}[\lambda(\theta_P,z_P)g_0(\theta,z,\theta_P,z_P)]_{z_P=z}\,\mathrm{d}\theta_P, \tag{3.19}$$

$$\begin{aligned} L_2^{\theta,z}[\lambda] = \frac{1}{4\pi\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}}\int_0^{2\pi}\Bigg\{&\mathrm{fp}\int_0^1\frac{\lambda(\theta_P,z_P)g_0(\theta,z,\theta_P,z_P)\,\mathrm{d}z_P}{|z_P-z|^3} \\ &-\frac{\mathrm{d}^2}{\mathrm{d}z_P^2}[\{1+\log[\tfrac{1}{2}H(\theta_P,z_P,\theta,z)]\}\lambda(\theta_P,z_P)g_0(\theta,z,\theta_P,z_P)]_{z_P=z} \\ &+\frac{\lambda(\theta_P,z)A(\theta_P,z,\theta,z)}{H^2(\theta_P,z,\theta,z)}\left[\frac{[f_z^1(\theta_P,z)]^2}{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z)} - \frac{[f_z^1(\theta,z)]^2}{1+(f^{-1}f_\theta^1)^2(\theta,z)}\right]\Bigg\}\mathrm{d}\theta_P. \end{aligned} \tag{3.20}$$

Observe that the last contribution on the right-hand side of (3.20) vanishes as soon as the body is of revolution.

(c) *Physical interpretation of quantity $L_0^{\theta,z}[\lambda]$*

If $\mathcal{P}(z')$ designates the plane $z'=z'_P$, then, for $z' \in]0,L[$, the closed path $\partial\mathcal{C}'(z') = \partial\mathcal{A}'\cap\mathcal{P}(z')$ is the boundary of the cross-section $\mathcal{C}'(z')$ of body $\mathcal{A}'$ at z' (see figure 2).

In plane $\mathcal{P}(z')$, the potential function $\phi_{2\mathrm{D}}$, due to a source distribution over $\partial\mathcal{C}'(z')$ and of lineic strength $q(P)$ for $P=(F(\theta_P,z'),\theta_P,z') \in \partial\mathcal{C}'(z')$, obeys

$$\phi_{2\mathrm{D}}(M) = \frac{1}{2\pi}\oint_{\partial\mathcal{C}'(z')} q(P)\log[PM]\,\mathrm{d}l'_P, \quad \text{for } M=(r',\theta,z')\in\mathcal{P}(z') \tag{3.21}$$

and the associated induced velocity $\boldsymbol{u}_{2\mathrm{D}}(M) = \mathrm{grad}_{2\mathrm{D}}[\phi_{2\mathrm{D}}(M)]$ satisfies, for $M \in \partial\mathcal{C}'(z')$ (and outside $\mathcal{C}'(z')$),

$$\boldsymbol{u}_{2\mathrm{D}}(M)\cdot\boldsymbol{n}_{2\mathrm{D}}(M) = \tfrac{1}{2}q(M) + \frac{1}{2\pi}\oint_{\partial\mathcal{C}'(z')}\frac{q(P)\boldsymbol{PM}\cdot\boldsymbol{n}_{2\mathrm{D}}(M)}{PM^2}\,\mathrm{d}l'_P, \quad M\in\partial\mathcal{C}'(z'), \tag{3.22}$$

where (see figure 2) $\boldsymbol{n}_{2\mathrm{D}}(M)$ is the unit vector normal to $\partial\mathcal{C}(z')$ and directed outwards $\mathcal{C}(z')$. Use of new coordinates with $P=(F(\theta_P,z'),\theta_P,z') = (ef(\theta_P,z),\theta_P,Lz)$ yields, for $w_{2\mathrm{D}}(M) = u_\infty^{-1}\boldsymbol{u}_{2\mathrm{D}}(M)\cdot\boldsymbol{n}_{2\mathrm{D}}(M)$ and $\lambda(P)=u_\infty^{-1}q(P)$, the relation

$$w_{2\mathrm{D}}(M) = \tfrac{1}{2}\lambda(\theta,z) + \frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda(\theta_P,z)f(\theta_P,z)\sqrt{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z)}\,\mathrm{d}\theta_P}{[A(\theta_P,z,\theta,z)]^{-1}H^2(\theta_P,z,\theta,z)\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}}. \tag{3.23}$$

Clearly, definitions (3.5) and (3.18) show that $w_{2\mathrm{D}}(M) = L_0^{\theta,z}[\lambda]$. According to Zabreyko (1975), the integral equation $L_0^{\theta,z}[\lambda]=0$ admits only the trivial solution $\lambda=0$.

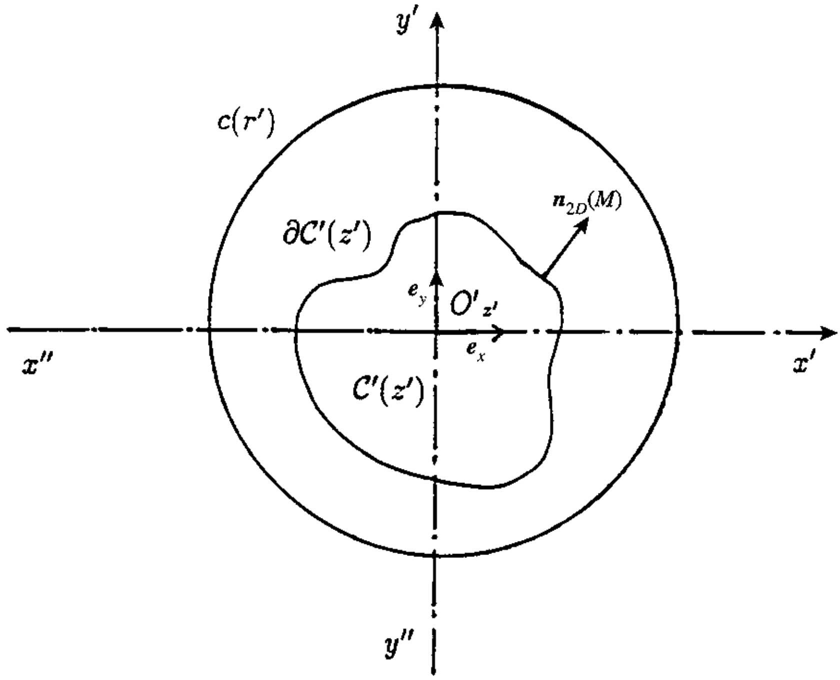

Figure 2.

4. Asymptotic solution for potential function

(*a*) *Building a formal asymptotic solution*

The key equation (3.15) suggests we seek the following asymptotic estimate for the unknown source distribution λ:

$$\lambda(\theta, z) = \lambda_1(\theta, z)\epsilon + \lambda_2(\theta, z)\epsilon^3 \log \epsilon + \lambda_3(\theta, z)\epsilon^3 + o(\epsilon^3), \tag{4.1}$$

with $|\lambda_i(\theta, z)| = O(1)$ for $i \in \{1, 2, 3\}$. As outlined in the last subsection, the homogeneous integral equation $L_0^{\theta,z}[v] = 0$ admits only the zero solution and this justifies why the first term on the right-hand side of (4.1) is of order ϵ. Similar arguments, combined with the form of equation (3.15), easily lead to sequence (4.1). Such a choice is also consistent with the link between the remainders, i.e. $o[\lambda(\theta, z)\epsilon^2] = o(\epsilon^3)$. By reintroducing the behaviour (4.1) in equality (3.15), one indeed easily deduces the following set of two-dimensional problems:

$$L_0^{\theta,z}[\lambda_1] = d_1(\theta, z), \tag{4.2}$$

$$L_0^{\theta,z}[\lambda_2] = -L_1^{\theta,z}[\lambda_1], \tag{4.3}$$

$$L_0^{\theta,z}[\lambda_3] = d_3(\theta, z) - L_2^{\theta,z}[\lambda_1]. \tag{4.4}$$

Thus, the solution is built by successively inverting at each order the two-dimensional operator $L_0^{\theta,z}$. The above system is actually triangular in the sense that it only requires, at each order, to know the previous corrections. In view of this feature, a crucial step consists in solving the two-dimensional problem $L_0^{\theta,z}[\lambda] = d(\theta, z)$, for given function d.

(b) A general property and its consequences

Thanks to (3.21)–(3.22), the problem to solve for given functions $f(\theta, z)$ and $d(\theta, z)$ is written

$$d(\theta,z) = \tfrac{1}{2}\lambda(\theta,z) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\lambda(\theta_P,z)f(\theta_P,z)\sqrt{1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z)}}{\sqrt{1+(f^{-1}f^1_\theta)^2(\theta,z)}} \times \frac{f(\theta,z) - f(\theta_P,z)[\cos(\theta_P-\theta) - \sin(\theta_P-\theta)(f^{-1}f^1_\theta)(\theta,z)]}{[f^2(\theta_P,z)+f^2(\theta,z)-2\cos(\theta_P-\theta)f(\theta,z)f(\theta_P,z)]}\,\mathrm{d}\theta_P. \quad (4.5)$$

One may always solve this Fredholm integral equation of the second kind by numerical methods. Such a treatment presents no special difficulties. Despite the fact that it remains impossible to derive a general formula in closed form for the solution $\lambda = \{L_0^{\theta,z}\}^{-1}[d]$, it is easy to find a general property satisfied by λ. For this purpose we consider, in plane $\mathcal{P}(z')$, a circle $c(r')$ (see figure 2) of radius r' and centred at $O'_{z'} = \mathcal{P}(z') \cap \mathcal{C}'(z')$. Remember (see § 3 *c*) that the two-dimensional problem $L_0^{\theta,z}[\lambda] = d$ can be written as $u_\infty d(M) = \boldsymbol{u}_{\mathrm{2D}}(M)\cdot\boldsymbol{n}_{\mathrm{2D}}(M)$ for $M \in \partial\mathcal{C}'(z')$, where $\boldsymbol{u}_{\mathrm{2D}}$ is the two-dimensional velocity induced by potential function ϕ_{2D} (see (3.21)). Since this velocity field is solenoidal, one gets

$$\oint_{\partial\mathcal{C}'(z')} \boldsymbol{u}_{\mathrm{2D}}(P)\cdot\boldsymbol{n}_{\mathrm{2D}}(P)\,\mathrm{d}l'_P = u_\infty \oint_{\partial\mathcal{C}'(z')} d(P)\,\mathrm{d}l'_P = \oint_{c(r')} \boldsymbol{u}_{\mathrm{2D}}(P)\cdot\frac{\boldsymbol{O'_{z'}P}}{O'_{z'}P}\,\mathrm{d}l'_P.$$

Moreover, application of the usual Gauss's theorem to potential function ϕ_{2D} yields the relation

$$\lim_{r'\to+\infty}\oint_{c(r')} \boldsymbol{u}_{\mathrm{2D}}(P)\cdot\frac{\boldsymbol{O'_{z'}P}}{O'_{z'}P}\,\mathrm{d}l'_P = \oint_{\partial\mathcal{C}'(z')} q(P)\,\mathrm{d}l'_P = u_\infty \oint_{\partial\mathcal{C}'(z')} \lambda(P)\,\mathrm{d}l'_P. \quad (4.6)$$

Thereafter, and by using non-dimensional variables, the next general property holds

$$S_\lambda(z) = \oint_{\partial\mathcal{C}(z)} \lambda(M)\,\mathrm{d}l_M = \int_0^{2\pi} \lambda(\theta,z)f(\theta,z)\sqrt{1+(f^{-1}f^1_\theta)^2(\theta,z)}\,\mathrm{d}\theta$$
$$= \oint_{\partial\mathcal{C}(z)} d(M)\,\mathrm{d}l_M = \int_0^{2\pi} d(\theta,z)f(\theta,z)\sqrt{1+(f^{-1}f^1_\theta)^2(\theta,z)}\,\mathrm{d}\theta. \quad (4.7)$$

Observe that $S_\lambda(z)$ is the total strength of non-dimensional sources of lineic density $\lambda(\theta, z)$ to spread on the boundary $\partial\mathcal{C}(z)$ of non-dimensional cross-section $\mathcal{C}(z)$. As a consequence, if $S_i(z) := S_{\lambda_i}(z)$ for $i \in \{1,2,3\}$, then previous equations (4.2)–(4.4) and definitions (3.16)–(3.17) allow us to give $S_i(z)$, whatever the shape of the non-dimensional body $\mathcal{A}$. More precisely, and for $i \in \{1,2\}$, one finds (see Appendix B for details) that

$$S_1(z) = S^{(1)}(z), \quad S_2(z) = (2\pi)^{-1}[S(z)S^{(2)}(z)]^{(1)}(z), \quad (4.8)$$

where $S(z)$ designates the non-dimensional area of cross-section $\mathcal{C}(z)$, i.e.

$$S(z) = \oint_{\mathcal{C}(z)} \mathrm{d}S_P = \int_0^{2\pi} \tfrac{1}{2}[f(\theta,z)^2]\,\mathrm{d}\theta.$$

These relations agree with usual results obtained by employing the method of matched asymptotic expansions at this order (see Euvrard 1983). Unfortunately, the part of $d_3(\theta, z)$ involving $[1+(f^{-1}f^1_\theta)^2(\theta,z)]^{-3/2}$ makes it impossible to derive a similar and general formula for $S_3(z)$ in terms of partial derivatives of function $f(\theta, z)$ when the shape of $\mathcal{A}$ is arbitrary.

(c) *Special case of a body of revolution*

As far as the author knows, the available theoretical studies in the field only consider an axially symmetric flow around a body of revolution (see Moran 1963; Handelsman & Keller 1967). Hence, it is worth focusing in this subsection on the case of a body of revolution. Nevertheless, we allow a non-zero incidence angle α, i.e. the flow is not necessarily axially symmetric. As shown below, such assumptions authorize an analytical treatment of the problem.

(i) *Inversion of two-dimensional problem*

In such circumstances, $f(\theta, z) = f(z)$ and the integral equation (4.5) takes the pleasant form

$$d(\theta, z) = \tfrac{1}{2}\lambda(\theta, z) + \frac{1}{4\pi}\int_0^{2\pi} \lambda(\theta_P, z)\,\mathrm{d}\theta_P. \tag{4.9}$$

Application of property (4.7) yields

$$\int_0^{2\pi} \lambda(\theta_P, z)\,\mathrm{d}\theta_P = \int_0^{2\pi} d(\theta_P, z)\,\mathrm{d}\theta_P.$$

Accordingly, the solution of the two-dimensional problem is

$$\lambda(\theta, z) = 2d(\theta, z) - \frac{1}{2\pi}\int_0^{2\pi} d(\theta_P, z)\,\mathrm{d}\theta_P. \tag{4.10}$$

(ii) *The asymptotic solution for source density $\lambda(\theta, z)$*

If functions $C_1(z)$ and $C_2(z)$ obey the following definitions:

$$C_1(z) = [f^{(1)}(z)]^2 + f(z)f^{(2)}(z), \tag{4.11}$$

$$C_2(z) = [f^{(1)}(z)]^3 + \tfrac{5}{2}f(z)f^{(1)}(z)f^{(2)}(z) + \tfrac{1}{2}[f(z)]^2 f^{(3)}(z), \tag{4.12}$$

then a careful combination of equations (4.2)–(4.4) and the above result (4.10) leads, after some algebra, to

$$\lambda_1(\theta, z) = f^{(1)}(z) - 2\alpha\cos(\theta), \quad \lambda_2(\theta, z) = C_2(z) + 2\alpha\cos(\theta)C_1(z), \tag{4.13}$$

and, on the other hand, after noting that the last contribution to $L_2^{\theta,z}$ vanishes, to

$$\begin{aligned}\lambda_3(\theta, z) = &- \mathrm{fp}\int_0^1 \frac{\{(z_P - z)f^{(1)}(z) + f(z)\}f(z_P)f^{(1)}(z_P)\,\mathrm{d}z_P}{2|z_P - z|^3} + \tfrac{1}{2}[f^{(1)}(z)]^3 \\ &+ (\log[f(z)/2] + \tfrac{3}{2})C_2(z) + f(z)f^{(1)}(z)f^{(2)}(z) - \alpha^2\tfrac{1}{2}f^{(1)}(z) \\ &+ 2\alpha\cos(\theta)\Big\{\tfrac{1}{6}\alpha^2 + \tfrac{7}{4}[f^{(1)}(z)]^2 + (\log[f(z)] + \tfrac{7}{4} - \tfrac{3}{2}\log 2)C_1(z) \\ &- \mathrm{fp}\int_0^1 \frac{[f(z_P)]^2\,\mathrm{d}z_P}{2|z_P - z|^3}\Big\},\end{aligned} \tag{4.14}$$

where $f^{(k)}(z) := \mathrm{d}^k f/\mathrm{d}z^k$, $k \in \mathbb{N}$. Note that when α is non-zero, each $\lambda_i(\theta, z)$ for $i \in \{1, 2\}$ also depends on θ via an additional term proportional to $\cos\theta$.

5. Estimate of the pressure coefficient on the body

The previous section allowed us to deduce the asymptotic expansion of potential function, and thereafter of fluid velocity, $\boldsymbol{u}(M)$ everywhere in $\mathbb{R}^3 \setminus \mathcal{A}'$. For applications, it is also of prime interest to give the dynamic pressure $p(M)$, especially on the body itself. Unfortunately, not so many works in the field of slender bodies provide such a result. For instance, Tuck (1992) exhibits an asymptotic expansion for pressure coefficient C_p on a body of revolution (with $\alpha = 0$) but Moran (1963) and Handelsman & Keller (1967) disregarded this question even though they achieved a beautiful piece of work. This section presents a method to build the asymptotic expansion of pressure coefficient on the body.

(a) *A convenient form for the pressure coefficient on the body*

For each point M of $\partial\mathcal{A}' \setminus \{O', E'\}$, we consider the set of three unit vectors $(\boldsymbol{t}_1(M), \boldsymbol{t}_2(M), \boldsymbol{n}(M))$ such that, if $\boldsymbol{n}(M)$ remains the normal vector already defined, $\boldsymbol{t}_1(M)$ and $\boldsymbol{t}_2(M)$ are tangential to surface $\partial\mathcal{A}'$ at point M with $\boldsymbol{t}_1(M) \cdot \boldsymbol{e}_z = 0$, $\boldsymbol{t}_1(M) \cdot \boldsymbol{e}_\theta > 0$ and $\boldsymbol{t}_2(M) := \boldsymbol{n}(M) \wedge \boldsymbol{t}_1(M)$. Under notation $u_i(M) = \boldsymbol{u}(M) \cdot \boldsymbol{t}_i(M)$ for $i \in \{1, 2\}$, observe that, for $M \in \partial\mathcal{A}' \setminus \{O', E'\}$, combination of property $\boldsymbol{t}_i(M) \cdot \boldsymbol{t}_j(M) = \delta_{ij}$ (where δ_{ij} denotes the Kronecker delta) and of flow-tangency condition $\boldsymbol{u}(M) \cdot \boldsymbol{n}(M) = 0$ yields the following equality:

$$C_\mathrm{p}(M) = 1 - \left[\frac{\boldsymbol{u}(M)}{u_\infty}\right]^2 = 1 - \left[\frac{u_1}{u_\infty}\right]^2 - \left[\frac{u_2}{u_\infty}\right]^2. \tag{5.1}$$

Keeping in mind that $\boldsymbol{u}(M) = \boldsymbol{u}_\infty + \mathrm{grad}_M[\phi]$ with ϕ given by (2.5), one easily obtains

$$u_i(M) = \boldsymbol{u}_\infty \cdot \boldsymbol{t}_i(M) + \mathrm{vp} \iint_{\partial\mathcal{A}'} q(P) \frac{\boldsymbol{PM} \cdot \boldsymbol{t}_i(M)}{4\pi PM^3}\, \mathrm{d}S'_P, \quad \text{for } i \in \{1, 2\}, \tag{5.2}$$

where the symbol vp indicates that the weakly singular integral arising on the right-hand side of (5.2) is to handle in the principal value sense of Cauchy. More precisely, if the kernel function $g(M, P)$ is singular for $P = M$ and $\mathcal{S}$ is a surface, then the integral $\mathrm{vp} \iint_\mathcal{S} g(M, P)\, \mathrm{d}S'_P$ is defined for $M \in \mathcal{S}$ as

$$\mathrm{vp} \iint_\mathcal{S} g(M, P)\, \mathrm{d}S'_P = \lim_{\epsilon \to 0, \epsilon > 0} \iint_{\mathcal{S} \setminus D_\epsilon(M)} g(M, P)\, \mathrm{d}S'_P, \tag{5.3}$$

where $D_\epsilon(M) = \{P \in \mathcal{S}, MP < \epsilon\}$ for $\epsilon > 0$. Sufficient conditions for the existence of the above integral are (consult, for instance, Kupradze 1963):

(1) $g(M, P) = h(M, P)/MP^2$, with function h regular on $\mathcal{S} \times \mathcal{S}$;
(2) $h(M, P) = s(M, \theta)$, where $\theta = \boldsymbol{MP}/MP$ for $P \neq M$;
(3) $\int_0^{2\pi} s(M, \theta)\, \mathrm{d}\theta = 0$.

In the present case, one may actually write

$$\mathrm{vp} \iint_{\partial\mathcal{A}'} q(P) \frac{\boldsymbol{PM} \cdot \boldsymbol{t}_i(M)}{4\pi PM^3}\, \mathrm{d}S'_P = \iint_{\partial\mathcal{A}'} \{q(P) - q(M)\} \frac{\boldsymbol{PM} \cdot \boldsymbol{t}_i(M)}{4\pi PM^3}\, \mathrm{d}S'_P + q(M)\, \mathrm{vp} \iint_{\partial\mathcal{A}'} \frac{\boldsymbol{PM} \cdot \boldsymbol{t}_i(M)}{4\pi PM^3}\, \mathrm{d}S'_P, \tag{5.4}$$

where the previous conditions are clearly satisfied for the second integral on the right-hand side of (5.4) under choice $g(M, P) = \boldsymbol{PM} \cdot \boldsymbol{t}_i(M)/PM^3$, whereas integration bearing on function $q(P) - q(M)$ is regular as soon as density q obeys an Hölder condition in a neighbourhood of M, i.e. there exists $(\delta, \gamma, K) \in \mathbb{R}_+^{*3}$ such that $|q(P) - q(M)| < MP^\gamma$ for $P \in D_\delta(M)$. According to previous sections, since q is actually assumed to be smooth enough in $\partial\mathcal{A}' \setminus \{O', E'\}$, this is indeed the case. As already employed for the integral equation (2.6), the next step consists in applying a change of variable $P = (r'_P, \theta_P, z'_P) = (\epsilon r_P, \theta_P, L z_P)$ to equation (5.2) in order to exhibit the small slenderness parameter ϵ. First, the reader may check that for $a(\theta, z) = \sin(\theta) - \cos(\theta)(f^{-1} f_\theta^1)(\theta, z)$, and also $b(\theta, z) = \cos(\theta) + \sin(\theta)(f^{-1} f_\theta^1)(\theta, z)$, vectors $\boldsymbol{t}_i(M)$ and $c_i(M, \epsilon) := \boldsymbol{u}_\infty \cdot \boldsymbol{t}_i(M)/u_\infty$ $(i \in \{1, 2\})$ satisfy

$$\boldsymbol{t}_1(M) = \frac{-a(\theta, z)\boldsymbol{e}_x + b(\theta, z)\boldsymbol{e}_y}{[1 + (f^{-1} f_\theta^1)^2(\theta, z)]^{1/2}}, \quad c_1(M, \epsilon) = -\frac{\sin(\alpha\epsilon) a(\theta, z)}{[1 + (f^{-1} f_\theta^1)^2(\theta, z)]^{1/2}}, \tag{5.5}$$

$$\boldsymbol{t}_2(M) = \frac{\epsilon f_z^1(\theta, z)\{b(\theta, z)\boldsymbol{e}_x + a(\theta, z)\boldsymbol{e}_y\} + [1 + (f^{-1} f_\theta^1)^2(\theta, z)]\boldsymbol{e}_z}{\{[1 + (f^{-1} f_\theta^1)^2(\theta, z)][1 + (f^{-1} f_\theta^1)^2(\theta, z) + [\epsilon f_z^1(\theta, z)]^2]\}^{1/2}}, \tag{5.6}$$

$$c_2(M, \epsilon) = \frac{\cos(\alpha\epsilon)[1 + (f^{-1} f_\theta^1)^2(\theta, z)] + \epsilon \sin(\alpha\epsilon) f_z^1(\theta, z) b(\theta, z)}{\{[1 + (f^{-1} f_\theta^1)^2(\theta, z)][1 + (f^{-1} f_\theta^1)^2(\theta, z) + [\epsilon f_z^1(\theta, z)]^2]\}^{1/2}}. \tag{5.7}$$

Thus, a change of variables applied to relation (5.2) yields, for $M \in \partial\mathcal{A} \setminus \{O, E\}$ and $v_i(M, \epsilon) = u_i(M)/u_\infty$,

$$v_i(M, \epsilon) = c_i(M, \epsilon) + \int_0^{2\pi} T_{i,\epsilon}^{\theta,z}(\theta_P)\, \mathrm{d}\theta_P \quad \text{for } i \in \{1, 2\}, \tag{5.8}$$

where the new functions $T_{i,\epsilon}^{\theta,z}$ are given later. Consequently, the work reduces to the establishment of the asymptotic expansion of $v_i(M, \epsilon)$. This is, once again, carried out by employing the method explained in § 3 *b*.

(*b*) *Treatment of term $v_1^2(M, \epsilon)$ up to order $O(\epsilon^4 \log \epsilon)$*

It is not difficult to get

$$T_{1,\epsilon}^{\theta,z}(\theta_P) = \frac{\epsilon^2}{4\pi} \operatorname{fp} \int_0^1 \frac{\lambda(\theta_P, z_P)\{1 + (f^{-1} f_{\theta_P}^1)^2(\theta_P, z_P) + [\epsilon f_{z_P}^1(\theta_P, z_P)]^2\}^{1/2}\, \mathrm{d}z_P}{[g_1(\theta_P, z_P, \theta, z)]^{-1}[(z_P - z)^2 + \epsilon^2 H^2(\theta_P, z_P, \theta, z)]^{3/2}},$$

where the new function $g_1(\theta_P, z_P, \theta, z)$ obeys

$$g_1(\theta_P, z_P, \theta, z) = \frac{f_\theta^1(\theta, z) - f(\theta_P, z_P)\{\sin(\theta_P - \theta) + \cos(\theta_P - \theta)(f^{-1} f_\theta^1)(\theta, z)\}}{[f(\theta_P, z_P)]^{-1}[1 + (f^{-1} f_\theta^1)^2(\theta, z)]^{1/2}}.$$

Such a quantity is expanded with respect to the small parameter ϵ by taking into account both basic result (3.11) and asymptotic solution (4.1). Consequently, one easily obtains

$$\int_0^{2\pi} T_{1,\epsilon}^{\theta,z}(\theta_P)\, \mathrm{d}\theta_P = \epsilon T_1^{\theta,z} + O(\epsilon^3 \log \epsilon),$$

with (see (3.12))

$$T_1^{\theta,z} = \int_0^{2\pi} I_{0,\theta_P}^{\theta,z}[\lambda_1(\theta_P, z) g_1(\theta_P, z, \theta, z)[1 + (f^{-1} f_{\theta_P}^1)^2(\theta_P, z)]^{1/2}]\, \mathrm{d}\theta_P. \tag{5.9}$$

Using the expansion of $c_1(M, \epsilon)$, and if $V_1(\theta, z) := -\alpha a(\theta, z)/[1 + (f^{-1} f_\theta^1)^2(\theta, z)]^{1/2}$, it follows that

$$v_1^2(M, \epsilon) = \{V_1(\theta, z) + T_1^{\theta,z}\}^2 \epsilon^2 + O(\epsilon^4 \log \epsilon). \tag{5.10}$$

(c) Treatment of term $v_2^2(M,\epsilon)$ up to order $O(\epsilon^4\log\epsilon)$

First, the equality (5.7) readily leads to $c_2(M,\epsilon)=1+V_2(\theta,z)\epsilon^2+O(\epsilon^4)$, with

$$V_2(\theta,z)=-\tfrac{1}{2}\alpha^2+\frac{f_z^1(\theta,z)}{1+(f^{-1}f_\theta^1)^2(\theta,z)}\{\alpha b(\theta,z)-\tfrac{1}{2}f_z^1(\theta,z)\}.$$

On the other hand, $T_{2,\epsilon}^{\theta,z}(\theta_P)$ may be shared into two contributions $U_{0,\epsilon}^{\theta,z}(\theta_P)$ and $U_{1,\epsilon}^{\theta,z}(\theta_P)$ with, for $i\in\{0,1\}$, the next definitions:

$$U_{i,\epsilon}^{\theta,z}(\theta_P)=\frac{\epsilon^{1+2i}}{4\pi}\,\mathrm{fp}\int_0^1\frac{\lambda(\theta_P,z_P)\{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z_P)+[\epsilon f_{z_P}^1(\theta_P,z_P)]^2\}^{1/2}\mathrm{d}z_P}{[G_{1+2i}^\epsilon(\theta_P,z_P,\theta,z)]^{-1}[(z_P-z)^2+\epsilon^2H^2(\theta_P,z_P,\theta,z)]^{3/2}},$$

and also, for functions G_1^ϵ and G_3^ϵ depending on (θ_P,z_P,θ,z),

$$G_1^\epsilon=-\frac{(z_P-z)f(\theta_P,z_P)[1+(f^{-1}f_\theta^1)^2(\theta,z)]^{1/2}}{[1+(f^{-1}f_\theta^1)^2(\theta,z)+[\epsilon f_z^1(\theta,z)]^2]^{1/2}},\tag{5.11}$$

$$G_3^\epsilon=\frac{f(\theta_P,z_P)}{[f_z^1(\theta,z)]^{-1}}\frac{f(\theta,z)-f(\theta_P,z_P)\{\cos(\theta_P-\theta)-\sin(\theta_P-\theta)(f^{-1}f_\theta^1)(\theta,z)\}}{\{[1+(f^{-1}f_\theta^1)^2(\theta,z)][1+(f^{-1}f_\theta^1)^2(\theta,z)+[\epsilon f_z^1(\theta,z)]^2]\}^{1/2}}.\tag{5.12}$$

Thanks to the systematic formula presented in Appendix A, it is easy to show that the next term in asymptotic behaviour (3.11) is of order $O(\epsilon^4\log\epsilon)$. Hence, another careful application of result (3.11) respectively leads to

$$\int_0^{2\pi}U_{0,\epsilon}^{\theta,z}(\theta_P)\,\mathrm{d}\theta_P=U_1^{\theta,z}\epsilon^2\log\epsilon+U_2^{\theta,z}\epsilon^2+U_3^{\theta,z}\epsilon^4\log^2\epsilon+O(\epsilon^4\log\epsilon)$$

and

$$\int_0^{2\pi}U_{1,\epsilon}^{\theta,z}(\theta_P)\,\mathrm{d}\theta_P=U_4^{\theta,z}\epsilon^2+O(\epsilon^4\log\epsilon),$$

with (see results (3.12) and (3.13)), for $i\in\{0,1\}$,

$$U_{1+2i}^{\theta,z}=\int_0^{2\pi}I_{1,\theta_P}^{\theta,z}[\lambda_{1+i}(\theta_P,z_P)G_1^0(\theta_P,z_P,\theta,z)\{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z_P)\}^{1/2}]\,\mathrm{d}\theta_P,\tag{5.13}$$

$$U_2^{\theta,z}=\int_0^{2\pi}I_{2,\theta_P}^{\theta,z}[\lambda_1(\theta_P,z_P)G_1^0(\theta_P,z_P,\theta,z)\{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z_P)\}^{1/2}]\,\mathrm{d}\theta_P,\tag{5.14}$$

$$U_4^{\theta,z}=\int_0^{2\pi}I_{0,\theta_P}^{\theta,z}[\lambda_1(\theta_P,z_P)G_3^0(\theta_P,z_P,\theta,z)\{1+(f^{-1}f_{\theta_P}^1)^2(\theta_P,z_P)\}^{1/2}]\,\mathrm{d}\theta_P.\tag{5.15}$$

Accordingly,

$$v_2^2(M,\epsilon)=1+2U_1^{\theta,z}\epsilon^2\log\epsilon+2[V_2(\theta,z)+U_2^{\theta,z}+U_4^{\theta,z}]\epsilon^2+2U_3^{\theta,z}\epsilon^4\log^2\epsilon+O(\epsilon^4\log\epsilon).\tag{5.16}$$

(d) Asymptotic expansion of pressure coefficient up to order $O(\epsilon^4\log\epsilon)$

Gathering previous results, one easily obtains the following asymptotic expansion of the pressure coefficient:

$$C_\mathrm{p}(M,\epsilon)=-2U_1^{\theta,z}\epsilon^2\log\epsilon-\{(V_1(\theta,z)+T_1^{\theta,z})^2+2V_2(\theta,z)+2U_2^{\theta,z}+2U_4^{\theta,z}\}\epsilon^2-2U_3^{\theta,z}\epsilon^4\log^2\epsilon+O(\epsilon^4\log\epsilon).\tag{5.17}$$

Thanks to the general property exhibited in § 4 *b*, it is possible to rewrite terms $U_1^{\theta,z}$, $U_3^{\theta,z}$ and $U_2^{\theta,z}$ as follows:

$$U_1^{\theta,z} = S^{(2)}(z)/[2\pi], \quad U_3^{\theta,z} = S_2^{(1)}(z)/[2\pi] = [S(z)S^{(2)}(z)]^{(2)}(z)/[4\pi^2], \tag{5.18}$$

$$U_2^{\theta,z} = -\operatorname{fp}\int_0^1 \frac{\operatorname{sgn}(z_P - z)S^{(1)}(z_P)}{4\pi(z_P - z)^2}\,\mathrm{d}z_P + (1 - \log 2)\frac{S^{(2)}(z)}{2\pi}$$
$$+\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{\lambda_1(\theta_P, z_P)\log[H(\theta_P, z_P, \theta, z)]f(\theta_P, z_P)}{\{1 + (f^{-1}f^1_{\theta_P})^2(\theta_P, z_P)\}^{-1/2}}\,\mathrm{d}\theta_P\right]_{z_P = z}. \tag{5.19}$$

Observe that, according to (5.18), the leading term $-2U_1^{\theta,z}\epsilon^2\log\epsilon$ arising in the asymptotic estimate of $C_\mathrm{p}(M)$ does not depend upon the angular coordinate θ of point M.

(*e*) *Case of a body of revolution*

In such circumstances $S(z) = \pi[f(z)]^2$ and, if R designates the last term on the right-hand side of (5.19), then the result $\lambda_1(\theta_P, z_P) = (f^{(1)})(z_P) - 2\alpha\cos(\theta_P)$ ensures the following equalities:

$$R = [ff^{(1)}]^{(1)}(z)\log[f(z)] + \tfrac{1}{2}[f^{(1)}(z)]^2 + \alpha\cos(\theta)f^{(1)}(z), \tag{5.20}$$

$$V_1(\theta, z) = -\alpha\sin(\theta), \quad T_1^{\theta,z} = -\int_0^{2\pi} \frac{\lambda_1(\theta_P, z)\sin(\theta_P - \theta)}{4\pi[1 - \cos(\theta_P - \theta)]}\,\mathrm{d}\theta_P = -\alpha\sin(\theta), \tag{5.21}$$

$$V_2(\theta, z) = \tfrac{1}{2}\alpha^2 + \alpha\cos(\theta)f^{(1)}(z) - \tfrac{1}{2}[f^{(1)}(z)]^2, \quad U_4^{\theta,z} = \tfrac{1}{2}[f^{(1)}(z)]^2. \tag{5.22}$$

Consequently, (5.17) takes the form

$$C_\mathrm{p}(M, \epsilon) = -2[ff^{(1)}]^{(1)}(z)\epsilon^2\log\epsilon - \Big\{4\alpha f^{(1)}(z)\cos(\theta) + \alpha^2[4\sin^2(\theta) - 1]$$
$$-[f^{(1)}(z)]^2 + 2[ff^{(1)}]^{(1)}(z)\log[\tfrac{1}{2}f(z)]$$
$$-\operatorname{fp}\int_0^1 \frac{\operatorname{sgn}(z_P - z)}{(z_P - z)^2}f(z_P)f^{(1)}(z_P)\,\mathrm{d}z_P + 2[ff^{(1)}]^{(1)}(z)\Big\}\epsilon^2$$
$$-[(ff^{(1)})^2 + f^3f^{(2)}]^{(2)}(z)\epsilon^4\log^2\epsilon + O(\epsilon^4\log\epsilon). \tag{5.23}$$

The reader may check that, for $\alpha = 0$, such a result (5.23) agrees with the expansion proposed by Tuck (1992) and, incidentally, gives the corrective term of order $\epsilon^4\log^2\epsilon$.

6. Concluding remarks

In this work, an alternative approach has been proposed to deal with a non-lifting and slender body which is not necessarily of circular cross-section. The employed boundary integral equation is free from the mathematical problems emphasized by Cade (1994). When expanding with respect to the small slenderness parameter ϵ, this integral equation—the concept of integration in the finite-part sense of Hadamard—appears quite powerful and not only allows one to avoid tedious matching rules, but also makes it possible to build high order approximations without further difficulties.

In this asymptotic framework, the source's density is found up to order $o(\epsilon^3)$ by successively solving bidimensional problems; special attention is also paid to the pressure coefficient on the body. Such a physical quantity of prime interest is expanded up to order $O(\epsilon^4\log\epsilon)$, whatever the shape of the body. Since it admits an analytical treatment, the special case of a slender body of revolution (with potential non-zero incidence) is detailed.

Appendix A.

This appendix briefly recalls the definition of an integration in the finite-part sense of Hadamard and also provides a general formula for the asymptotic expansion of the integral $M_h(\epsilon)$ defined by (3.8).

A complex function f is of the first kind if and only if there exist $\eta > 0$, a positive integer N, a family of positive integers $(M(n))$, two complex families (γ_n), (f_{nm}) and a complex function F such that $\mathrm{Re}(\gamma_N) < \cdots < \mathrm{Re}(\gamma_0) := 0$, $f_{00} = 0$ if $\gamma_0 = 0$, $\lim_{\epsilon \to 0, \epsilon > 0} F(\epsilon)$ exists and $\forall \epsilon \in]0, \delta[$ then

$$f(\epsilon) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} f_{nm} \epsilon^{\gamma_n} \log^m \epsilon + F(\epsilon).$$

Then $\mathrm{fp}[f(\epsilon)] = \lim_{\epsilon \to 0, \epsilon > 0} F(\epsilon)$ is called the finite part in the Hadamard sense of the quantity $f(\epsilon)$. For $\mathcal{D}$, a subset of $\mathbb{R}$, and $g \in L^1_{\mathrm{loc}}(\mathcal{D} \setminus \{x_0\}, C)$ obeying an adequate behaviour near potential singularity x_0 on the left and on the right (see Lavoine 1959; Schwartz 1966), it is possible to introduce the integral

$$\mathrm{fp} \int_{\mathcal{D}} g(u)\, \mathrm{d}u := \mathrm{fp} \left[\int_{\mathcal{D} \setminus]x_0 - \epsilon, x_0 + \epsilon[} g(u)\, \mathrm{d}u \right]. \tag{A 1}$$

The above concept turns out to be very fruitful when expanding integrals with respect to a small, or a large, parameter (see Sellier 1994). More precisely, if $\mathcal{D}$ is an open subset of $\mathbb{R}$ containing zero, K is a kernel obeying $K(\alpha x, \alpha y) = \mathrm{sgn}(\alpha) \alpha^Q K(x, y)$, where Q is an integer, and functions w and h are smooth enough that the following result holds (for a derivation consult Sellier (1996, theorem 16)):

$$\begin{aligned}
\mathrm{fp} \int_{\mathcal{D}} w(u) K[u, \epsilon h(u)]\, \mathrm{d}u &= \sum_{n=0}^{N} \frac{\partial_n K(1,0)}{n!} \left[\mathrm{fp} \int_{\mathcal{D}} \mathrm{sgn}(u) w(u) u^{Q-n} [h(u)]^n\, \mathrm{d}u \right] \epsilon^n \\
&+ \sum_{m=0}^{N-Q-1} \sum_{j=0}^{m} \frac{w^{(j)}(0)}{j!} \left[\sum_{l=0}^{m-j} \frac{a^l_{m-j-l}}{l!}\, \mathrm{fp} \int_{-\infty}^{\infty} \partial_l K(t, h(0)) t^m\, \mathrm{d}t \right] \epsilon^{Q+m+1} \\
&-2 \sum_{n=\max[0, Q+1]}^{[\![R]\!]} \sum_{l=0}^{n-Q-1} \sum_{i=0}^{n-Q-l-1} \frac{[h(0)]^{n-i} w^{(l)}(0)}{i!(n-i)!l!} \\
&\times a^i_{n-Q-l-i-1} \partial_n K(1,0) \epsilon^n \log \epsilon + o(\epsilon^N),
\end{aligned} \tag{A 2}$$

where N is an integer such that $N \geqslant \max(Q+1, 0)$, $\partial_n K[u, v] := \partial^n K / \partial v^n [u, v]$ and the coefficients a^l_i are defined by $a^0_0 := 1$, $a^0_i := 0$, for $i \geqslant 1$, and, if $l \geqslant 1$, then

$$\left[\frac{h(u) - h(0)}{u} \right]^l = \sum_{i \in \mathbb{N}} a^l_i t^i, \quad l \in \mathbb{N}. \tag{A 3}$$

Observe that the only difficulty in employing such a systematic formula consists in carefully calculating the integrals $I_{lm}[h(0)] = \mathrm{fp} \int_{-\infty}^{\infty} \partial_l K(t, h(0)) t^m\, \mathrm{d}t$ arising on the right-hand side of (A 2). Indeed, for these kind of integrals, the change of scale

$t = h(0)u$ may induce extra terms (see Sellier 1994, lemma 2). For instance, if $h(0) > 0$,

$$I_{02}[h(0)] = \mathrm{fp}\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{[t^2+h^2(0)]^{3/2}} = \mathrm{fp}\int_{-\infty}^{\infty} \frac{u^2\,\mathrm{d}u}{[1+u^2]^{3/2}} - 2\log[h(0)]. \qquad \text{(A 4)}$$

A careful application of general formula (A 2) with $\mathcal{D} :=]-z, 1-z[$, $K[u,v] := (u^2+v^2)^{-3/2}$, $Q = -3$ and $h(u) > 0$ for $u \in\,]-z, 1-z[$ allows us to expand $M_h(\epsilon) = \mathrm{fp}\int_{-z}^{1-z} w(u)\,\mathrm{d}u/[u^2+\epsilon^2 h^2(u)]^{3/2}$ up to large enough order. Hence, one obtains

$$M_h(\epsilon) = \frac{2w(0)}{[h(0)]^2}\epsilon^{-2} - w^{(2)}(0)\log\epsilon + \mathrm{fp}\int_{-z}^{1-z} \frac{\mathrm{sgn}(u)w(u)\,\mathrm{d}u}{u^3} - A + o(1), \qquad \text{(A 5)}$$

with the following relation:

$$A + (\log 2 - 1)w^{(2)}(0) = \log[h(0)]w^{(2)}(0) + \frac{h^{(1)}(0)}{h(0)}w^{(1)}(0) + \frac{h^{(1)}(0)}{h(0)}w^{(1)}(0) + \left\{\frac{h^{(2)}(0)}{h(0)} - h^{(1)}(0)\frac{h^{(1)}(0)}{[h(0)]^2}\right\}w(0) = [w(u)\log[h(u)]]^{(2)}(0). \qquad \text{(A 6)}$$

Appendix B.

In this appendix, results (4.8) are established by using for the area $S(z)$ of the non-dimensional cross-section $\mathcal{C}(z)$, the relation $S(z) = \int_0^{2\pi}\frac{1}{2}[f(\theta,z)]^2\,\mathrm{d}\theta$ and formula (4.7):

$$S_{\lambda_i}(z) = S_i(z) := \int_0^{2\pi} s_i(\theta,z)f(\theta,z)\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}\,\mathrm{d}\theta, \qquad \text{(B 1)}$$

with $s_1(\theta,z) = d_1(\theta,z), s_2(\theta,z) = -L_1^{\theta,z}[\lambda_1]$. For convenience, the functions

$$E(\theta,z) = \cos(\theta)f(\theta,z) + \sin(\theta)f_\theta^1(\theta,z) = \partial_\theta^1[\sin(\theta)f(\theta,z)],$$
$$F(\theta,z) = -\sin(\theta)f(\theta,z) + \cos(\theta)f_\theta^1(\theta,z) = \partial_\theta^1[\cos(\theta)f(\theta,z)],$$

are introduced. Since $f(0,z) = f(2\pi,z)$, one finds that

$$\int_0^{2\pi} E(\theta,z)\,\mathrm{d}\theta = \int_0^{2\pi} F(\theta,z)\,\mathrm{d}\theta = 0.$$

For $k \in \mathbb{N}$, we set $S_i^{(k)}(z) := \mathrm{d}^k S_i/\mathrm{d}z^k$.

(a) *Treatment of* $S_1(z)$

Thanks to (3.16), one immediately obtains

$$S_1(z) = \int_0^{2\pi} f_z^1(\theta,z)f(\theta,z)\,\mathrm{d}\theta - \alpha\int_0^{2\pi}[f(\theta,z)\cos(\theta) + f_\theta^1(\theta,z)\sin(\theta)]\,\mathrm{d}\theta. \qquad \text{(B 2)}$$

Since $\int_0^{2\pi} E(\theta,z)\,\mathrm{d}\theta = 0$, it follows that $S_1(z) = \mathrm{d}S(z)/\mathrm{d}z = S^{(1)}(z)$.

(b) *Treatment of* $S_2(z)$

The introduction of $a_2'(\theta,z)$ and $a_2''(\theta,z)$, such that

$$\sqrt{1+(f^{-1}f_\theta^1)^2(\theta,z)}\,s_2(\theta,z) = [a_2'(\theta,z) + a_2''(\theta,z)]/[4\pi],$$

leads, when taking into account definitions (3.5) and (3.19), to the following relations:

$$a_2'(\theta,z) = \frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi} \frac{\lambda_1(\theta_P,z_P)f(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\frac{\partial A}{\partial z_P}(\theta_P,z_P,\theta,z)\,\mathrm{d}\theta_P\right]_{z_P=z}, \quad \text{(B 3)}$$

$$a_2''(\theta,z) = \frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}z_P}\left[\frac{\lambda_1(\theta_P,z_P)f(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\right] A(\theta_P,z_P,\theta,z)\,\mathrm{d}\theta_P\right]_{z_P=z}. \quad \text{(B 4)}$$

According to the definitions of $A(\theta_P,z_P,\theta,z)$ and $S_1(z)$, one finds that

$$\begin{aligned} a_2'(\theta,z) = {} & f^1_z(\theta,z)\frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi} \frac{\lambda_1(\theta_P,z_P)f(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\,\mathrm{d}\theta_P\right]_{z_P=z} \\ & -\frac{E(\theta,z)}{f(\theta,z)}\frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi} \frac{\cos(\theta_P)\lambda_1(\theta_P,z_P)f(\theta_P,z_P)f^1_{z_P}(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\,\mathrm{d}\theta_P\right]_{z_P=z} \\ & +\frac{F(\theta,z)}{f(\theta,z)}\frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi} \frac{\sin(\theta_P)\lambda_1(\theta_P,z_P)f(\theta_P,z_P)f^1_{z_P}(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\,\mathrm{d}\theta_P\right]_{z_P=z}, \end{aligned} \quad \text{(B 5)}$$

$$\begin{aligned} a_2'(\theta,z) = {} & \frac{\mathrm{d}}{\mathrm{d}z_P}[(z_P-z)f^1_z(\theta,z)S_1^{(1)}(z_P) + f(\theta,z)S_1^{(1)}(z_P)]_{z_P=z} \\ & -\frac{E(\theta,z)}{f(\theta,z)}\frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi}\cos(\theta_P)f(\theta_P,z_P)\right. \\ & \left.\times\frac{\mathrm{d}}{\mathrm{d}z_P}\left\{\frac{\lambda_1(\theta_P,z_P)f(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\right\}\mathrm{d}\theta_P\right]_{z_P=z} \\ & +\frac{F(\theta,z)}{f(\theta,z)}\frac{\mathrm{d}}{\mathrm{d}z_P}\left[\int_0^{2\pi}\sin(\theta_P)f(\theta_P,z_P)\right. \\ & \left.\times\frac{\mathrm{d}}{\mathrm{d}z_P}\left\{\frac{\lambda_1(\theta_P,z_P)f(\theta_P,z_P)}{[1+(f^{-1}f^1_{\theta_P})^2(\theta_P,z_P)]^{-1/2}}\right\}\mathrm{d}\theta_P\right]_{z_P=z}. \end{aligned} \quad \text{(B 6)}$$

When calculating

$$S_2'(z) = \int_0^{2\pi} a_2'(\theta,z)f(\theta,z)\,\mathrm{d}\theta \quad \text{and} \quad S_2''(z) = \int_0^{2\pi} a_2''(\theta,z)f(\theta,z)\,\mathrm{d}\theta,$$

the properties

$$\int_0^{2\pi} E(\theta,z)\,\mathrm{d}\theta = 0 \quad \text{and} \quad \int_0^{2\pi} F(\theta,z)\,\mathrm{d}\theta = 0$$

again play a role and show that only the first term on the right-hand side of (B 5) or (B 6) is non-zero. More precisely, one easily gets

$$S_2'(z) = \int_0^{2\pi} f^1_z(\theta,z)f(\theta,z)S^{(2)}(z)\,\mathrm{d}\theta = S^{(1)}(z)S^{(2)}(z) \quad \text{(B 7)}$$

and also

$$\begin{aligned} S_2''(z) &= \int_0^{2\pi} [f^1_z(\theta,z)S_1^{(1)}(z) + f(\theta,z)S_1^{(2)}(z)]f(\theta,z)\,\mathrm{d}\theta \\ &= S^{(1)}(z)S^{(2)}(z) + 2S(z)S^{(3)}(z). \end{aligned} \quad \text{(B 8)}$$

Accordingly, $S_2(z) = [S_2'(z) + S_2''(z)]/[4\pi] = [S(z)S^{(2)}(z)]^{(1)}(z)/[2\pi]$.

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