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Asymptotic behaviour of a class of singular Fourier Integrals

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Abstract

This paper presents the asymptotic expansion, with respect to the large real parameter λ, of the singular Fourier integral

$$I(\lambda) := fp \int_0^b K(x, \lambda x)\, e^{i\lambda x}\, dx,$$

where $0 < b \leqslant +\infty$, the symbol fp designates an integration in the finite part sense of Hadamard and complex function $K(x, u)$ belongs to a specific set of pseudo-functions which may present a singular behaviour of logarithmic nature at the endpoints.

1. *Introduction*

For a real value $0 < b \leqslant +\infty$, the question of finding, as real parameter λ tends to infinity, the asymptotic behaviour of Lebesgue's integral

$$L(\lambda) := \int_0^b f(x)\, H(\lambda x)\, e^{i\lambda x}\, dx, \tag{1·1}$$

is widely encountered when solving problems arising in many fields of physics and mechanics.

If $H(u) := 1$ then $L(\lambda) = \int_0^b f(x)\, e^{i\lambda x}\, dx$, i.e. it reduces to a Fourier integral. As soon as f is analytic in an open subset of C (the set of complex numbers) containing $[0, b)$, combination of Cauchy's theorem and results obtained for Laplace integrals (for instance consult Bleistein & Handelsman [**2**], Estrada & Kanwal [**8**] and Wong [**19**]) leads to the sought asymptotic behaviour. Unfortunately, function f is often defined only for real values in $[0, b)$ and this method is not valid. In such circumstances, the asymptotic expansion of $L(\lambda)$ is usually obtained for complex and measurable functions f which are smooth enough in $]0, b[$ and offer more or less restrictive behaviours near zero on the right and near point b on the left. Among the methods employed, one may think not only of the integration by parts procedure (see Erdelyi [**5**] and also the standard textbooks: Bleistein & Handelsman [**2**], Erdelyi [**4**]) but also of the summability method (see Wong & Lin [**18**, **19**]) and of the distributional approach developed by Estrada & Kanwal [**6**, **8**]. Finally, if fp denotes an integration in the finite part sense of Hadamard (see Section 2), note that the singular Fourier integral

$$F(\lambda) := fp \int_0^b f(x)\, e^{i\lambda x}\, dx, \tag{1·2}$$

has been briefly handled by Estrada & Kanwal[7] for $0 < b < +\infty$, f smooth in $]0, b[$ and admitting a specific and singular behaviour near zero or b (see Proposition 3 with $a = 0$, $J(i) := 1$, $\forall i \in \{0, \dots, I\}$) and $K(l) = 0$, $\forall l \in \{0, \dots, L\}$.

Armstrong & Bleistein[1] also dealt with the general case of $L(\lambda)$ when complex kernel $H(x)$ and function $f(x)$ present respectively at infinity and near zero on the right general expansions involving complex powers of x and $\log x$. Unfortunately, the case of singular integral

$$J(\lambda) := fp \int_0^b f(x)\, H(\lambda x)\, e^{i\lambda x}\, dx, \tag{1·3}$$

was not investigated.

As an extension, consider the singular and Fourier integral

$$I(\lambda) := fp \int_0^b K(x, \lambda x)\, e^{i\lambda x}\, dx, \tag{1·4}$$

which will be said to be general since it includes previous cases of $F(\lambda)$ and $J(\lambda)$ (respectively obtained for $K(x, u) = f(x)$ and $K(x, u) = f(x)\, H(u)$). The aim of this paper is to derive the asymptotic expansion of $I(\lambda)$ with respect to the large real parameter λ and for weak assumptions regarding complex pseudofunction $K(x, u)$. Observe that the case of $N(\lambda) := fp \int_0^b K(x, \lambda x)\, e^{-i\lambda x}\, dx$ may be deduced by replacing throughout this study the complex number i by $-i$. Actually, this work may be seen as an extension to the case of oscillatory integrands of the approach developed by Sellier[17] which yields to the asymptotic expansion of $M(\lambda) := fp \int_0^b K(x, \lambda x)\, dx$. More precisely, the study is organized as follows. In Section 2 below, basic definitions and results are presented. The asymptotic behaviour of $F(\lambda)$ is considered in Section 3 whereas Section 4 deals with the general case of $I(\lambda)$. Finally, Section 5 exhibits several examples or applications.

2. *Definitions and mathematical framework*

This section introduces two kinds of integrations for specific pseudofunctions and also important and related results. For further details regarding the concept of integration in the finite part sense of Hadamard the reader is referred to Hadamard[9], Schwartz[16] and Sellier[17].

Definition 1. The complex function f belongs to the set $\mathscr{P}(]0, +\infty[, C)$ if and only if there exist positive reals η_f and A_f, two functions $F^0 \in L^1_{\text{loc}}([0, \eta_f], C)$ and $F^\infty \in L^1_{\text{loc}}([A_f, +\infty[, C)$, two families of positive integers $(J(i)), (K(l))$ and complex families (α_i), (f^0_{ij}), (γ_l), (f^∞_{lk}) such that $f \in L^1([\eta_f, A_f], C)$, $(f^0_{ij}) = (0)$ otherwise $\forall j \in \{0, \dots, J(i)\}$: $(f^0_{ij}) \neq (0)$; $(f^\infty_{lk}) = 0$ otherwise $\forall k \in \{0, \dots, K(l)\}$: $(f^\infty_{lk}) \neq 0$ and also

$$\left.\begin{aligned} f(x) &= \sum_{i=0}^{I} \sum_{j=0}^{J(i)} f^0_{ij}\, x^{\alpha_i} \log^j x + F^0(x), \text{ a.e. in }]0, \eta_f], \\ Re(\alpha_I) &< Re(\alpha_{I-1}) < \dots < Re(\alpha_1) < Re(\alpha_0) := -1; \end{aligned}\right\} \tag{2·1}$$

$$\left.\begin{aligned} f(x) &= \sum_{l=0}^{L} \sum_{k=0}^{K(l)} f^\infty_{lk}\, x^{-\gamma_l} \log^k x + F^\infty(x), \text{ a.e. in } [A_f, +\infty[, \\ Re(\gamma_L) &< Re(\gamma_{L-1}) < \dots < Re(\gamma_1) < Re(\gamma_0) := 1. \end{aligned}\right\} \tag{2·2}$$

In this definition, 'a.e.' means almost everywhere. Clearly, if $f\in L^1([0,+\infty[,C)$ then $f\in\mathscr{P}(]0,+\infty[,C)$ with $(f^0_{ij})=(f^\infty_{lk})=(0)$. According to Definition 1, the set $\mathscr{P}(]0,+\infty[,C)$ also contains complex pseudofunctions which are not measurable on the sets $]0,\eta_f]$ or $[A_f,+\infty[$. More precisely, if $(f^0_{ij})\neq(0)$, f is singular at zero (on the right) and $S_0(f):=Re(\alpha_I)\leqslant -1$ and if $(f^\infty_{lk})\neq(0)$ then f is singular at infinity and $S_\infty(f):=Re(\gamma_L)\leqslant 1$. Nevertheless, and for $a\in\mathbb{R}^*_+$, it is both possible and fruitful to introduce the two particular integrals $\mathrm{fp}\int_0^\infty f(x)\,dx$ and $\mathrm{fp}^{k*}\int_0^\infty f(x)\,e^{iax}\,dx$. This is achieved by the following steps.

Definition 2. For $r>0$, the complex function h is of the second kind on the set $]0,r[$ if and only if there exist a complex function H, a family of positive integers $(M(n))$, and two complex families (β_n) and (h_{nm}) such that

$$\forall\epsilon\in]0,r[,\quad h(\epsilon)=\sum_{n=0}^{N}\sum_{m=K(n)}^{M(n)}h_{nm}\,\epsilon^{\beta_n}\log^m(\epsilon)+H(\epsilon), \tag{2·3}$$

$$\left.\begin{aligned}&Re(\beta_N)<Re(\beta_{N-1})<\ldots<Re(\beta_1)<Re(\beta_0):=0,\\&\lim_{\epsilon\to0}H(\epsilon)\in C\text{ and }h_{00}:=0\text{ for }\beta_0=0.\end{aligned}\right\} \tag{2·4}$$

Extending Hadamard's concept (see Hadamard[**9**], Schwartz[**16**]) the finite part in the Hadamard sense of the quantity $h(\epsilon)$, noted $\mathrm{fp}[h(\epsilon)]$, is the complex $\lim_{\epsilon\to0}H(\epsilon)$.

Proposition 1. *For $j\in\mathbb{N}$, $\alpha\in C$ and two real values c and d with $0<c<d$, then*

$$\int_c^d x^\alpha\log^j x\,dx=P^j_\alpha(d)-P^j_\alpha(c), \tag{2·5}$$

with $$P^j_{-1}(t):=\frac{\log^{j+1}(t)}{j+1},\quad\text{else } P^j_\alpha(t):=t^{\alpha+1}\sum_{k=0}^{j}\frac{(-1)^{j-k}j!}{k!(\alpha+1)^{1+j-k}}\log^k(t). \tag{2·6}$$

Definition 3. If $f\in\mathscr{P}(]0,+\infty[,C)$, then the integration in the finite part sense of Hadamard of f is defined as

$$\mathrm{fp}\int_0^\infty f(x)\,dx:=\mathrm{fp}\left[\int_\epsilon^{1/\epsilon}f(x)\,dx\right]=\int_\delta^B f(x)\,dx$$
$$+\int_0^\delta F^0(x)\,dx+\int_B^\infty F^\infty(x)\,dx+\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f^0_{ij}P^j_{\alpha_i}(\delta)-\sum_{l=0}^{L}\sum_{k=0}^{K(l)}f^\infty_{lk}P^k_{-\gamma_l}(B), \tag{2·7}$$

for any pair (δ,B) such that $0<\delta\leqslant\eta_f$ and $A_f\leqslant B<+\infty$.

For $0<\delta\leqslant\eta_f$ and $A_f\leqslant B<+\infty$, the assumption $f\in L^1([\eta_f,A_f],C)$ and decompositions (2·1) and (2·2) ensure that $f\in L^1([\delta,B],C)$. Equality (2·7) is thereafter obtained by applying definition 2 to the complex function h defined for $0<\epsilon<\mathrm{Min}\{\delta,B^{-1}\}$ as

$$h(\epsilon):=\int_\epsilon^{\epsilon^{-1}}f(x)\,dx=\int_\delta^B f(x)\,dx+\int_\epsilon^\delta F^0(x)\,dx+\int_B^{\epsilon^{-1}}F^\infty(x)\,dx$$
$$+\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f^0_{ij}\int_\epsilon^\delta x^{\alpha_i}\log^j x\,dx+\sum_{l=0}^{L}\sum_{k=0}^{K(l)}f^\infty_{lk}\int_B^{\epsilon^{-1}}x^{-\gamma_l}\log^k x\,dx,$$

which is indeed of the second kind, thanks to Proposition 1.

Proposition 2. *Consider $a\in\mathbb{R}_+$ and $f\in\mathscr{P}(]0,+\infty[,C)$. If $a>0$ then*

$$fp^*\int_0^\infty f(x)\,e^{iax}\,dx := \lim_{\mu\to 0^+}\left[fp\int_0^\infty f(x)\,e^{(ia-\mu)x}\,dx\right] = \int_B^\infty F^\infty(x)\,e^{iax}\,dx$$

$$+fp\int_0^B\left[f(x)-\sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^\infty x^{-\gamma_l}\log^k x\right]e^{iax}\,dx+\sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^\infty fp^*\int_0^\infty x^{-\gamma_l}\log^k x\,e^{iax}\,dx, \quad (2\cdot 8)$$

for any real value $B\geqslant A_f$ and if $a\geqslant 0$ and $(\int_{lk}^\infty)=(0)$ then

$$fp^*\int_0^\infty f(x)\,e^{iax}\,dx := \lim_{\mu\to 0^+}\left[fp\int_0^\infty f(x)\,e^{(ia-\mu)x}\,dx\right] = fp\int_0^\infty f(x)\,e^{iax}\,dx. \quad (2\cdot 9)$$

For $a\in\mathbb{R}_+$ and $f\in\mathscr{P}(]0,+\infty[,C)$ one has to justify the existence of the complex $fp^*\int_0^\infty f(x)\,e^{iax}\,dx$, introduced by (2·8) or (2·9). If $\mu>0$, then clearly

$$f(x)\,e^{(ia-\mu)x}\in\mathscr{P}(]0,+\infty[,C)$$

and use of behaviours (2·1), (2·2) yields for $0<\delta\leqslant\eta_f$, $B\geqslant A_f$ and $0<\epsilon<\operatorname{Min}\{\delta,B^{-1}\}$

$$h_\mu^a(\epsilon) := \int_\epsilon^{\epsilon^{-1}} f(x)\,e^{(ia-\mu)x}\,dx = \Delta_\mu^a(\epsilon)+\sum_{i=0}^{I}\sum_{j=0}^{J(i)} f_{ij}^0 L_{ij}^0(\mu,\epsilon)+\sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^\infty L_{lk}^\infty(\mu,\epsilon)$$

with the following definitions

$$\Delta_\mu^a(\epsilon) = \int_\delta^B f(x)\,e^{(ia-\mu)x}\,dx+\int_\epsilon^\delta F^0(x)\,e^{(ia-\mu)x}\,dx+\int_B^{\epsilon^{-1}} F^\infty(x)\,e^{(ia-\mu)x}\,dx,$$

$$L_{ij}^0(\mu,\epsilon) = \int_\epsilon^\delta x^{\alpha_i}\log^j x e^{(ia-\mu)x}\,dx; \quad L_{lk}^\infty(\mu,\epsilon) = \int_B^{\epsilon^{-1}} x^{-\gamma_l}\log^k x e^{(ia-\mu)x}\,dx.$$

By means of Lebesgue's theorem, it is straightforward to deduce that

$$\lim_{\mu\to 0^+}\{fp[\Delta_\mu^a(\epsilon)]\} = \int_\delta^B f(x)\,e^{iax}\,dx+\int_0^\delta F^0(x)\,e^{iax}\,dx+\int_B^\infty F^\infty(x)\,e^{iax}\,dx. \quad (2\cdot 10)$$

If $(f_{ij}^0)\neq(0)$, then $Re(\alpha_i)\leqslant -1$ and $P_i := [\![-Re(\alpha_i)]\!]\in\mathbb{N}$ if $[\![d]\!]$ denotes the integer part of real d. Moreover, function $R_{ij}^a(x,\mu) := x^{\alpha_i}\log^j x[e^{(ia-\mu)x}-\sum_{p=0}^{P_i}(ia-\mu)^p x^p/p!]$ is bounded on $[0,\delta]$ for $0<\mu<\mu_1$ with $\lim_{\mu\to 0^+}R_{ij}^a(x,0),\forall x\in[0,\delta]$. Hence

$$L_{ij}^0(\mu,\epsilon) = \sum_{p=0}^{P_i}\frac{(ia-\mu)^p}{p!}\int_\epsilon^\delta x^{\alpha_i+p}\log^j x\,dx+\int_\epsilon^\delta R_{ij}^a(x,\mu)\,dx. \quad (2\cdot 11)$$

If $L_{ij}^0 := \lim_{\mu\to 0^+}\{fp[L_{ij}^0(\mu,\epsilon)]\}$, application of Lebesgue's theorem leads to

$$L_{ij}^0 = \sum_{p=0}^{P_i}\frac{(ia)^p}{p!}fp\int_0^\delta x^{\alpha_i+p}\log^j x\,dx+\int_0^\delta R_{ij}^a(x,0)\,dx = fp\int_0^\delta x^{\alpha_i}\log^j x e^{iax}\,dx. \quad (2\cdot 12)$$

Combination of equalities (2·10) and (2·12) shows that for $a\geqslant 0$ and $(f_{lk}^\infty)=(0)$ then

$fp^*\int_0^\infty f(x)\,e^{iax}\,dx$ exists and is equal to $fp\int_0^\infty f(x)\,e^{iax}\,dx$. Assume now that $(f_{lk}^\infty) \neq (0)$ and $a > 0$. In such circumstances,

$$fp[L_{lk}^\infty(\mu,\epsilon)] = -fp\int_0^B x^{-\gamma_l}\log^k x e^{(ia-\mu)x}\,dx + fp\int_0^\infty x^{-\gamma_l}\log^k x e^{(ia-\mu)x}\,dx. \quad (2\cdot13)$$

Noting that the function g defined by $g(x) := x^{-\gamma_l}\log^k(x)$ if $0 < x \leqslant B$, else $g(x) := 0$ belongs to $\mathscr{P}(]0,+\infty[,C)$ with $(g_{pq}^\infty) = (0)$, the first integral on the right-hand side of (2·13) tends, as $\mu \to 0^+$, to $-fp\int_0^B x^{-\gamma_l}\log^k x e^{iax}\,dx$. Hence, $\lim_{\mu\to0^+}\{fp[L_{lk}^\infty(\mu,\epsilon)]\}$ exists if and only if the complex

$$S_a(-\gamma_l,k) := fp^*\int_0^\infty x^{-\gamma_l}\log^k(x)\,e^{iax}\,dx \quad (2\cdot14)$$

also exists for $a > 0$. As shown in Lemma 2 below this is indeed the case and $fp^*\int_0^\infty f(x)\,e^{iax}\,dx$ admits a sense for $f\in\mathscr{P}(]0,+\infty[,C)$ with $(f_{lk}^\infty) \neq (0)$ and $a > 0$.

These notions suggest some remarks.

(*a*) Of course, previous results ensure for $f\in L^1([0,+\infty[,C)$ that $\int_0^\infty f(x)\,e^{iax}\,dx = fp\int_0^\infty f(x)\,e^{iax}\,dx = fp^*\int_0^\infty f(x)\,e^{iax}\,dx$.

(*b*) The introduction for $a > 0$ and $f\in\mathscr{P}(]0,+\infty[,C)$ of the new integration $fp^*\int_0^\infty f(x)\,e^{iax}\,dx$ is very important for this paper. It may be seen as an extension to the integration in the finite part sense of Hadamard of the work developed by Hardy[**10**, **11**] and further employed by Olver[**14**]. Another point of view is to consider this new integration as a generalization of the Abel limit of the function $f(x)\,e^{iax}$ (see Wong[**19**]).

(*c*) If Lemma 2 proves the existence of complex $S_a(\alpha,j)$ (see (2·14)) for $a\in\mathbb{R}_+^*$, $\alpha\in C$ and $j\in\mathbb{N}$ it is also worth giving the extra terms arising when applying changes of variable for these two kinds of particular integrations. If any shift induces no additional term, the case of change of scale $x = wt$ with $w\in\mathbb{R}_+^*$ is provided by Lemma 1 and Lemma 3.

LEMMA 1. *If $f\in\mathscr{P}(]0,+\infty[,C)$ and $w\in\mathbb{R}_+^*$, then (see Sellier*[**17**])

$$fp\int_0^\infty f(x)\,dx = fp\int_0^\infty f(wt)\,d(wt) + \sum_{j=0}^{J(0)}\delta_{(-1,\alpha_0)}f_{0j}^0\frac{\log^{j+1}w}{j+1} - \sum_{k=0}^{K(0)}\delta_{(1,\gamma_0)}f_{0k}^\infty\frac{\log^{k+1}w}{k+1}, \quad (2\cdot15)$$

where for complex values z and z'; $\delta_{(z,z')} := 1$ if $z = z'$, else $\delta_{(z,z')} := 0$.

LEMMA 2. *For $a\in\mathbb{R}_+^*$, $\alpha\in C$ and $j\in\mathbb{N}$ complex $S_a(a,j)$ admits a sense and*

$$S_a(\alpha,j) := fp^*\int_0^\infty x^\alpha\log^j x e^{iax}\,dx = \left\{\frac{E(\alpha)(-1)^{-\alpha+j}}{(-\alpha-1)!\,(j+1)}\log^{j+1}[ae^{-i\pi/2}]\right.$$
$$\left. + \sum_{l=0}^{j}\sum_{k=0}^{l} C_j^l C_l^k(-1)^{l-k}\left(i\frac{\pi}{2}\right)^{j-l}\left[fp\int_0^\infty u^\alpha\log^k u e^{-u}\,du\right]\log^{l-k}(a)\right\} e^{i\frac{\pi}{2}(\alpha+1)}a^{-(\alpha+1)}, \quad (2\cdot16)$$

where $C_m^n := m!/[n!(m-n)!]$ for integers $0 \leqslant n \leqslant m$ and $E(\alpha) := 1$ if $-1-\alpha\in\mathbb{N}$, else $E(\alpha) := 0$.

Proof. For real values $a\in\mathbb{R}_+^*$, $0 < \epsilon < 1$, $\mu > 0$ and $A \geqslant 1$, we introduce $w_\mu := (a^2+\mu^2)^{1/2}$, $\theta_\mu \in\,]-\pi/2, 0[$ such that $\tan(\theta_\mu) = -a/\mu$ and also the paths in complex set

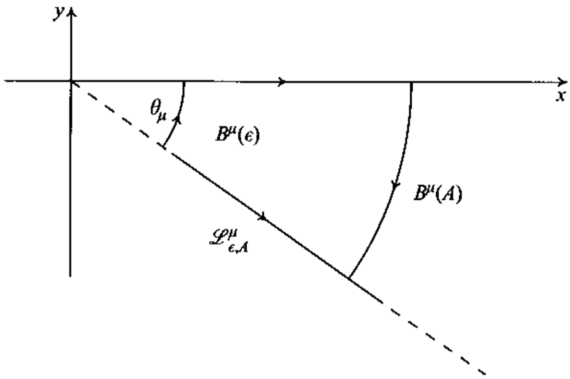

Fig. 1

C (see Fig. 1 above): $\mathscr{L}^\mu_{\epsilon,A} := \{z\in C; z = te^{i\theta_\mu} \text{ for } \epsilon \leqslant t \leqslant A\}$, $\mathscr{L}^\mu_A := \lim_{\epsilon\to 0^+} \mathscr{L}^\mu_{\epsilon,A}$, $\mathscr{L}^\mu := \lim_{A\to+\infty} \mathscr{L}^\mu_A$ and also for $b\in\mathbb{R}^*_+$, $B^\mu(b) := \{z\in C,\ z = be^{i\theta} \text{ with } \theta_\mu \leqslant \theta \leqslant 0\}$.

Since $\mu > 0$, if $s^\mu_a(x) := x^\alpha \log^j x e^{(ia-\mu)x}$ then $s^\mu_a \in \mathscr{P}(]0,+\infty[,C)$, i.e. $S^\mu_a(\alpha,j) := fp\int_0^\infty s^\mu_a(x)\,dx$ admits a sense and according to Proposition 2 if $\lim_{\mu\to 0^+} S^\mu_a(\alpha,j)$ exists then $S_a(\alpha,j)$ is the limit. Clearly, $s^\mu_a \in L^1_{\text{loc}}([1,+\infty[,C)$ and $S^\mu_a(\alpha,j) = \lim_{A\to+\infty} S^\mu_a(\alpha,j,A)$ if $S^\mu_a(\alpha,j,A) := fp\int_0^A s^\mu_a(x)\,dx$. After some algebra, one gets

$$S^\mu_a(\alpha,j,A) = fp\int_0^A x^\alpha \log^j x\, e^{-w_\mu x e^{i\theta_\mu}}\,dx = e^{-a(\alpha+1)\theta_\mu}\sum_{l=0}^{j} C^l_j(-i\theta_\mu)^{j-l} E^A_{\alpha,l}(w_\mu), \quad (2\cdot17)$$

where each complex quantity $E^A_{\alpha,l}(w_\mu)$ satisfies

$$E^A_{\alpha,l}(w_\mu) := fp\int_0^A (x\,e^{i\theta_\mu})^\alpha \log^l[x\,e^{i\theta_\mu}]\, e^{-w_\mu x e^{i\theta_\mu}}\,d(x\,e^{i\theta_\mu}) = fp\int_{\mathscr{L}^\mu_A} z^\alpha \log^l z\, e^{-w_\mu z}\,dz. \quad (2\cdot18)$$

The next step consists in giving the link between $E^A_{\alpha,l}(w_\mu)$ and $fp\int_0^A x^\alpha \log^l x\, e^{-w_\mu x}\,dx$ (this latter integral admitting a sense even when $A\to+\infty$ since $w_\mu > 0$). For any $\epsilon > 0$, observe that complex function $F(z) := z^\alpha \log^l z\, e^{-w_\mu z}$ turns out to be analytic in $D_{\epsilon,A} := \{z\in C\backslash\mathbb{R}_-, \epsilon/2 < |z| < 2A\}$. Thus, application of the usual Cauchy theorem ensures that

$$h(\epsilon) := \int_{\mathscr{L}^\mu_{\epsilon,A}} F(z)\,dz - \int_\epsilon^A F(x)\,dx = \int_{B^\mu(\epsilon)} F(z)\,dz - \int_{B^\mu(A)} F(z)\,dz. \quad (2\cdot19)$$

For given α, the first integral on the right-hand side of (2·19), denoted $L^\mu(\epsilon)$, is treated by introducing integer $N := [\![-Re(\alpha)]\!]$ and function

$$R^\mu_{\alpha,l}(z) := z^\alpha \log^l z\left[e^{-w_\mu z} - \sum_{n=0}^{N}(-w_\mu)^n z^n/n!\right],$$

where it is understood that $\sum_{n=0}^N := 0$ for $N < 0$. Clearly, $R^\mu_{\alpha,l}(z)$ goes to zero with $|z|$ in $C\backslash\mathbb{R}_-$. Thus,

$$L^\mu(\epsilon) = \sum_{n=0}^{N}\frac{(-w_\mu)^n}{n!}\int_{B^\mu(\epsilon)} z^{\alpha+n}\log^l z\,dz + O^\mu(\epsilon) \quad (2\cdot20)$$

with $O^\mu(\epsilon)\to 0$, as $\epsilon\to 0^+$. Each integration on the right-hand side of (2·20) is performed by using a legitimate extension of function $P^j_{\alpha+n}$ in $C\backslash\mathbb{R}_-$ (see (2·6)). Thereafter and by expanding $P^j_{\alpha+n}[\epsilon\, e^{i\theta_\mu}]$, one obtains

$$fp\int_{B^\mu(\epsilon)} z^{\alpha+n}\log^l z\,dz = fp\{P^l_{\alpha+n}[\epsilon]-P^l_{\alpha+n}[\epsilon\, e^{i\theta_\mu}]\} = -\delta_{(\alpha+n,-1)}\frac{(i\theta_\mu)^{l+1}}{l+1}. \qquad (2\cdot 21)$$

Combination of equalities (2·19), (2·20), (2·21) and application of definition 2 yield

$$E^A_{\alpha,l}(w_\mu) = fp\int_0^A x^\alpha \log^l x\, e^{-w_\mu x}\,dx - E(\alpha)\frac{(i\theta_\mu)^{l+1}(-w_\mu)^{-\alpha-1}}{(l+1)(-\alpha-1)!} - \int_{B^\mu(A)} F(z)\,dz. \quad (2\cdot 22)$$

Since $\mu>0$ then $\cos(\theta)>0$ for $\theta_\mu \leqslant \theta \leqslant 0$ and $\lim_{A\to+\infty}\int_{B^\mu(A)}F(z)\,dz = 0$. Note also that $E := \lim_{A\to+\infty} fp\int_0^A x^\alpha\log^l x\, e^{-w_\mu x}\,dx = fp\int_0^\infty x^\alpha \log^l x\, e^{-w_\mu x}\,dx$ becomes

$$E = w_\mu^{-(\alpha+1)}\sum_{k=0}^{l} C^k_l \log^{l-k}[w_\mu^{-1}]fp\int_0^\infty (w_\mu x)^\alpha \log^k[w_\mu x]\,e^{-w_\mu x}\,d(w_\mu x). \qquad (2\cdot 23)$$

Use of Lemma 1 to apply change of scale $u = w_\mu x$ leads to

$$E^\infty_{\alpha,l}(w_\mu) := \lim_{A\to+\infty} E^A_{\alpha,l}(w_\mu) = w_\mu^{-(\alpha+1)}\sum_{k=0}^{l} C^k_l\log^{l-k}[w_\mu^{-1}]fp\int_0^\infty u^\alpha \log^k u\, e^{-u}\,du$$
$$-\frac{E(\alpha)(-w_\mu)^{-\alpha-1}}{(-\alpha-1)!}\left\{\left[\sum_{k=0}^{l}\frac{C^k_l(-1)^{l-k}}{k+1}\right]\log^{l+1}(w_\mu)+\frac{(i\theta_\mu)^{l+1}}{(l+1)}\right\}. \quad (2\cdot 24)$$

Observe that, in equality (2·24), $\sum_{k=0}^{l} C^k_l(-1)^{l-k}/[k+1] = (-1)^l/(l+1)$. Thus, after some algebra and by gathering the terms such as $E(\alpha)(i\theta_\mu)^{1+j-l-1}\log^{l+1}[w_\mu]$ for $-1\leqslant l\leqslant j+1$, one gets

$$S^\mu_a(\alpha,j) := \lim_{A\to+\infty} S^\mu_a(\alpha,j,A) = w_\mu^{-(\alpha+1)}\,e^{-i(\alpha+1)\theta_\mu}\left\{\frac{E(\alpha)(-1)^{-\alpha+j}}{(-\alpha-1)!\,(j+1)}\log^{j+1}[w_\mu\, e^{i\theta_\mu}]\right.$$
$$\left.+\sum_{l=0}^{j}\sum_{k=0}^{l} C^l_j C^k_l(-1)^{l-k}(-i\theta_\mu)^{j-l}\left[fp\int_0^\infty u^\alpha\log^k u\, e^{-u}\,du\right]\log^{l-k}(w_\mu)\right\}. \qquad (2\cdot 25)$$

Finally, result (2·16) is deduced by letting μ go to 0^+ in (2·25) with $\theta_\mu\to-\pi/2$ and $w_\mu\to a$.

Observe that for $\beta\in C$, $k\in\mathbb{N}$ and a function $g^\beta_k(v) := v^{\beta-1}\log^k(v)\,e^{-v}$, then $g^\beta_k\in\mathscr{P}(]0, +\infty[, C)$ and thereafter $N_k(\beta) := fp\int_0^\infty g^\beta_k(v)\,dv$ exists. If $Re(\beta)>0$, $N_0(\beta) = \int_0^\infty v^{\beta-1}e^{-v}\,dv := \Gamma(\beta)$ where Γ denotes the usual gamma function. Moreover, using integration by parts (always valid when dealing with an integration in the finite part sense of Hadamard (see Schwartz [**16**])), one obtains

$$\beta N_0(\beta) = fp[v^\beta e^{-v}]_0^\infty + N_0(\beta+1) \quad \text{for} \quad \beta \neq 0. \qquad (2\cdot 26)$$

$$N_0(0) = fp[\ln(v)\,e^{-v}]_0^\infty + A \quad \text{with} \quad A = \int_0^\infty \ln(v)\,e^{-v}\,dv := -C_e, \qquad (2\cdot 27)$$

where C_e designates Euler's constant. Here, $fp[\ln(v)\,e^{-v}]_0^\infty = -fp[\ln(\epsilon)\,e^{-\epsilon}] = 0$ and according to definition 2, $fp[v^\beta e^{-v}]_0^\infty = -fp[\epsilon^\beta e^{-\epsilon}] = 0$ if β is not a negative integer, else for $\beta = -p$ with $p \in \mathbb{N}^*$, $fp[v^{-p} e^{-v}] = (-1)^{p+1}/p!$. Consequently for $Re(\beta) \leqslant 0$, if β is not a negative integer $N_0(\beta) = \Gamma(\beta+m)/[\beta(\beta+1)\dots(\beta+m-1)]$ where m is any positive integer such that $Re(\beta) > -m$, else for $p \in \mathbb{N}$ the equality $N_0(0) = -C_e$ and induction relation (2·26), i.e. $(p+1)N_0(-p-1) = (-1)^{p+1}/(p+1)! - N_0(-p)$, show that $N_0(-p) = (-1)^p \psi(p+1)/p!$, where $\psi(p+1) := \sum_{l=1}^{p} l^{-1} - C_e$ is the usual digamma function. For instance, if $Re(\alpha) > -1$, $j = 0$ then Lemma 2 leads to $\lim_{\mu\to 0^+} \int_0^\infty x^\alpha e^{(ia-\mu)x}\,dx = \Gamma(\alpha+1)\, e^{\frac{i\pi}{2}(\alpha+1)} a^{-(\alpha+1)}$.

Lemma 3. *For* $(a, w) \in \mathbb{R}_+^{*2}$ *and* $f \in \mathscr{P}(]0, +\infty[, C)$ *then*

$$pf^* \int_0^\infty f(x)\, e^{iax}\, dx = fp^* \int_0^\infty f(wt)\, e^{iawt}\, d(wt) + \sum_{i=0}^{I} \sum_{j=0}^{J(i)} f_{ij}^0 \frac{E(\alpha_i)\,(ia)^{-1-\alpha_i}}{(-1-\alpha_i)!} \frac{\log^{j+1} w}{j+1}.$$

Proof. According to Proposition 2, $I := pf^* \int_0^\infty f(x)\, e^{iax}\, dx$ becomes for $B \geqslant A_f$,

$$I = fp \int_0^B f(x)\, e^{iax}\, dx + \int_B^\infty F^\infty(x)\, e^{iax}\, dx + \sum_{l=0}^{L} \sum_{k=0}^{K(l)} f_{lk}^\infty\, pf^* \int_B^\infty x^{-\gamma_l} \log^k x\, e^{iax}\, dx. \quad (2.28)$$

Lemma 1 provides the corrective terms arising for the first integral on the right hand side of (2.28). If $(a, \delta) \in \mathbb{R}_+^{*2}$, $\alpha \in C$ with $Re(\alpha) \geqslant -1$, $j \in \mathbb{N}$ and $L_\delta(\mu) := \int_B^\infty x^{-\alpha} \log^j x\, e^{(ia-\delta\mu)x}\, dx$, then use of $m := [\![Re(\alpha)]\!] + 2 \geqslant 1$ integrations by parts yields

$$L_\delta(\mu) = \sum_{l=0}^{m-1} \frac{f^{(l)}(B)\, e^{(ia-\delta\mu)B} (-1)^{l+1}}{(ia-\delta\mu)^{l+1}} + (-1)^m \int_B^\infty f^{(m)}(x) \frac{e^{(ia-\delta\mu)x}}{(ia-\delta\mu)^m}\, dx, \quad (2·29)$$

where $f(x) := x^\alpha \log^j x$ and $f^{(m)} \in L^1([B, +\infty[, C)$. Application of the usual Lebesgue's theorem gives $\lim_{\mu\to 0^+} L_\delta(\mu) = \sum_{l=0}^{m-1} f^{(l)}(B)\, e^{iaB} (ia^{-1})^{l+1} + i^m a^{-m} \int_B^\infty f^{(m)}(x)\, e^{iax}\, dx$ for any $\delta > 0$. Consequently, for $w \in \mathbb{R}_+^*$

$$\begin{aligned} fp^* \int_B^\infty x^{-\gamma_l} \log^k x\, e^{iax}\, dx &= \lim_{\mu\to 0^+} \int_B^\infty x^{-\gamma_l} \log^k x\, e^{(ia-w^{-1}\mu)x}\, dx \\ &= \lim_{\mu\to 0^+} \int_{Bw^{-1}}^\infty (wt)^{-\gamma_l} \log^k[wt]\, e^{(iaw-\mu)t}\, d(wt) \\ &= fp^* \int_{Bw^{-1}}^\infty (wt)^{-\gamma_l} \log^k[wt]\, e^{(iaw-\mu)t}\, d(wt), \end{aligned}$$

and this latter relation leads to the stated result. It is also easy to show that definition (2·8) may be replaced by: $fp^* \int_0^\infty f(x)\, e^{iax}\, dx := \lim_{\mu\to 0^+} [\int_0^\infty f(x)\, e^{(ia-\delta\mu)x}\, dx]$ for any $\delta \in \mathbb{R}_+^*$.

3. *Treatment of* $F(\lambda)$

Since it will be useful to tackle the general case of $I(\lambda)$, this section is restricted to the establishing of the asymptotic expansion, with respect to the large and real parameter λ, of the singular Fourier integral

$$F(\lambda) := fp \int_0^b f(x)\, e^{i\lambda x}\, dx, \quad (3·1)$$

for $0 < b \leqslant +\infty$ and complex pseudofunction f belonging to $\mathscr{F}_E(]0, b[, C)$, a specific subset of $\mathscr{P}(]0,+\infty[,C)$. By the way, the proposed results will also apply to the case of pseudofunctions f admitting singular behaviour near a finite number of points in $[0,b]$ for $b<+\infty$, else in $[0,+\infty[$.

Definition 4. For $E\in\mathbb{N}$, $d\in\mathbb{R}_+^*$ the sets $\mathscr{L}_E(d)$ and $\mathscr{D}_E^0(]0,+\infty[,C)$ are defined by:

(*a*) $\mathscr{L}_E(d):=\{h, h(x)=x^sH(x)$ with $s>E-1$ and H bounded in $[0,d]$ and also if $E\geqslant 1$ then $h^{(E)}\in L^1([0,d],C)$ and $\forall e\in\{0,\dots,E-1\}$: $\lim_{x\to 0^+}h^{(e)}(x)=0\}$,

(*b*) $\mathscr{D}_E^0(]0,+\infty[,C)$ is the set of complex pseudofunctions f such that

1. $f\in\mathscr{P}(]0,+\infty[,C)$ with, for expansion (2·2), $f_{lk}^\infty:=0$ if $Re(\gamma_l)\leqslant 0$;
2. decomposition (2·1) takes the form

$$f(x)=\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f_{ij}^0x^{\alpha_i}\log^j x+h(x),\quad \text{in }]0,\eta_f] \tag{3·2}$$

with $h\in\mathscr{L}_E(\eta_f)$ and $Re(\alpha_0)<\dots<Re(\alpha_I)\leqslant E-1$;

3. if $E=0$ then f is continuous on $]0,+\infty[$ and $I_0(\lambda):=\int_1^\infty f(x)\,e^{i\lambda x}\,dx$ converges uniformly for λ large enough, else f is E times continuously differentiable in $]0,+\infty[$, $I_E(\lambda):=\int_1^\infty f^{(E)}(x)\,e^{i\lambda x}\,dx$ converges uniformly for λ large enough and $\forall\mu>0$, $\forall e\in\{0,\dots,E-1\}$: $\lim_{x\to+\infty}e^{-\mu x}f^{(e)}(x)=0$.

Observe that in this definition, assumption 1 with $f_{lk}^\infty=0$ for $Re(\gamma_l)\leqslant 0$ implies that $I_0(\lambda)=\int_1^\infty f(x)\,e^{i\lambda x}\,dx$ converges uniformly.

Theorem 1. *If* $E\in\mathbb{N}$ *and* $f\in\mathscr{D}_E^0(]0,+\infty[,C)$ *then* $F(\lambda):=fp\int_0^\infty f(x)\,e^{i\lambda x}\,dx$ *admits the following asymptotic behaviour, as* $\lambda\to+\infty$,

$$F(\lambda)=\lambda^{-E}R_E(\lambda)+\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f_{ij}^0\left\{\frac{E(\alpha_i)(-1)^{-\alpha_i+j}}{(-\alpha_i-1)!\,(j+1)}\log^{j+1}[\lambda\, e^{-i\pi/2}]\right.$$
$$\left.+\sum_{l=0}^{j}\sum_{k=0}^{l}C_j^lC_l^k(-1)^{l-k}\left(i\frac{\pi}{2}\right)^{j-l}\left[fp\int_0^\infty u^{\alpha_i}\log^k u\,e^{-u}\,du\right]\log^{l-k}\lambda\right\}e^{i\frac{\pi}{2}(\alpha_i+1)}\lambda^{-(\alpha_i+1)}, \tag{3·3}$$

where complex function $R_E(\lambda):=i^E\int_0^\infty h^{(E)}(x)\,e^{i\lambda x}\,dx=o(1)$, *as* $\lambda\to+\infty$.

Proof. Case 1: $(f_{lk}^\infty)=(0)$.

Since $f\in\mathscr{P}(]0,+\infty[,C)$ with $(f_{lk}^\infty)=(0)$, Proposition 2 yields to the important relation: $F(\lambda)=\lim_{\mu\to 0^+}F_\mu(\lambda)$ if $F_\mu(\lambda):=fp\int_0^\infty f(x)\,e^{(i\lambda-\mu)x}\,dx$. If the function h obeys $h(x):=f(x)-\sum_{i=0}^I\sum_{j=0}^{J(i)}f_{ij}^0x^{\alpha_i}\log^jx$ for $x>0$, then the new integral $F_\mu(\lambda)$ rewrites

$$F_\mu(\lambda)=\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f_{ij}^0fp\int_0^\infty x^{\alpha_i}\log^jx\,e^{(i\lambda-\mu)x}\,dx+\int_0^\infty h(x)\,e^{(i\lambda-\mu)x}\,dx, \tag{3·4}$$

where function $\lambda^{-E}R_E^\mu(\lambda):=\int_0^\infty h(x)\,e^{(i\lambda-\mu)x}\,dx$ exists because $h\in L^1([0,\eta_f],C)$, h is continuous in $]0,+\infty[$ and $\mu>0$. Moreover for $E\geqslant 1$, h is E times continuously differentiable in $]0,+\infty[$ and thereafter applying E times a legitimate integration by parts procedure, one gets

$$\lambda^{-E}R_E^\mu(\lambda)=\sum_{e=0}^{E-1}\left[\frac{h^{(e)}(x)\,e^{(i\lambda-\mu)x}(-1)^e}{(i\lambda-\mu)^{e+1}}\right]_0^\infty+\left(\frac{-1}{i\lambda-\mu}\right)^E\int_0^\infty h^{(E)}(x)\,e^{i\lambda x}\,e^{-\mu x}\,dx. \tag{3·5}$$

Due to assumption 3 and since $h \in \mathscr{L}_E(\eta)$ the sum on the right-hand side of (3·5) is zero and if $h^{(0)} := h$, this ensures for $E \geqslant 0$,

$$\lambda^{-E} R_E^{\mu}(\lambda) = (\mu - i\lambda)^{-E} \int_0^{\infty} h^{(E)}(x)\, e^{i\lambda x}\, e^{-\mu x}\, dx. \tag{3·6}$$

Observe that $g_E(x) := h^{(E)}(x)\, e^{i\lambda x}$ is piecewise continuous in $]0, +\infty[$. According to Olver[**14**, lemma 2], $\lim_{\mu\to 0^+} \int_0^{\infty} g_E(x)\, e^{-\mu x}\, dx = \int_0^{\infty} g_E(x)\, dx$ as soon as this latter integral converges. Here, $h^{(E)} \in L^1([0, \eta_f], C)$ whereas near infinity

$$h^{(E)}(x) = f^{(E)}(x) - \sum_{i=0}^{I} \sum_{j=0}^{J_E(i)} f_{ij}^0\, a_E(j)\, x^{\alpha_i - E} \log^j x$$

with $Re(\alpha_i - E) \leqslant Re(\alpha_I - E) \leqslant 0$. Thus, the integral $\int_0^{\infty} h^{(E)}(x)\, e^{i\lambda x}\, dx$ converges uniformly for λ large enough and $\lim_{\mu\to 0^+} R_E^{\mu}(\lambda) = R_E(\lambda) = i^E \int_0^{\infty} h^{(E)}(x)\, e^{i\lambda x}\, dx$. Moreover, use of an extension of the Riemann–Lebesgue lemma (cf. Olver[**14**, theorem 4·1, p. 73]) applied to the piecewise continuous function $h^{(E)}$ ensures that $R_E(\lambda) = o(1)$, as $\lambda \to +\infty$. To conclude, each contribution $S_\lambda(\alpha_i, j) = \lim_{\mu\to 0^+} fp \int_0^{\infty} x^{\alpha_i} \log^j x\, e^{(i\lambda - \mu)x}\, dx$, arising in relation (3·4) when μ goes to 0^+, is provided by equality (2·16).

Case 2: $(f_{lk}^{\infty}) \neq (0)$.

In such a case, $\mathscr{E} := \{l \text{ such that } (f_{lk}^{\infty}) \neq (0)\} \neq \varnothing$ and for $l \in \mathscr{E}: 0 < Re(\gamma_l) \leqslant 1$. If $0 < B < +\infty$, thanks to Proposition 2 and Olver's result [**15**, lemma 2], one gets

$$F(\lambda) = fp \int_0^{B} f(x)\, e^{i\lambda x}\, dx + \int_B^{\infty} f(x)\, e^{i\lambda x}\, dx = \lim_{\mu\to 0^+} \left[fp \int_0^{\infty} f(x)\, e^{(i\lambda - \mu)x}\, dx \right].$$

Moreover, the new pseudofunction $u(x) := f(x) - \sum_{l\in\mathscr{E}} \sum_{k=0}^{K(l)} f_{lk}^{\infty} x^{-\gamma_l} \log^k x$ turns out to belong to $\mathscr{D}_E^0(]0, +\infty[, C)$ with $(u_{lk}^{\infty}) = (0)$ and also for $K_\mu^0(\lambda) := fp \int_0^{\infty} f(x)\, e^{i(\lambda - \mu)x}\, dx$

$$K_\mu^0(\lambda) = fp \int_0^{\infty} u(x)\, e^{i(\lambda - \mu)x}\, dx + \sum_{l\in\mathscr{E}} \sum_{k=0}^{K(l)} f_{lk}^{\infty} fp \int_0^{\infty} x^{-\gamma_l} \log^k x\, e^{i(\lambda - \mu)x}\, dx. \tag{3·7}$$

Application of Lemma 2 and of previous treatment for case 1, leads to result (3·3).

It is worth noting that the asymptotic behaviour of $F(-\lambda) = fp \int_0^{\infty} f(x)\, e^{-i\lambda x}\, dx$ may be obtained by replacing throughout this paper the complex number i by $-i$. Observe also that Theorem 1 provides an explicit form for the remainder $\lambda^{-E} R_E(\lambda) = o(\lambda^{-E})$ but assumes the smoothness of certain derivatives of pseudo-function f everywhere in $]0, +\infty[$. When these assumptions break down at a finite number of points in $[0, +\infty[$, the next extension is possible.

Definition 5. For $N \in \mathbb{N}^*$ and $E \in \mathbb{N}$, $\mathscr{D}_E^N(]0, +\infty[, C)$ is the set of complex pseudo-functions f such that there exist a real family $(x_n)_{n\in\{0,\ldots,N+1\}}$ with $0 := x_0 < x_1 < \ldots < x_N < x_{N+1} := +\infty$, real values $0 < \eta_f < \mathrm{Min}\,[(x_{n+1} - x_n)/2]$, $A_f > 0$ and $F^{\infty} \in L^1_{\mathrm{loc}}([A_f, +\infty[, C)$ such that:

1. assumptions 1 and 2 of definition 4 for $\mathscr{D}_E^0(]0, +\infty[, C)$ hold;
2. $\forall n \in \{1, \ldots, N\}$ there exist positive integers $I_{\pm}^n$, families of positive integers

$(J^n_\pm(i))$, complex families $(f^{n\pm}_{ij})$, $(\beta^{n\pm}_i)$ and complex functions $h^\pm_n \in \mathscr{L}_E(\eta_f)$ such that for $s\in\{-,+\}$

$$f(x_n+su) = \sum_{i=0}^{I^n_s}\sum_{j=0}^{J^n_s(i)} f^{ns}_{ij}\, u^{\beta^{ns}_i} \log^j u + h^s_n(u), \quad \text{for} \quad u\in\,]0,\eta_f[, \tag{3·8}$$

$$Re(\beta^{ns}_0) < \ldots < Re(\beta^{ns}_{I^n_s}) \leqslant E-1, \quad (f^{ns}_{0j}) = (0) \quad \text{else } S^s_{x_n} := Re(\beta^{ns}_0); \tag{3·9}$$

3. for $n\in\{0,\ldots,N\}$, $f^{(E)}$ exists and is continuous on $]x_n, x_{n+1}[$;

4. $I_E(\lambda) := \int_{x_{N+1}}^\infty f^{(E)}(x)\, e^{i\lambda x}\, dx$ converges uniformly for λ large enough and if $E \geqslant 1$ then $\lim_{x\to+\infty} e^{-\mu x} f^{(e)}(x) = 0$, $\forall\mu>0$, $\forall e\in\{0,\ldots,E-1\}$.

Since each element of $\mathscr{D}^N_E(],+\infty[,C)$ may be singular at each point x_n (as soon as $S^+_{x_n} \leqslant -1$ or $S^-_{x_n} \leqslant -1$) one has to define the integral $F(\lambda) := fp\int_0^\infty f(x)\, e^{i\lambda x}\, dx$. For $0 \leqslant c < b < +\infty$, $g\in\mathscr{P}(]c,b[,C)$ if and only if $G\in\mathscr{P}(]0,+\infty[,C)$ where $G(x) := g(x)$ for $c<x<b$, else $G(x) := 0$ and $fp\int_c^b g(x)\,dx := fp\int_0^\infty G(x)\,dx$. When $a\in\mathbb{R}_+$ and $f\in\mathscr{D}^N_E(]0,+\infty[,C)$, observe that $fp\int_{x_N}^\infty f(x)\, e^{iax}\,dx$ exists and

$$g(x) := f(x)\, e^{iax} \in \mathscr{P}(]x_n, x_{n+1}[, C), \forall n\in\{0,\ldots,N\}.$$

Consequently, it is legitimate to introduce

$$F(\lambda) = fp\int_0^\infty f(x)\, e^{i\lambda x}\, dx := fp\int_{x_N}^\infty f(x)\, e^{i\lambda x}\, dx$$
$$+\sum_{n=0}^{N-1}\left\{e^{i\lambda x_n} fp\int_0^{\frac{x_{n+1}-x_n}{2}} f(x_n+u)\, e^{i\lambda u} du + e^{i\lambda x_{n+1}} fp\int_0^{\frac{x_{n+1}-x_n}{2}} f(x_{n+1}-u)\, e^{-i\lambda u}\, du\right\}. \tag{3·10}$$

THEOREM 2. *If $N\in\mathbb{N}^*$, $E\in\mathbb{N}$ and $f\in\mathscr{D}^N_E(]0,+\infty[,C)$ then the integral $F(\lambda)$ above presented, as $\lambda\to+\infty$, the following asymptotic expansion*

$$F(\lambda) = o(\lambda^{-E}) + \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^0_{ij}\left\{\frac{E(\alpha_i)(-1)^{-\alpha_i+j}}{(-\alpha_i-1)!\,(j+1)}\log^{j+1}[\lambda\, e^{-i\pi/2}]\right.$$
$$\left.+\sum_{l=0}^{j}\sum_{k=0}^{l} C^l_j C^k_l (-1)^{l-k}\left(i\frac{\pi}{2}\right)^{j-l}\left[fp\int_0^\infty u^{\alpha_i}\log^k u\, e^{-u}\, du\right]\log^{l-k}\lambda\right\} e^{i\frac{\pi}{2}(\alpha_i+1)}\,\lambda^{-(\alpha_i+1)}$$
$$+\sum_{n=1}^{N}\sum_{s\in\{-,+\}}\sum_{i=0}^{I^n_s}\sum_{j=0}^{J^n_s(i)} f^{ns}_{ij}\, e^{i\lambda x_n}\left\{\frac{E(\beta^{ns}_i)(-1)^{-\beta^{ns}_i+j}}{(-\beta^{ns}_i-1)!\,(j+1)}\log^{j+1}[\lambda\, e^{-is\pi/2}]\right.$$
$$\left.+\sum_{l=0}^{j}\sum_{k=0}^{l} C^l_j C^k_l(-1)^{l-k}\left(is\frac{\pi}{2}\right)^{j-l}\left[fp\int_0^\infty u^{\beta^{ns}_i}\log^k u\, e^{-u}\, du\right]\log^{l-k}\lambda\right\}$$
$$\times e^{is\frac{\pi}{2}(\beta^{ns}_i+1)}\,\lambda^{-(\beta^{ns}_i+1)}. \tag{3·11}$$

Proof. For $f\in\mathscr{D}^N_E(]0,+\infty[,C)$, $F(\lambda)$, defined by (3·10), may be written

$$F(\lambda) = fp\int_{x_N}^\infty f(x)\, e^{i\lambda x}\, dx + \sum_{n=0}^{N-1} fp\int_{x_n}^{x_{n+1}} f(x)\, e^{i\lambda x}\, dx. \tag{3·12}$$

Step 1. Introduction of function g such that $g(u) := e^{i\lambda x_N} f(x_N+u)$ for $u>0$, yields $A_N(\lambda) := fp\int_{x_N}^\infty f(x)\, e^{i\lambda x}\, dx = fp\int_0^\infty g(u)\, e^{i\lambda u}\, du$. Moreover, $g\in\mathscr{D}^0_E(]0,+\infty[,C)$ since it satisfies the properties (*a*) and (*b*) following.

(*a*) $g\in\mathscr{P}(]0,+\infty[,C)$ and

$$g(u) = \textstyle\sum_{l=0}^L \sum_{k=0}^{K(l)} \sum_{m=0}^k C^m_K f^\infty_{lk}\, e^{i\lambda x_N} u^{-\gamma_l} \log^m u H_{lkm}(u^{-1}) + e^{i\lambda x_N} F^\infty(x_N+u)$$

for $u\in[A_f, +\infty[$ (see decomposition (2·2)) with $0 < Re(\gamma_l) \leqslant 1$ and each function $H_{lkm}(t) := (1+x_N t)^{-\gamma_l}\log^{k-m}[1+x_N t]$ is smooth near zero on the right. Hence, $H_{lkm}(u^{-1}) = H_{lkm}(0)+u^{-1}R_{lkm}(u)$ with $|R_{lkm}(u)|$ bounded for u large enough. This ensures that, near infinity,

$$g(u) = \sum_{l=0}^{L}\sum_{k=0}^{K(l)}\sum_{m=0}^{k} C_k^m f_{lk}^{\infty} H_{lkm}(0)\, e^{i\lambda x_N} u^{-\gamma_l}\log^m u + G^{\infty}(u) \tag{3·13}$$

where $G^{\infty}\in L^1_{\text{loc}}([A_f, +\infty[, C)$.

(*b*) Decomposition (3·2) holds for g with $h(x) := h_N^+(x)$ and the reader may check that property 3 of definition 4 is also fulfilled. Consequently, application of Theorem 1 provides the asymptotic behaviour of integral $A_N(\lambda)$, as $\lambda\to+\infty$.

Step 2. For $0<\delta'<\delta<\eta_f$, consider a function ν defined on $[0, +\infty[$ and E times continuously differentiable on $[0, +\infty]$ with $\nu(x) := 1$ for $0\leqslant x\leqslant \delta'$ and $\nu(x) := 0$ if $x\geqslant\delta$. Such a function exists and is called a neutralizer by Van der Corput [**3**] (see also Erdelyi [**4**] and Jones [**12**]). For $n\in\{0,\dots,N-1\}$, we introduce $A_n(\lambda) := fp\int_{x_n}^{x_{n+1}} f(x)\, e^{i\lambda x}\, dx$ and also the real functions $\nu_n^+(x) := \nu(x-x_n)$, $\nu_{n+1}^-(x) := \nu(x_{n+1}-x)$ and $\nu_t := \nu_n^+ + \nu_{n+1}^-$. Noting that $1_{[x_n, x_{n+1}]} = (1-\nu_t)+\nu_t$, one obtains $A_n(\lambda) = A_n^1(\lambda)+A_n^2(\lambda)+A_n^3(\lambda)$ with $A_n^1(\lambda) = \int_{x_n}^{x_{n+1}}(1-\nu_t)(x)f(x)\, e^{i\lambda x}\, dx$ and

$$A_n^2(\lambda) = fp\int_{x_n}^{x_{n+1}} \nu_n^+(x)f(x)\, e^{i\lambda x}\, dx; \quad A_n^3(\lambda) = fp\int_{x_n}^{x_{n+1}} \nu_{n+1}^-(x)f(x)\, e^{i\lambda x}\, dx. \tag{3·14}$$

If $E\geqslant 1$, since $(1-\nu_t)^{(e)}(x_n) = (1-\nu_t)^{(e)}(x_{n+1}) = 0$ for $e\in\{0,\dots,E-1\}$, application of E times integrations by parts yields (thanks to the Riemann–Lebesgue lemma)

$$A_n^1(\lambda) = (i\lambda^{-1})^E\int_{x_n}^{x_{n+1}}[(1-\nu_t)f]^{(E)}(x)\, e^{i\lambda x}\, dx = o(\lambda^{-E}), \quad \text{as} \quad \lambda\to+\infty. \tag{3·15}$$

Regarding integrals $A_n^2(\lambda)$ and $A_n^3(\lambda)$, respectively, changes of variables $x = x_n+u$ or $x = x_{n+1}-u$ combined with the definition of ν_n^+ and of ν_{n+1}^- lead to

$$A_n^2(\lambda) = e^{i\lambda x_n} fp\int_0^{\infty} g_2(u)\, e^{i\lambda u}\, du; \quad A_n^3(\lambda) = e^{i\lambda x_{n+1}} fp\int_0^{\infty} g_3(u)\, e^{-i\lambda u}\, du \tag{3·16}$$

with $g_2(u) := \nu(u)f(x_n+u)$ and $g_3(u) := \nu(u)f(x_{n+1}-u)$. For $f\in\mathscr{D}_E^N(]0, +\infty[, C)$, then $g_2\in\mathscr{D}_E^0(]0, +\infty[, C)$ and application of Theorem 1 gives the asymptotic expansion of $A_n^2(\lambda)$. Finally, the behaviour of $A_n^3(\lambda)$ is obtained by replacing i by $-i$ in Theorem 1 after noting that g_3 satisfies the associated and modified Definition 4.

Theorem 2 has many applications. For instance, it ensures:

PROPOSITION 3. *For $E\in\mathbb{N}$ and real values $-\infty<a<b<+\infty$, $f\in\mathscr{D}_E^0(]a,b[, C)$ if and only if pseudofunction f is E times continuously differentiable in $]a,b[$ and there exists $0<\eta_f<b-a$, $h_a\in\mathscr{L}_E(\eta_f)$, $h_b\in\mathscr{L}_E(\eta_f)$ such that for $u\in]0,\eta_f]$*

$$f(a+u) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f_{ij}^a u^{\alpha_i}\log^j u + h_a(u); \quad Re(\alpha_0)<\dots<Re(\alpha_I)\leqslant E-1, \tag{3·17}$$

$$f(b-u) = \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^b u^{\beta_l}\log^k u + h_b(u); \quad Re(\beta_0)<\dots<Re(\beta_L)\leqslant E-1. \tag{3·18}$$

For $f \in \mathscr{D}^0_E(]a,b[,C)$, *then* $L(\lambda) := fp\int_a^b f(x)\,e^{i\lambda x}\,dx$ *exists and admits, as* $\lambda \to +\infty$, *the following expansion*:

$$L(\lambda) = o(\lambda^{-E}) + \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^a_{ij}\,e^{i\lambda a}\left\{\frac{E(\alpha_i)(-1)^{-\alpha_i+j}}{(-\alpha_i-1)!\,(j+1)}\log^{j+1}[\lambda\,e^{-i\pi/2}]\right.$$
$$+\sum_{l=0}^{j}\sum_{k=0}^{l} C^l_j C^k_l (-1)^{l-k}\left(i\frac{\pi}{2}\right)^{j-l}\left[fp\int_0^\infty u^{\alpha_i}\log^k u\,e^{-u}\,du\right]\left.\log^{l-k}\lambda\right\}e^{i\frac{\pi}{2}(\alpha_i+1)}\,\lambda^{-(\alpha_i+1)}$$
$$+\sum_{l=0}^{L}\sum_{k=0}^{K(l)} f^b_{lk}\,e^{i\lambda b}\left\{\frac{E(\beta_l)(-1)^{-\beta_l+k}}{(-\beta_l-1)!\,(k+1)}\log^{k+1}[\lambda\,e^{i\pi/2}]\right.$$
$$+\sum_{n=0}^{k}\sum_{m=0}^{n} C^n_k C^m_n (-1)^{k-m}\left(i\frac{\pi}{2}\right)^{k-n}\left[fp\int_0^\infty u^{\beta_l}\log^m u\,e^{-u}\,du\right]\left.\log^{n-m}\lambda\right\}$$
$$\times e^{i\frac{\pi}{2}(\beta_l+1)}\,\lambda^{-(\beta_l+1)}. \tag{3·19}$$

Equality (3·19) is easily obtained by introducing pseudofunction g such that $g(u) := e^{i\lambda a} f(a+u)$ for $0 < u < b-a$, else $g(u) := 0$. Noting that $g \in \mathscr{D}^1_E(]0,+\infty[,C)$ with $x_1 = b-a$, application of Theorem 2 indeed ensures expansion of $L(\lambda) = fp\int_0^\infty g(u)\,e^{i\lambda u}\,du$.

If $\mathscr{F}_E(]0,+\infty[,C) := \mathscr{D}^0_E(]0,+\infty[,C)$ and for $0 < b < +\infty$, $\mathscr{F}_E(]0,b[,C) := \{f \in \mathscr{D}^0_E(]0,b[,C)$ such that $f \in L^1([b-\eta_f,b],C)\}$ then use of Theorem 1 or Proposition 3 provides for $0 < b \leqslant +\infty$ and $f \in \mathscr{F}_E(]0,b[,C)$ the asymptotic behaviour of $F(\lambda) := fp\int_0^b f(x)\,e^{i\lambda x}\,dx$, where symbol fp is only needed for the potential singularity at zero.

Example 1. For $E \in \mathbb{N}$, $-\infty < a < b < +\infty$, $(\alpha,\beta) \in C^2$, $(J,K) \in \mathbb{N}^2$ and a function g which is E times continuously differentiable in $]a,b[$, N_a (resp. N_b) times continuously differentiable in $[a,a+\eta]$ (resp. in $[b-\eta,b]$) for some $0 < \eta < b-a$ if $N_a := \mathrm{Max}\,\{E, [\![E-Re(\alpha)]\!]\}$ and $N_b := \mathrm{Max}\,\{E,[\![E-Re(\beta)]\!]\}$ then (see Section 2 for definition of $N_k(\beta)$ and $\sum_{n=0}^{M} := 0$ if $M < 0$)

$$L(\lambda) = fp\int_a^b (x-a)^\alpha (b-x)^\beta \log^J(x-a)\log^K(b-x)\,g(x)\,e^{i\lambda x}\,dx = o(\lambda^{-E})$$
$$+\sum_{n=0}^{N_a-1}\frac{e^{i\lambda a}}{n!}\frac{d^n}{dx^n}[(b-x)^\beta\log^K(b-x)\,g(x)](a)\left\{\frac{E(\alpha+n)(-1)^{-\alpha-n+J}}{(-\alpha-n-1)!\,(J+1)}\log^{J+1}[\lambda\,e^{-i\pi/2}]\right.$$
$$+\sum_{l=0}^{J}\sum_{k=0}^{l} C^l_J C^k_l(-1)^{l-k}\left(i\frac{\pi}{2}\right)^{J-l} N_k(\alpha+n+1)\left.\log^{l-k}\lambda\right\}e^{i\frac{\pi}{2}(\alpha+n+1)}\,\lambda^{-(\alpha+n+1)}$$
$$+\sum_{n=0}^{N_b-1}\frac{e^{i\lambda b}}{n!}\frac{d^n}{dx^n}[(x-a)^\alpha\log^J(x-a)\,g(x)](b)\left\{\frac{E(\beta+n)(-1)^{-\beta+K}}{(-\beta-n-1)!\,(K+1)}\log^{K+1}[\lambda\,e^{i\pi/2}]\right.$$
$$+\sum_{l=0}^{K}\sum_{k=0}^{l} C^l_K C^k_l(-1)^{l-k+n}\left(-i\frac{\pi}{2}\right)^{K-l} N_k(\beta+n+1)\left.\log^{l-k}\lambda\right\}e^{-i\frac{\pi}{2}(\beta+n+1)}\,\lambda^{-(\beta+n+1)}. \tag{3·20}$$

Proof. Here, pseudofunction $f(x) := (x-a)^\alpha (b-x)^\beta \log^J(x-a)\log^K(b-x)\,g(x)$ is E times continuously differentiable in $]a,b[$ and one has to check expansions (3·17) and (3·18). For instance, if $x \to a^+$ it is possible to write $f(x) = (x-a)^\alpha \log^J(x-a)\,w(x)$ with

$w(x) := (b-x)^\beta \log^K(b-x)\, g(x)$ and thanks to definition of N_a to obtain, for $u \to 0^+$, (and $\sum_{n=0}^{N_a-1} := 0$ if $N_a < 1$)

$$\left.\begin{aligned} f(a+u) &= \sum_{n=0}^{N_a-1} \frac{w^{(n)}(a)}{n!} u^{\alpha+n} \log^J u + h_a(u);\ Re(\alpha+n) \leqslant Re(\alpha+N_a-1) \leqslant E-1, \\ h_a(u) &= u^\alpha \log^J u \left[w(a+u) - \sum_{n=0}^{N_a-1} \frac{w^{(n)}(a)}{n!} u^{\alpha+n} \right] := [u^\alpha \log^J u]\, W(u), \quad u \geqslant 0. \end{aligned}\right\} \tag{3.21}$$

For $q \in \{0, \ldots, E\}$, since $E \leqslant N_a$, function $w^{(q)}$ is $N_a - q$ times continuously differentiable for $0 \leqslant u < \eta$ and satisfies $w^{(q)}(a+u) = \sum_{m=0}^{N_a-1-q} [(q+m)!]^{-1} w^{(q+m)}(a)\, u^m + u^{N_a-q} R_q(u)$ where R_q is bounded in $[0, \eta]$. Observe that this latter relation ensures $W^{(q)}(u) = u^{N_a-q} R_q(u)$ if function W is introduced by (3·21). For $q = 0$, this shows that $h_a(u) = [u^{\alpha+N_a} \log^J u] R_0(u)$ with $Re(\alpha) + N_a > E-1$, i.e. $h_a(u) = u^s H_a(u)$ with $s > E-1$, H_a bounded in $[0, \eta]$. Moreover, if $E \geqslant 1$, then for $e \in \{0, \ldots, E\}$ and $u > 0$ there exists complex $a^J_{eq}(\alpha)$ such that

$$h_a^{(e)}(u) = \sum_{q=0}^{e} C_e^q [u^\alpha \log^J u]^{(e-q)}\, W^{(q)}(u) = \sum_{q=0}^{e} \sum_{m=0}^{J} C_e^q\, a^J_{eq}(\alpha)\, [u^{\alpha+q-e} \log^m u]\, u^{N_a-q} R_q(u).$$

Since $N_a + Re(\alpha) > E-1$ and R_q bounded near zero, for $e \in \{0, \ldots, E-1\}$ then $\lim_{u \to 0^+} h_a^{(e)}(u) = 0$ and $h_a^{(E)} \in L^1([0, \eta], C)$. Hence, $h_a \in \mathscr{L}_E(\eta)$. For

$$h_b(u) := u^\beta \log^K u \left[t(b-u) - \sum_{n=0}^{N_b-1} (n!)^{-1}\, t^{(n)}(b)\, (-1)^n \right]$$

if $t(x) := (x-a)^\alpha \log^J(x-a)\, g(x)$, similar arguments lead to $h_b \in \mathscr{L}_E(\eta)$. To conclude, Proposition 3 yields (3·20).

For instance, if $J = K = 0$, $\alpha = \gamma - 1$, $\beta = \mu - 1$ with $0 < \gamma \leqslant 1$, $0 < \mu \leqslant 1$ then $N_a = N_b = E$, $E(\alpha+n) = E(\beta+n) = 0$, $N_0(\alpha+n+1) = \Gamma(\gamma+n)$ and it follows that

$$\begin{aligned} \int_a^b (x-a)^{\gamma-1} (b-x)^{\mu-1} g(x)\, e^{i\lambda x}\, dx &= \sum_{n=0}^{E-1} \left\{ e^{i\lambda a} \frac{\Gamma(\gamma+n)}{n!} \frac{d^n}{dx^n} [(b-x)^{\mu-1} g(x)]\, (a) \right. \\ &\quad \times e^{i\frac{\pi}{2}(\gamma+n)} \lambda^{-(\gamma+n)} + e^{i\lambda b} \frac{\Gamma(\mu+n)}{n!} \frac{d^n}{dx^n} [(x-a)^{\gamma-1} g(x)]\, (b)\, e^{i\frac{\pi}{2}(n-\mu)} \lambda^{-(\mu+n)} \Big\} + o(\lambda^{-E}), \end{aligned}$$

i.e. it agrees (with the improved remainder $o(\lambda^{-E})$ instead of $O(\lambda^{-E})$) with Erdelyi [**4**, **5**] and Bleistein & Handelsman [**2**].

4. *Asymptotic behaviour of* $I(\lambda)$

Before dealing with $I(\lambda)$, it is convenient to introduce the set $\mathscr{E}^{r_2}_{r_1}(]0, b[, C)$ and real $S_0(f)$, $S_\infty(f)$ for $0 < b \leqslant +\infty$, two real values r_1, r_2 and pseudofunction f. More precisely:

(*a*) $\mathscr{E}^{r_2}_{r_1}(]0, b[, C) :=$ {pseudofunction f such that there exist (α_i), (f^0_{ij}), $(J(i))$ with $Re(\alpha_0) < \ldots < Re(\alpha_i) < \ldots < Re(\alpha_I) \leqslant r_1$, real $s_1 > r_1$ and function $F^0_{r_1}$ bounded in a neighbourhood of zero on the right where $f(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} f^0_{ij} x^{\alpha_i} \log^j x + x^{s_1} F^0_{r_1}(x)$; if $b < +\infty$ then $f \in L^1_{loc}(]0, b], C)$ else $f \in L^1_{loc}(]0, b[, C)$ and there exist (γ_l), (f^∞_{lk}), $(K(l))$ such that $Re(\gamma_0) < \ldots < Re(\gamma_l) < \ldots < Re(\gamma_L) \leqslant r_2$, real $s_2 > r_2$ and function $F^{\infty}_{r_2}$ bounded in a neighbourhood of infinity where $f(x) = \sum_{l=0}^{L} \sum_{k=0}^{K(l)} f^{\infty}_{lk} x^{-\gamma_l} \log^k x + x^{-s_2} F^{\infty}_{r_2}(x)\}$.

(*b*) For f defined on $]0, \eta]$ and $\mathscr{F}(x) := f(x)$ if $x \in]0, \eta]$, else $\mathscr{F}(x) := 0$ then if there exist (r_1, r_2) with $\mathscr{F} \in \mathscr{E}^{r_2}_{r_1}(]0, +\infty[, C)$ and $(f^0_{0j}) \neq (0)$ then $S_0(f) := Re(\alpha_0)$.

(*c*) For f defined on $[A, +\infty[$ and $\mathscr{F}(x) := f(x)$ if $x \in [A, +\infty[$, else $\mathscr{F}(x) := 0$ then if there exist (r_1, r_2) with $\mathscr{F} \in \mathscr{E}^{r_2}_{r_1}(]0, +\infty[, C)$ and $(f^{\infty}_{0k}) \neq (0)$ then $S_{\infty}(f) := Re(\gamma_0)$.

The next definition presents a specific set of pseudofunctions $K(x, u)$ for which the asymptotic expansion of $I(\lambda)$ will be provided by Theorem 3.

Definition 6. For real values $r_1, r_2, 0 < b \leqslant +\infty$, pseudofunction $K(x, u)$ belongs to the set $\mathscr{A}^{r_2}_{r_1}(]0, b[, C)$ if and only if for λ large enough $f(x) := K(x, \lambda x)$ for $0 < x < b$, else $f(x) := 0$ belongs to $\mathscr{P}(]0, +\infty[, C)$ with for decomposition (2·2) $f^{\infty}_{lk} = 0$ if $Re(\gamma_l) \leqslant 0$ and all the next assumptions are fulfilled:

1. There exist positive integers N, I, two complex families $(\gamma_n), (\alpha_i)_{i \in \{0,\ldots,I\}}$ with $Re(\gamma_0) < \ldots < Re(\gamma_N) := r_2$, $Re(\alpha_0) < \ldots < Re(\alpha_I) := r_1$, two families of positive integers $(M(n))$, $(J(i))_{i \in \{0,\ldots,I\}}$ and of complex pseudofunctions $(K_{nm}(x)), (h^{ij}(u))$, complex functions $G_{r_2}(x, u), H_{r_1}(x, u)$ and real values $s_2 > r_2$, $s_1 > r_1$, $B_2 \geqslant 0$, $B_1 \geqslant 0$, $0 < \eta \leqslant A < +\infty$, $W > 0$ such that

$$K(x, u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}(x)\, u^{-\gamma_n} \log^m u + u^{-s_2} G_{r_2}(x, u), \quad (x, u) \in]0, b[\times [\eta, +\infty[, \tag{4·1}$$

$$K(x, u) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} h^{ij}(u)\, x^{\alpha_i} \log^j x + x^{s_1} H_{r_1}(x, u), \quad (x, u) \in]0, W[\times]0, +\infty[, \tag{4·2}$$

$$\int_{\eta}^{b} x^{-s_2} |G_{r_2}(x, \lambda x)|\, dx \leqslant B_1 < +\infty, \quad \int_{0}^{A} u^{s_1} |H_{r_1}(u/\lambda, u)|\, du \leqslant B_2 < +\infty. \tag{4·3}$$

2. If for $n \in \{0, \ldots, N\}$ positive integers E_n, V_n and V obey $E_n - 1 < r_2 - Re(\gamma_n) \leqslant E_n$, $V_n - 1 < E_n + Re(\gamma_n) \leqslant V_n$ if $Re(\gamma_n) > 0$, else $V_n := E_n$, $V := \mathrm{Max}_{n \in \{0,\ldots,N\}}(V_n, r_1 + 1)$ then there exist positive integer I', family of positive integers $(J(i))_{i \in \{0,\ldots,I'\}}$, complex families $(K^{ij}_{nm}), (\alpha_i)_{i \in \{0,\ldots,I'\}}$ with $Re(\alpha_0) < \ldots < Re(\alpha_{I'}) := V - 1$ such that:

2·1. For $i \in \{0, \ldots, I\}$, $j \in \{0, \ldots, J(i)\}$ then $h^{ij} \in \mathscr{E}^{1+Re(\alpha_i)}_{-1-Re(\alpha_i)}(]0, +\infty[, C)$ and there exist $A_{ij} > 0$, $\eta_{ij} > 0$, $t_{ij} > -Re(\alpha_i) - 1$, complex families $(H^{ij}_{pq}), (\beta_q)$ with $Re(\beta_0) < \ldots < Re(\beta_P) \leqslant -Re(\alpha_i) - 1$ and

$$h^{ij}(u) = \sum_{p=0}^{P} \sum_{q=0}^{Q(p)} H^{ij}_{pq}\, u^{\beta_p} \log^q u + u^{t_{ij}} R_{ij}(u); \quad R_{ij} \text{ bounded in } [0, \eta_{ij}], \tag{4·4}$$

$$h^{ij}(u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K^{ij}_{nm}\, u^{-\gamma_n} \log^m u + u^{-s_2} O_{ij}(u); \quad O_{ij} \text{ bounded in } [A_{ij}, +\infty[. \tag{4·5}$$

2·2. $\forall n \in \{0, \ldots, N\}$, $\forall m \in \{0, \ldots, M(n)\}$ there exist $\eta_{nm} > 0$ and $h_{nm} \in \mathscr{L}_{V_n}(\eta_{nm})$ such that if $I'_n = \mathrm{Max}\{i \in \{0, \ldots, I'\};\, Re(\alpha_i) \leqslant V_n - 1\}$ then

$$K_{nm}(x) = \sum_{i=0}^{I'_n} \sum_{j=0}^{J(i)} K^{ij}_{nm}\, x^{\alpha_i} \log^j x + h_{nm}(x); \quad x \in]0, \eta_{nm}], \tag{4·6}$$

$$K_{nm}(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K^{ij}_{nm}\, x^{\alpha_i} \log^j x + x^{s_1} L_{nm}(x), \quad L_{nm} \text{ bounded in } [0, \eta]. \tag{4·7}$$

Moreover if $b = +\infty$ then $\forall l \in \{0, \ldots, m\}\, K_{nm}(x)\, x^{-\gamma_n} \log^l x \in \mathscr{D}^0_{E_n}(]0, +\infty[, C)$ else K_{nm}

is E_n times continuously differentiable in $]0,b[$ and there exist complex families $(B^{nm}_{es}), (\delta^{nm}_e)$ such that $-1 < Re(\delta^{nm}_0) < \ldots < R(\delta^{nm}_{\mathscr{E}}) \leqslant E_n - 1$, $K^{nm}_b \in \mathscr{L}_{E_n}(\eta_{nm})$ with

$$K_{nm}(b-u) = \sum_{e=0}^{\mathscr{E}} \sum_{s=0}^{S(e)} B^{nm}_{es}\, u^{\delta^{nm}_e} \log^s u + K^{nm}_b(u), \quad u \in]0, \eta_{nm}]. \tag{4·8}$$

3. Function W_{r_1,r_2} such that $x^{s_1}u^{-s}W_{r_1,r_2}(x,u) := K(x,u) - \sum_{n=0}^{N}\sum_{m=0}^{M(n)} K_{nm}(x) \times u^{-\gamma_n}\log^m u - \sum_{i=0}^{I}\sum_{j=0}^{J(i)}[h^{ij}(u) - \sum_{n=0}^{N}\sum_{m=0}^{M(n)} K^{ij}_{nm}\, u^{-\gamma_n}\log^m u]\, x^{\alpha_i}\log^j x$ is bounded in $]0,\eta] \times [A, +\infty[$.

For $n \in \{0,\ldots,N\}$, definition of V_n yields $V_n \geqslant r_2 \geqslant Re(\gamma_n)$. Observe that relation (4·6) implies an adequate behaviour near zero on the right for pseudofunction $K_{nm}(x)\, x^{-\gamma_n}\log^{m-l}(x)$. By the way, if $V_n \geqslant r_1 + 1$ then (4·6) also ensures (4·7). By now, if families of positive integers $(M(n))$ and of complex $(\alpha_n), (a_{nm})$ are such that $m \in \{0,\ldots,M(n)\}$, $Re(\alpha_0) < \ldots < Re(\alpha_n) < \ldots$ then for

$$(r', r, t) \in \mathbb{R}^2 \times \mathbb{R}^*_+,\ S_r(t) = \sum_{m,\, Re(\alpha) \leqslant r} a_{nm}\, t^{\alpha_n}\log^m t := \sum_{n=0}^{N}\sum_{m=0}^{M(n)} a_{nm}\, t^{\alpha_n}\log^m t,$$

where $N := \mathrm{Sup}\{\text{n, such that } Re(\alpha_n) \leqslant r\}$ and if $r' < r$ then $\sum_{m,\, r' < Re(\alpha) \leqslant r} a_{nm}\, t^{\alpha_n}\log^m t := S_r(t) - S_{r'}(t)$.

THEOREM 3. *For $r \in \mathbb{R}$ and $K(x,u) \in \mathscr{A}^{r_2}_{r_1}(]0,b[,C)$ with $r \leqslant \mathrm{Max}\{r_1+1, r_2\}$ then, as $\lambda \to +\infty$, $I(\lambda)$ admits the following asymptotic behaviour*

$$\begin{aligned} I(\lambda) = fp\int_0^b K(x,\lambda x)\, e^{i\lambda x}\, dx = &\sum_{j,\, Re(\alpha) \leqslant r-1}\ \sum_{l=0}^{j} C^l_j(-1)^l \Bigg\{ fp^* \int_0^\infty h^{ij}(u)\, u^{\alpha_i}\log^{j-l}(u)\, e^{iu}\, du \\ &- \sum_{p=0}^{P}\sum_{q=0}^{Q(p)} i^{-1-\beta_p-\alpha_i}\frac{H^{ij}_{pq} E(\beta_p+\alpha_i)}{(-1-\beta_p-\alpha_i)!}\frac{\log^{1+j+q-l}\lambda}{1+j+q-l}\Bigg\}\lambda^{-(\alpha_i+1)}\log^l\lambda \\ &+ \Delta(b) \sum_{m,\, Re(\gamma) \leqslant r}\ \sum_{l=0}^{m}\ \sum_{Re(\delta^{nm}_e+\gamma_n+k) \leqslant r-1} C^l_m B^{nm}_{es}\frac{e^{i\lambda b}}{k!}\frac{d^k}{du^k}[G^{m-l}_{\gamma_n}]\,(b) \\ &\times \Bigg\{\sum_{v=0}^{s}\sum_{p=0}^{v} C^v_s C^p_v(-1)^{s-p+k}\left(i\frac{\pi}{2}\right)^{s-v} N_p(\delta^{nm}_e+k+1)\log^{v-p}\lambda \\ &+ \frac{E(\delta^{nm}_e+k)(-1)^{-\delta^{nm}_e+s}}{(-\delta^{nm}_e-k-1)!\,(s+1)}\log^{s+1}[\lambda e^{i\frac{\pi}{2}}]\Bigg\} e^{-i\frac{\pi}{2}(\delta^{nm}_e+k+1)}\lambda^{-(\delta^{nm}_e+\gamma_n+k+1)}\log^l\lambda + o(\lambda^{-r}), \end{aligned} \tag{4·9}$$

where $\Delta(b) := 1$ for $0 < b < +\infty$, $\Delta(\infty) := 0$ and $G^j_\alpha(u) := u^{-\alpha}\log^j u$ for $(\alpha, j) \in C \times \mathbb{N}$.

Proof. According to Olver[15, lemma 2], for $Re(\gamma_l) > 0$ and also for $d > 0$ then $\int_d^\infty x^{-\gamma_l}\log^k x\, e^{i\lambda x}\, dx = \lim_{\mu\to 0^+}\int_d^\infty x^{-\gamma_l}\log^k x\, e^{(i\lambda-\mu)x}\, dx$. Hence, combination of Proposition 2 and of property $f(x) = K(x,\lambda x) \in \mathscr{P}(]0,+\infty[,C)$ with $f^\infty_{lk} = 0$ if $Re(\gamma_l) \leqslant 0$, implies the basic relation

$$I(\lambda) = \lim_{\mu\to 0^+} I_\mu(\lambda) \quad \text{with} \quad I_\mu(\lambda) := fp\int_0^b K(x,\lambda x)\, e^{(i\lambda-\mu)x}\, dx. \tag{4·10}$$

Thus, the question reduces to the derivation of the asymptotic behaviour of $I_\mu(\lambda)$ for $\mu > 0$. Under the properties and notation proposed by definition 6, this is achieved in several steps. For a sake of concision and since further explanations are available in Sellier[17, theorem 1], the first part of the derivation will be briefly reported in steps 1 and 2 whereas the second part will be detailed by step 3 and the Appendix.

For large λ, real δ such that $\lambda^{-1}\eta \leqslant \delta \leqslant \eta$ is introduced in order to take into account decompositions (4·1) and (4·2). Indeed one writes $I_\mu(\lambda) = I'_\mu(\lambda) + I''_\mu(\lambda)$, where $I'_\mu(\lambda) := \int_\delta^b K(x, \lambda x)\, e^{(i\lambda-\mu)x}\, dx$ and $I''_\mu(\lambda) := fp \int_0^\delta K(x, \lambda x)\, e^{(i\lambda-\mu)x}\, dx$ are treated separately in steps 1 and 2.

Step 1. Since $\lambda x \geqslant \lambda\delta \geqslant \eta$ for $I'_\mu(\lambda)$, expansion (4·1) holds and if $T'_\mu(\lambda) := \int_\delta^b (\lambda x)^{-s_2} G_{r_2}(x, \lambda x)\, e^{(i\lambda-\mu)x}\, dx$ it yields

$$I'_\mu(\lambda) = \sum_{m,\, Re(\gamma) \leqslant r_2} \left[fp \int_0^b - fp \int_0^\delta\right] K_{nm}(x)\,(\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x}\, dx + T'_\mu(\lambda). \quad (4·11)$$

For $0 \leqslant x \leqslant \delta \leqslant \eta$, relation (4·7) is valid for each pseudofunction K_{nm} and allows us to cast $I'_\mu(\lambda)$ into the following form

$$I'_\mu(\lambda) = -\sum_{m,\, Re(\gamma) \leqslant r_2}\ \sum_{j,\, Re(\alpha) \leqslant r_1} K^{ij}_{nm}\, fp \int_0^\delta (\lambda x)^{-\gamma_n} x^{\alpha_i} \log^m(\lambda x) \log^j x\, e^{(i\lambda-\mu)x}\, dx$$

$$+ \sum_{m,\, Re(\gamma) \leqslant r_2} fp \int_0^b K_{nm}(x)\,(\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x}\, dx + T'_\mu(\lambda) + U'_\mu(\lambda), \quad (4·12)$$

where each integral occurring on the right-hand side of (4·12) and complex $U'_\mu(\lambda) := -\sum_{m,\, Re(\gamma) \leqslant r_2} fp \int_0^\delta x^{s_1} L_{nm}(x)\,(\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x}\, dx$ admit a sense since, under proposed assumptions 2·2, integral $fp \int_0^b g(x)\, e^{(i\lambda-\mu)x}\, dx$ exists for $g(x) = x^{\alpha_i} \log^j x$ and $g(x) = K_{nm}(x)\, x^{-\gamma_n} \log^{m-l} x$.

Step 2. Regarding $I''_\mu(\lambda)$, successive use of behaviours (4·2) and (4·5) for h^{ij} easily ensures that

$$I''_\mu(\lambda) = \sum_{j,\, Re(\alpha) \leqslant r_1} \left[fp \int_0^\infty - fp \int_\delta^\infty\right] h^{ij}(\lambda x)\, x^{\alpha_i} \log^j x\, e^{(i\lambda-\mu)x}\, dx + T''_\mu(\lambda)$$

$$= -\sum_{m,\, Re(\gamma) \leqslant r_2}\ \sum_{j,\, Re(\alpha) \leqslant r_1} K^{ij}_{nm}\, fp \int_\delta^\infty (\lambda x)^{-\gamma_n} x^{\alpha_i} \log^m(\lambda x) \log^j(x)\, e^{(i\lambda-\mu)x}\, dx$$

$$+ \sum_{j,\, Re(\alpha) \leqslant r_1} fp \int_0^\infty h^{ij}(\lambda x)\, x^{\alpha_i} \log^j x\, e^{(i\lambda-\mu)x}\, dx + T''_\mu(\lambda) + U''_\mu(\lambda), \quad (4·13)$$

where $T''_\mu(\lambda) := \int_0^\delta x^{s_1} H_{r_1}(x, \lambda x)\, e^{(i\lambda-\mu)x}\, dx$, each integral arising on the right-hand side of (4·13) exists since $h^{ij} \in \mathscr{E}^{1+Re(\alpha_i)}_{-1-Re(\alpha_i)}(]0, +\infty[, C)$ and

$$U''_\mu(\lambda) = -\sum_{j,\, Re(\alpha) \leqslant r_1} \int_\delta^\infty (\lambda x)^{-s_2} O_{ij}(\lambda x)\, x^{\alpha_i} \log^j x\, e^{(i\lambda-\mu)x}\, dx$$

has a sense for $\mu > 0$.

Step 3. By adding equalities (4·12) and (4·13), one gets $I_\mu(\lambda) = I^1_\mu(\lambda) + I^2_\mu(\lambda) + I^3_\mu(\lambda) + R_\mu(\lambda)$ with $R_\mu(\lambda) = T'_\mu(\lambda) + T''_\mu(\lambda) + U'_\mu(\lambda) + U''_\mu(\lambda)$ and the next definitions

$$I^1_\mu(\lambda) = -\sum_{m,\, Re(\gamma) \leqslant r_2}\ \sum_{j,\, Re(\alpha) \leqslant r_1} K^{ij}_{nm}\, fp \int_0^\infty (\lambda x)^{-\gamma_n} x^{\alpha_i} \log^m(\lambda x) \log^j(x)\, e^{(i\lambda-\mu)x}\, dx, \quad (4·14)$$

$$I^2_\mu(\lambda) = \sum_{m,\, Re(\gamma) \leqslant r_2} fp \int_0^b K_{nm}(x)\,(\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x}\, dx, \quad (4·15)$$

$$I^3_\mu(\lambda) = \sum_{j,\, Re(\alpha) \leqslant r_1} fp \int_0^\infty h^{ij}(\lambda x)\, x^{\alpha_i} \log^j(x)\, e^{(i\lambda-\mu)x}\, dx. \quad (4·16)$$

For $l\in\{1,2,3\}$, the asymptotic behaviour of $I^l(\lambda):=\lim_{\mu\to 0^+} I_\mu(\lambda)$ is obtained as follows:

(*a*) Note that $I^3(\lambda)=\sum_{j,\,Re(\alpha)\leqslant r_1}\sum_{l=0}^{j} C_j^l(-1)^l D_l^{ij}(\lambda)\,\lambda^{-\alpha_i+1}\log^l\lambda$ where $D_l^{ij}(\lambda):= fp^*\int_0^\infty h^{ij}(\lambda x)(\lambda x)^{\alpha_i}\log^{j-l}(\lambda x)\,e^{i\lambda x}d(\lambda x)$. Since $h^{ij}\in\mathscr{E}^{1+Re(\alpha_i)}_{-1-Re(\alpha_i)}(]0,+\infty[,C)$, change of variable $u=\lambda x$ is legitimate for each integral $D_l^{ij}(\lambda)$. Taking into account expansion (4·4), application of Lemma 3 immediately leads to

$$I^3(\lambda)=\sum_{j,\,Re(\alpha)\leqslant r_1}\sum_{l=0}^{j} C_j^l(-1)^l\left\{fp^*\int_0^\infty h^{ij}(u)\,u^{\alpha_i}\log^{j-l}(u)\,e^{iu}\,du\right.$$
$$\left.-\sum_{p=0}^{P}\sum_{q=0}^{Q(p)}E(\alpha_i+\beta_p)\frac{i^{-1-\beta_p-\alpha_i}H_{pq}^{ij}}{(-1-\beta_p-\alpha_i)!}\frac{\log^{1+j+q-l}\lambda}{1+j+q-l}\right\}\lambda^{-(\alpha_i+1)}\log^l\lambda. \quad (4{\cdot}17)$$

(*b*) Thanks to Proposition 2 (see (2·8)) and Lemma 2, it is straightforward to cast $I^1(\lambda)$ into the form

$$I^1(\lambda)=-\sum_{m,\,Re(\gamma)\leqslant r_2}\sum_{j,\,Re(\alpha)\leqslant r_1}\sum_{l=0}^{m} C_m^l K_{nm}^{ij} S_\lambda(\alpha_i-\gamma_n, j+m-l)\,\lambda^{-\gamma_n}\log^l\lambda$$
$$=-\sum_{m,\,Re(\gamma)\leqslant r_2}\sum_{j,\,Re(\alpha)\leqslant r_1}\sum_{l=0}^{m} C_m^l K_{nm}^{ij}\left\{\frac{E(\alpha_i-\gamma_n)(-1)^{\gamma_n-\alpha_i+j+m-l}}{(\gamma_n-\alpha_i-1)!\,(j+m-l+1)}\log^{j+m-l+1}[\lambda e^{-i\frac{\pi}{2}}]\right.$$
$$\left.+\sum_{v=0}^{j+m-l}\sum_{k=0}^{v} C_{j+m-l}^{v} C_v^k\left(i\frac{\pi}{2}\right)^{j+m-l-v} N_k(\alpha_i-\gamma_n+1)\log^{v-k}[\lambda^{-1}]\right\}$$
$$\times e^{i\frac{\pi}{2}(\alpha_i-\gamma_n+1)}\lambda^{-(\alpha_i+1)}\log^l\lambda. \quad (4{\cdot}18)$$

(*c*) If $b=+\infty$, remind that $f_{nml}(x):=K_{nm}(x)\,x^{-\gamma_n}\log^{m-l}(x)\in\mathscr{D}^0_{E_n}(]0,+\infty[,C)$ for $l\in\{0,\ldots,m\}$. Accordingly, each Fourier integral $F^l_{nm}(\lambda)$ such that

$$F^l_{nm}(\lambda):=fp^*\int_0^b K_{nm}(x)\,x^{-\gamma_n}\log^{m-l}(x)\,e^{i\lambda x}\,dx=fp\int_0^b K_{nm}(x)\,x^{-\gamma_n}\log^{m-l}(x)\,e^{i\lambda x}\,dx,$$

exists and thereafter $I^2(\lambda)=\sum_{m,\,Re(\gamma)\leqslant r_2}\sum_{l=0}^{m} C_m^l F^l_{nm}(\lambda)\,\lambda^{-\gamma_n}\log^l\lambda$. Thanks to relation (4·6), for $x\in[0,\eta_{nm}]$ it is possible to rewrite pseudofunction f_{nml} as $f_{nml}(x)=\sum_{j,\,Re(\alpha)\leqslant E_n+Re(\gamma_n)-1}K^{ij}_{nm}\,x^{\alpha_i-\gamma_n}\log^{j+m-l}(x)+H_{nml}(x)$ where the reader may check that the assumption $h_{nm}\in\mathscr{L}_{V_n}(\eta_{nm})$ combined with definition of V_n ensures that $H_{nml}(x):=\sum_{j,\,E_n+Re(\gamma_n)<Re(\alpha)+1\leqslant V_n}K^{ij}_{nm}\,x^{\alpha_i-\gamma_n}\log^{j+m-l}(x)\,x^{-\gamma_n}\log^{m-l}(x)+h_{nm}(x)$ belongs to $\mathscr{L}_{E_n}(\eta_{nm})$. At this stage, two cases arise regarding the asymptotic behaviour of $F^l_{nm}(\lambda)$.

(i) If $b=+\infty$, then previous remarks show that theorem 1 applies to $F^l_{nm}(\lambda)$. More precisely, one gets $F^l_{nm}(\lambda)=F^l_{nm}(\lambda,0)+o(\lambda^{-E_n})$ where complex $F^l_{nm}(\lambda,0)$ is related to the behaviour of pseudofunction f_{nml} near zero and obeys (see (3·3))

$$F^l_{nm}(\lambda,0)=\sum_{j,\,Re(\alpha)\leqslant E_n+Re(\gamma_n)-1}K^{ij}_{nm}\left\{\frac{E(\alpha_i-\gamma_n)(-1)^{\gamma_n-\alpha_i+j+m-l}}{(\gamma_n-\alpha_i-1)!\,(j+m-l+1)}\log^{j+m-l+1}[\lambda e^{-i\frac{\pi}{2}}]\right.$$
$$\left.+\sum_{v=0}^{j+m-l}\sum_{k=0}^{v} C_{j+m-l}^{v} C_v^k\left(i\frac{\pi}{2}\right)^{j+m-l-v} N_k(\alpha_i-\gamma_n+1)\log^{v-k}[\lambda^{-1}]\right\}$$
$$\times e^{i\frac{\pi}{2}(\alpha_i-\gamma_n+1)}\lambda^{-(\alpha_i-\gamma_n+1)}.$$

(ii) If $b < +\infty$, f_{nml} is E_n times continuously differentiable in $]0,b[$ and behaviour of $f_{nml}(b-u)$ is needed for $u \to 0^+$. Assumption (4·8) with $-1 < Re(\delta_0^{nm}) < \ldots < Re(\delta_{\delta}^{nm}) \leqslant E_n - 1$ and notations $G_{\gamma_n}^{m-l}(x) := x^{-\gamma_n}\log^{m-l}(x)$, $[G_{\gamma_n}^{m-l}]^{(k)}(x) := d^k[G_{\gamma_n}^{m-l}]/[du^k](x)$ indeed yield for $u \in]0,\eta_{nm}]$, $f_{nml}(b-u) = \sum_{s,\,0<Re(\delta_e^{nm}+k+1)\leqslant E_n}[k!]^{-1}\times(-1)^k B_{es}^{nm}[G_{\gamma_n}^{m-l}]^{(k)}(b)\,u^{\delta_e^{nm}+k}\log^s u + f_{nml}^b(u)$ where function f_{nml}^b satisfies

$$f_{nml}^b(u) = \sum_{e=0}^{\delta}\sum_{s=0}^{S(e)} B_{es}^{nm}\,u^{\delta_e^{nm}+k}\log^s(u)\left[G_{\gamma_n}^{m-l}(b-u) - \sum_{k=0}^{E_n}[G_{\gamma_n}^{m-l}]^{(k)}(b)\,[k!]^{-1}(-1)^k u^k\right]$$

$$+K_b^{nm}(u)\,G_{\gamma_n}^{m-l}(b-u) + \sum_{s,\,E_n-1<Re(\delta^{nm})+k\leqslant 2E_n-1}\frac{(-1)^k}{k!}B_{es}^{nm}[G_{\gamma_n}^{m-l}]^{(k)}(b)\,u^{\delta_e^{nm}+k}\log^s u,$$

where for the last sum in the above equality: $0 \leqslant k \leqslant E_n$. It is thereafter easy to check that $f_{nml}^b \in \mathscr{L}_{E_n}(\eta_{nm})$. Consequently, proposition 3 applies to $F_{nm}^l(\lambda)$ and allows us to write $F_{nm}^l(\lambda) = F_{nm}^l(\lambda,0) + F_{nm}^l(\lambda,b) + o(\lambda^{-E_n})$ with $F_{nm}^l(\lambda,b)$ associated to the behaviour of f_{nml} near point b on the left and given by (see result (3·11))

$$F_{nm}^l(\lambda,b) = \sum_{s,\,Re(\delta_e^{nm}+k)\leqslant E_n-1} B_{es}^{nm}\frac{e^{i\lambda b}}{k!}[G_{\gamma_n}^{m-l}]^{(k)}(b)\left\{\frac{E(\delta_e^{nm}+k)(-1)^{-\delta_e^{nm}+s}}{(-\delta_e^{nm}-k-1)!\,(s+1)}\log^{s+1}[\lambda e^{i\frac{\pi}{2}}]\right.$$

$$\left.+\sum_{v=0}^{s}\sum_{p=0}^{v} C_s^v C_v^p(-1)^{s-p+k}\left(i\frac{\pi}{2}\right)^{s-v} N_p(\delta_e^{nm}+k+1)\log^{v-p}\lambda\right\} e^{-i\frac{\pi}{2}(\delta_e^{nm}+k+1)}\,\lambda^{-(\delta_e^{nm}+k+1)}.$$

For the above sum it is important to note that $Re(\delta_e^{nm}+k) > -1$ because $Re(\delta_e^{nm}) \geqslant Re(\delta_0^{nm}) > -1$. Hence, $F_{nm}^l(\lambda,b) = o(1)$, as $\lambda \to +\infty$. Keeping in mind this property, assumption $r \leqslant \mathrm{Max}\{r_1+1, r_2\}$ and definitions of integers E_n and V_n with $E_n - 1 < r_2 - Re(\gamma_n) \leqslant E_n$ and $V_n \geqslant r_2$ clearly lead (after cancellation, up to order $o(\lambda^{-r})$, of terms of $I^1(\lambda)$ and of $\sum_{m,\,Re(\gamma)\leqslant r_2}\sum_{l=0}^{m} C_m^l F_{nm}^l(\lambda,0)\,\lambda^{-\gamma_n}\log^l\lambda$) to $I^1(\lambda)+I^2(\lambda) = \Delta(b)\sum_{m,\,Re(\gamma)\leqslant r}\sum_{l=0}^{m} C_m^l F_{nm}^l(\lambda,b,r)\,\lambda^{-\gamma_n}\log^l\lambda + o(\lambda^{-r})$ if $F_{nm}^l(\lambda,b,r)$ is obtained by replacing in above definition of $F_{nm}^l(\lambda,b)$ sum $\sum_{s,\,-1<Re(\delta_e^{nm}+k)\leqslant E_n-1}$ by the sum $\sum_{s,\,-1<Re(\delta_e^{nm}+k)\leqslant r-Re(\gamma_n)-1}$. To conclude, the appendix shows that in such circumstances $R(\lambda) := \lim_{\mu\to 0^+} R_\mu(\lambda) = o(\lambda^{-r})$ and the asymptotic expansion of $I^3(\lambda)$, up to order $o(\lambda^{-r})$, turns out to be the first sum on the right-hand side of (4·9).

5. *Applications*

This last section exhibits a few applications or examples which are treated by the proposed approach. Example 2 below consists in an application of Theorem 3.

Example 2. For $N \in \mathbb{N}^*$ and $0 < b < +\infty$, assume that pseudofunction f and integer N obey $N \geqslant \mathrm{Max}\{1+S_0(f), 1\}$, f is $N-1$ times continuously differentiable in $]0,b[$ and there exist $\eta > 0$, complex families (δ_l), (β_p) such that $S_0(f) := Re(\delta_0) < \ldots < Re(\delta_l) < \ldots \leqslant N-1$, $-1 < Re(\beta_0) < \ldots < Re(\beta_p) \leqslant \ldots \leqslant N-1$ and functions f_0, f_b belonging to $\mathscr{L}_N(\eta)$ with

$$f(x) = \sum_{j,\,Re(\delta_l)\leqslant N-1} f_{lj}^0 x^{\delta_l}\log^j x + f_0(x); \quad x \in]0,\eta], \tag{5·1}$$

$$f(b-u) = \sum_{s,\,Re(\beta_p)\leqslant N-1} f_{es}^b u^{\beta_p}\log^s u + f_b(u); \quad u \in]0,\eta]. \tag{5·2}$$

In such circumstances, one obtains:

$$fp\int_0^b \frac{f(x)\,e^{i\lambda x}\,dx}{1+x+\lambda x}$$

$$= \sum_{j,\,Re(\delta_l+k)\leqslant N-1}\ \sum_{m=0}^{j} C_j^m f_{lj}^0(-1)^{m+k}\Big\{fp^*\int_0^\infty \frac{u^{\delta_l+k}\log^{j-m}(u)}{(1+u)^{k+1}}e^{iu}\,du$$

$$-E(\delta_l)\sum_{p=0}^{-1-\delta_l-k}\frac{(-1)(p+k)!\,i^{-1-\delta_l-k-p}}{p!\,k!(-1-\delta_l-k-p)!}\frac{\log^{1+j-m}\lambda}{1+j-m}\Big\}\lambda^{-(\delta_l+k+1)}\log^m\lambda$$

$$+\sum_{n=0}^{N-1}\ \sum_{s,\,Re(\beta_e+m+n+k+1)\leqslant N-1}^{m\leqslant n} C_n^m(-1)^{m+n+k}\frac{(n+k)!}{n!\,k!}f_{es}^b\,e^{i\lambda b}(1+b)^{n-m}\,b^{-(n+k+1)}$$

$$\times\Big\{\sum_{v=0}^{s}\sum_{p=0}^{v} C_s^v C_v^p(-1)^{s-p+k}\left(i\frac{\pi}{2}\right)^{s-v} N_p(\beta_e+m+k+1)\log^{v-p}\lambda$$

$$+\frac{E(\beta_e+m+k)(-1)^{-\beta_e-m+s}}{(-\beta_e-m-k-1)!\,(s+1)}\log^{s+1}[\lambda e^{i\frac{\pi}{2}}]\Big\}e^{-i\frac{\pi}{2}(\beta_e+m+k+1)}\lambda^{-(\beta_e+m+k+n+2)}+o(\lambda^{-N}).$$

Proof. This result is provided by Theorem 3. Under the proposed assumptions the reader may indeed check that $K(x,u):=f(x)/[1+x+u]\in\mathscr{A}_{N-1}^N(]0,b[,C)$ with $f\in L^1_{\mathrm{loc}}(]0,b],C)$ and for expansion (4·1): $r_2=N$, $s_2=N+1$, $\forall n\in\{0,\ldots,N-1\}$ then $M(n)=m=0, K_{n0}(x)=(-1)^n f(x)(1+x)^n, \gamma_n=n+1, E_n=N-n-1, V_n=N$ and also $G_{r_2}(x,u)=(-1)^N f(x)(1+x)^N/[1+u^{-1}(1+x)]$. This shows that $\int_\eta^b x^{-s_2}|G_{r_2}(x,\lambda x)|\,dx<+\infty$. Use of behaviour (5·1) ensures relation (4·2) with $r_1=N-1$ and for $x\to 0^+$,

$$\frac{f(x)}{1+x+u}=\sum_{j,\,S_0(f)\leqslant Re(\delta_l)+k\leqslant N-1} f_{lj}^0(-1)^k(1+u)^{-(k+1)}x^{\delta_l+k}\log^j x+x^{s_1}H_{r_1}(x,u),$$

where, if $K:=[\![N-S_0(f)-1]\!]$, then

$$x^{s_1}H_{r_1}(x,u)=\sum_{j,\,N-1<Re(\delta_l)+k\leqslant N+K-1} f_{lj}^0(-1)^k(1+u)^{-(k+1)}x^{\delta_l+k}\log^j x$$

$$+[f_0(x)+(-1)^{K+1}(1+u)^{-(K+1)}x^{K+1}f(x)]/[1+x+u].$$

Thus, $\alpha_{lk}=\delta_l+k$, $h^{lkj}(u)=(-1)^k f_{lj}^0(1+u)^{-(k+1)}$. Accordingly, h^{lkj} belongs to $\mathscr{E}_{-1-Re(\alpha_{lk})}^{1+Re(\alpha_{lk})}(]0,+\infty[,C)$ with for behaviour (4·4) $Q(p)=q=0$, $\beta_p=p$, $H_{p0}^{lkj}=f_{lj}^0(-1)^{p+k}(p+k)!/[p!\,k!]$ and for relation (4·5) $K_{n0}^{lkj}=0$ if $0\leqslant n\leqslant k-1$, else for $k\leqslant n\leqslant N-1$, $K_{n0}^{lkj}=f_{lj}^0(-1)^n n!/[k!(n-k)!]$. Definition of $K_{n0}(x)$ and (5·1) yield expansions (4·6) and (4·7) with

$$h_{n0}(x)=(-1)^n(1+x)^n f_0(x)+\sum_{j,\,N-1<Re(\delta_l+k)\leqslant N+n-1} f_{lj}^0\frac{(-1)^n n!}{k!(n-k)!}x^{\delta_l+k}\log^j x,$$

i.e. $h_{n0}\in\mathscr{L}_N(\eta)$. Finally, for $E_n=N-n-1$, assumption (5·2) provides for $u\to 0^+$

$$K_{n0}(b-u)=\sum_{s,\,Re(\beta_e+k)\leqslant N-n-2} C_n^k(-1)^{n+k}f_{es}^b(1+b)^{n-k}u^{\beta_e+k}\log^s u+K_b^{n0}(u),$$

$$K_b^{n0}(u)=\frac{f_b(u)(-1)^n}{(1+b-u)^{-n}}+\sum_{s,\,Re(\beta_e+k)>N-n-2} C_n^k(-1)^{n+k}f_{es}^b(1+b)^{n-k}u^{\beta_e+k}\log^s u,$$

where it is easy to show that $K_b^{n0} \in \mathscr{L}_N(\eta)$.

The case of singular integral $J(\lambda)$ (see (1·3)) is also widely encountered for applications. The next theorem states, under specific assumptions, the asymptotic behaviour of $J(\lambda)$, as $\lambda \to +\infty$.

THEOREM 4. *For $0 < b \leqslant +\infty$ assume that real value t and pseudofunctions f and H satisfy, for λ large enough, $f(x)\,H(\lambda x) \in L^1_{loc}(]0, b[, C)$ and the following properties:*

(*a*) *$t \geqslant \mathrm{Max}\,[-S_0(H), 1+S_0(f), S_\infty(H)]$, there exists $w \geqslant \mathrm{Max}\,[S_0(H), -1-S_0(f)]$ such that $H \in \mathscr{E}^t_w(]0, +\infty[, C)$ with*

$$H(u) = \sum_{q,\, Re(\beta) \leqslant w} H^0_{pq}\, u^{\beta_p} \log^q u + u^{w'} H^0_w(u), \quad w' > w, u \to 0; \tag{5·3}$$

$$H(u) = \sum_{m,\, Re(\gamma) \leqslant t} H^\infty_{nm}\, u^{-\gamma_n} \log^m u + u^{-t'} H^\infty_t(u), \quad t' > t, u \to +\infty, \tag{5·4}$$

where functions H^0_w and H^∞_t are bounded, respectively near zero and at infinity;

(*b*) *if $\forall n \in \{0, \dots, N\}$ positive integers E_n, V_n, E and V obey $E-1 < t-Re(\gamma_n) \leqslant E_n$, $V_n - 1 < E_n + Re(\gamma_n) \leqslant V_n$ for $Re(\gamma_n) > 0$, else $V_n := E_n$, $E-1 < t-S_\infty(H) \leqslant E$ and also $V := \mathrm{Max}\,(V_n)$ then f is E times continuously differentiable in $]0, b)$ and there exists $h \in \mathscr{L}_V(\eta)$ with*

$$f(x) = \sum_{j,\, Re(\alpha) \leqslant V-1} f^0_{ij}\, x^{\alpha_i} \log^j x + h(x), \quad x \in]0, \eta]; \tag{5·5}$$

(*c*) *moreover, if $b < +\infty$, then for u near zero and also $Re(\delta_e) > -1$,*

$$f(b-u) = \sum_{s,\, Re(\delta_e) \leqslant E-1} f^b_{es}\, u^{\delta_e} \log^s u + h_b(u), \quad h \in \mathscr{L}_E(\eta) \tag{5·6}$$

and if $b = +\infty$ then $f \in \mathscr{E}^{1-S_\infty(H)}_{V-1}(]0, +\infty[, C)$, $t \geqslant 1-S_\infty(f)$, $S_\infty(f)+S_\infty(H) > 0$ and $\forall \in \{0, \dots, m\}$ then $f(x)\, x^{-\gamma_n} \log^{m-l}(x) \in \mathscr{D}^0_{E_n}(]0, +\infty[, C)$.

In such circumstances, the following asymptotic behaviour holds for any real $r \leqslant t$

$$\begin{aligned}
&fp \int_0^b f(x)\, H(\lambda x)\, e^{i\lambda x}\, dx \\
&= \sum_{j,\, Re(\alpha) \leqslant r-1} \sum_{l=0}^{j} C^l_j (-1)^l f^0_{ij} \Bigg\{ fp^* \int_0^\infty H(u)\, u^{\alpha_i} \log^{j-l}(u)\, e^{iu}\, du \\
&\quad - \sum_{p=0}^{P} \sum_{q=0}^{Q(p)} i^{-1-\beta_p-\alpha_i} \frac{H^0_{pq}\, E(\beta_p+\alpha_i)}{(-1-\beta_p-\alpha_i)!} \frac{\log^{1+j+q-l}\lambda}{1+j+q-l} \Bigg\} \lambda^{-(\alpha_i+1)} \log^l \lambda \\
&\quad + \Delta(b) \sum_{m,\, Re(\gamma) \leqslant r} \sum_{l=0}^{m} H^\infty_{nm} \sum_{s,\, -1 < Re(\delta_e+k) \leqslant r-Re(\gamma_n)-1} C^l_m f^b_{es} \frac{e^{i\lambda b}}{k!} \frac{d^k}{du^k} [G^{m-l}_{\gamma_n}]\,(b) \\
&\quad \times \Bigg\{ \sum_{v=0}^{s} \sum_{p=0}^{v} C^v_s C^p_v (-1)^{s-p+k} \left(i\frac{\pi}{2} \right)^{s-v} N_p(\delta_e+k+1) \log^{v-p}\lambda \\
&\quad + \frac{E(\delta_e+k)\,(-1)^{-\delta_e+s}}{(-\delta_e-k-1)!\,(s+1)} \log^{s+1}[\lambda e^{i\frac{\pi}{2}}] \Bigg\} e^{-i\frac{\pi}{2}(\delta_e+k+1)}\, \lambda^{-(\delta_e+\gamma_n+k+1)} \log^l \lambda + o(\lambda^{-r}).
\end{aligned} \tag{5·7}$$

Proof. Result (5·7) is obtained by applying Theorem 3 to $K(x, u) := f(x)\, H(u)$ as soon as it belongs to $\mathscr{A}^t_{t-1}(]0, b[, C)$. Due to definition of V_n it is clear that $V_n \geqslant t$, i.e. $V \geqslant t$ and under the proposed assumptions the reader may check the following statements:

(*a*) If $g(x) := f(x)H(\lambda x)$ for $0 < x < b$, else $g(x) := 0$ then $g \in \mathscr{P}(]0, +\infty[, C)$ with $g_{lk}^{\infty} = 0$ for $Re(\gamma_l) \leqslant 0$ because $f(x)H(\lambda x) \in L_{\text{loc}}^1(]0, b[, C)$, near zero behaviours (5·3) and (5·5) hold with $w \geqslant -1 - S_0(f)$, $V - 1 + S_0(H) \geqslant -1$ (with $V \geqslant t \geqslant -S_0(H)$) and if $b := +\infty$ not only $S_\infty(f) + S_\infty(H) > 0$, $t + S_\infty(f) \geqslant 1$ but also $f \in \mathscr{E}_{V-1}^{1-S_\infty(H)}(]0, +\infty[, C)$.

(*b*) Relations (4·1) and (4·2) are satisfied with $r_2 := t$, $s_2 := t'$, $G_{r_2}(x, u) := f(x) H_t^{\infty}(u)$, $K_{nm}(x) = H_{nm}^{\infty} f(x)$, $r_1 := t-1$, $h^{ij}(u) := f_{ij}^0 H(u)$ for i such that $S_0(f) \leqslant Re(\alpha_i) \leqslant t-1$ and finally $x^{s_1} H_{r_1}(x, u) = F_{r_1}(x) H(u)$ if $F_{r_1}(x) := f(x) - \sum_{j, Re(\alpha) \leqslant t-1} f_{ij}^0 x^{\alpha_i} \log^j x$. Note that $\int_\eta^b x^{-s_2} |G_{r_2}(x, \lambda x)| \, dx = \int_\eta^b x^{-t'} |f(x) H_t^{\infty}(\lambda x)| \, dx$ is bounded since for $b < +\infty$, $f \in L_{\text{loc}}^1([\eta, b[, C)$ (as a continuous function on $[\eta, b[$ and, see expansion (5·6), measurable in a neighbourhood of b on the left), else $-t' - S_\infty(f) + 1 < 0$ thanks to $t' > t \geqslant 1 - S_\infty(f)$. Moreover, (5·5) authorizes to write $F_{r_1}(x) = \sum_{j, t-1 < Re(\alpha) \leqslant V-1} f_{ij}^0 x^{\alpha_i} \log^j x + h(x)$ with $h(x) = x^s F(x)$, $s > V-1$ and F bounded near zero on the right. If $s' := \text{Min}\{Re(\alpha_i)$ for i such that $t-1 < Re(\alpha_i) \leqslant V-1\}$, $s'' := \text{Min}\{s, s'\}$ and $s_1 := t - 1 + [s'' + 1 - t]/2$, then $H_{r_1}(x, u) = F_{t-1}^0(x) H(u)$ with

$$F_{t-1}^0(x) = \sum_{j, s' \leqslant Re(\alpha) \leqslant V-1} f_{ij}^0 x^{\alpha_i - s_1} \log^j x + x^{s - s_1} F(x)$$

is bounded near zero. Thus,

$$\int_0^A u^{s_1} |H_{r_1}(u/\lambda, u)| \, du \leqslant \lambda^{s_1 - s} \int_0^A u^s |F(u/\lambda) H(u)| \, du + \sum_{j, s' \leqslant Re(\alpha) \leqslant V-1} \lambda^{s_1 - Re(\alpha_i)} \int_0^A u^{Re(\alpha_i)} |F(u/\lambda) H(u)| \, du. \quad (5·8)$$

Finally, $\text{Min}\{s, s'\} > t - 1 \geqslant -S_0(H) - 1$ shows that (4·3) is true.

(*c*) Since $H \in \mathscr{E}_w^t(]0, +\infty[, C)$ with $w \geqslant -1 - S_0(f)$ and $S_0(f) \leqslant Re(\alpha_i) \leqslant t-1$ then $h^{ij} \in \mathscr{E}_{-1-Re(\alpha_i)}^{1+Re(\alpha_i)}(]0, +\infty[, C)$ and relations (4·4) and (4·5) are fulfilled with $H_{pq}^{ij} := f_{ij}^0 H_{pq}^0$, $K_{nm}^{ij} := f_{ij}^0 H_{nm}^{\infty}$, $O_{ij}(u) := f_{ij}^0 H_t^{\infty}(u)$.

(*d*) Because $V \geqslant t = r_1 + t$, $V := \text{Max}(V_n) = \text{Max}(V_n, r_1 + 1)$ and behaviour (5·5) ensures for $K_{nm}(x) = H_{nm}^{\infty} f(x)$ relations (4·6) and (4·7) with $h_{nm}(x) := H_{nm}^{\infty}[\sum_{j, V_n - 1 < Re(\alpha) \leqslant V-1} f_{ij}^0 x^{\alpha_i} \log^j x + h(x)] \in \mathscr{L}_{V_n}(\eta)$. If $b < +\infty$, (4·8) is provided by assumption (5·6). To conclude, one finds that $W_{r_1, r_2}(x, u) = F_{t-1}^0(x) H_t^{\infty}(u)$ is bounded for $(x, u) \in]0, \eta] \times [A, +\infty[$.

Example 3. Consider $\alpha \in C$, R a real value such that $R \geqslant \text{Max}\{2 - Re(\alpha), 2\}$, and $0 < b < +\infty$. The following asymptotic behaviour holds

$$\begin{aligned} fp \int_0^b \frac{\sin(x) \, e^{i\lambda x} \, dx}{x^\alpha (1 + \lambda x)^2} &= \sum_{l=0}^{[R + Re(\alpha)/2 - 1]} \frac{(-1)^l}{(2l+1)!} \left[fp^* \int_0^\infty \frac{u^{-\alpha + 2l + 1}}{(1+u)^2} e^{iu} \, du \right] \lambda^{-(2l+2-\alpha)} \\ &\quad - P(\alpha) \sum_{l=0}^{[\alpha/2 - 1]} \sum_{p=0}^{\alpha - 2(l+1)} \frac{(-1)^{l+p} (p+1) \, i^{\alpha - p - 2(l+1)}}{(2l+1)! \, [\alpha - p - 2(l+1)]!} \lambda^{-(2l+2-\alpha)} \log \lambda \\ &\quad + \sum_{n=0}^{[R-2]} \sum_{m+k \leqslant R-n+2} \frac{(-1)^{n+m+k} (n+1)}{m! \, k!} \sin^{(m)}(b) \, A_k(-\alpha - n - 2) \, b^{-\alpha - n - 2 - k} \\ &\quad \times (m+k)! \, e^{-i\frac{\pi}{2}(m+k+1)} \, \lambda^{-(m+k+n+3)} + o(\lambda^{-R}), \end{aligned}$$

where $\sin^{(m)}(b)$ refers to the mth derivative of sin at point b, $P(\alpha) := 1$ if $\alpha \in \mathbb{N}\backslash\{0,1\}$, else $P(\alpha) := 0$ and for $(\beta, k) \in C \times \mathbb{N}$ then $A_0(\beta) := 1$, $A_k(\beta) = \beta(\beta-1)\dots(\beta-k+1)$. Thereafter the asymptotic expansion of $I(\lambda)$ involves logarithmic sequence $\lambda^{-(2l+2-\alpha)} \log\lambda$ if and only if $\alpha \in \mathbb{N}\backslash\{0,1\}$.

Proof. By choosing $H(u) := u^{-\alpha}(1+u)^{-2}$ and $f(x) := \sin(x)$, then $fp\int_0^b \sin(x) \times e^{i\lambda x}\,dx/[x^\alpha(1+x)^2] = \lambda^\alpha J(\lambda)$ where integral $J(\lambda) = fp\int_0^b f(x)\,H(\lambda x)\,e^{i\lambda x}\,dx$ is expanded by applying Theorem 4 for $t = R + Re(\alpha)$. More precisely, $H \in \mathscr{E}^t_w(]0,+\infty[, C)$ for any real w and (5·3), (5·4) hold with $Q(p) = q = 0$, $\beta_p = p - \alpha$, $H^0_{p0} = (-1)^p(p+1)$, $S_0(H) = -Re(\alpha)$ and also $n \in \{0,\dots,N\}$, $N = [\![R-2]\!]$, $\gamma_n = a+n+2$, $H^\infty_{nm} = (-1)^n(n+1)$. Moreover, for positive integer M, f is M times continuously differentiable in $[0,b]$ and its Taylor expansion at zero or at b provides relation (5·5) or (5·6).

Example 4. For $(\alpha, J) \in C \times \mathbb{N}$ such that $Re(\alpha) > -1$, $0 < b \leqslant +\infty$ and $R \geqslant \operatorname{Max}\{1 - Re(\alpha), 0\}$ one gets the following asymptotic expansion

$$
\begin{aligned}
fp\int_0^b \frac{\log^J(x)\arctan(\lambda x)}{x^\alpha(1+x)}e^{i\lambda x}\,dx = {} & \sum_{l \leqslant Re(\alpha)+R-1}\sum_{j=0}^{J} C^j_J(-1)^{l+j}\Bigg\{fp^*\int_0^\infty \frac{\arctan(u)\,e^{iu}\,du}{u^{\alpha-l}\log^{j-J}(u)} \\
& -P(\alpha)\sum_{p=0}^{[\![\frac{\alpha-l}{2}-1]\!]} \frac{i^{\alpha-l-2(p+1)}(-1)^p}{(2p+1)[\alpha-l-2(p+1)]!} \\
& \times \frac{\log^{1+J-j}\lambda}{1+J-j}\Bigg\}\lambda^{-(l-\alpha)}\log^j\lambda \\
& -i\Delta(b)\,e^{i\lambda b}\sum_{0\leqslant e\leqslant R-1}\sum_{v=0}^{J}\sum_{p=0}^{v} C^v_J C^p_v(-1)^{J+e-p} \\
& \times\left(i\frac{\pi}{2}\right)^{J+1-v} f^{(e)}(b)\,e^{-i\frac{\pi}{2}(e+1)}\log^{v-p}\lambda \\
& +\Delta(b)\,e^{i\lambda b}\sum_{n=0}^{[\![\frac{R+1}{2}]\!]}\sum_{e+k\leqslant R-2(n+1)}\sum_{v=0}^{J}\sum_{p=0}^{v} C^v_J C^p_v \\
& \times\frac{(-1)^{J+n+k+e+1-p}}{(2n+1)\,e!\,k!}A_k(-2n-1)f^{(e)}(b) \\
& \times(e+k)!\,b^{-1-2n-k}\left(i\frac{\pi}{2}\right)^{J-v} e^{-i\frac{\pi}{2}(e+k+1)} \\
& \times\lambda^{-(e+k+2(n+1))}\log^{v-p}\lambda + o(\lambda^{-R}),
\end{aligned}
$$

where notations $P(\alpha)$ and $A_k(\beta)$ keep the meaning introduced by example 3 and $\sum_{0\leqslant e\leqslant R-1} := 0$ if $R < 1$.

Proof. Here $f(x) = x^{-\alpha}\log^J(x)/[1+x]$ and $H(u) = \arctan(u) = \pi/2 - \arctan(u^{-1})$ for $u \geqslant 0$. Consequently, expansions (5·3) or (5·4) hold with $t = R$, $Q(p) = q = 0$, $\beta_p = 2p+1$, $H^0_{p0} = (-1)^p/(2p+1)$, $S_0(H) = 1$ and also $H^{\infty'}_{00} = \pi/2$ with $\gamma'_0 = 0$ and for $n \in \{0,\dots,N\}$ then $\gamma_n = 2n+1$, $H^\infty_{n0} = (-1)^n/(2n+1)$, $S_\infty(H) = 0$. For $V \in \mathbb{N}^*$, $L := [\![V-1+Re(\alpha)]\!]$ and x near zero, then (5·5) is true with

$$
f(x) = \sum_{l=0}^{L'}(-1)^l x^{-\alpha+l}\log^J(x) + h(x)
$$

where function $h(x) := (-1)^{L+1} x^{L+1-\alpha} \log^J(x)/[1+x] \in \mathscr{L}_V(\eta)$. For $b < +\infty$, the same argument shows that f satisfies (5·6). If $b = +\infty$, $S_\infty(f) = Re(\alpha)+1$ and for $g_n(x) := x^{-\gamma_n} f(x)$ introduction of $L'_n := [\![E_n + Re(\alpha) + Re(\gamma_n) - 1]\!]$ leads, as $u \to +\infty$, to $g_n(x) = \sum_{l=0}^{L'_n} (-1)^l x^{-\alpha-\gamma_n+l} \log^J(x) + h_n(x)$ where $h_n(x) = (-1)^{L'_n+1} x^{-\alpha-\gamma-n+L'_n+1} \log^J(x)/[1+x] \in \mathscr{L}_{E_n}(\eta)$. To conclude, it is straightforward to prove that $g_n \in \mathscr{D}^0_{E_n}(]0, +\infty[, C)$.

Appendix

The aim of this appendix is to show that for $r \leqslant \text{Max}\{r_1+1, r_2\}$ and $R(\lambda) := \lim_{\mu\to 0^+}[R_\mu(\lambda)]$ then $R(\lambda) = o(\lambda^{-r})$, as $\lambda \to +\infty$.

Thanks to the definition of W_{r_1,r_2} and after some algebra $R_\mu(\lambda) = T'_\mu(\lambda) + T''_\mu(\lambda) + U'_\mu(\lambda) + U''_\mu(\lambda)$ rewrites $R_\mu(\lambda) = R^1_\mu(\lambda) + R^2_\mu(\lambda)$ with (see (4·5))

$$R^1_\mu(\lambda) = -\sum_{j,\, Re(\alpha)\leqslant r_1} \int_b^\infty (\lambda x)^{-s_2} O_{ij}(\lambda x)\, x^{\alpha_i} \log^j(x)\, e^{(i\lambda-\mu)x}\, dx, \tag{5·9}$$

$$R^2_\mu(\lambda) = fp\int_0^b x^{s_1}(\lambda x)^{-s_2} W_{r_1,r_2}(x, \lambda x)\, e^{(i\lambda-\mu)x}\, dx. \tag{5·10}$$

If $b = +\infty$ then $R^1_\mu(\lambda) = 0$, else assumption $h^{ij} \in \mathscr{E}^{1+Re(\alpha_i)}_{-1-Re(\alpha_i)}(]0, +\infty[, C)$ guarantees the existence of $s_{ij} > 1 + Re(\alpha_i)$ and of complex function V^{ij} bounded near infinity where $h^{ij}(\lambda x) = \sum_{m,\, Re(\gamma)\leqslant 1+Re(\alpha_i)} D^{ij}_{nm} (\lambda x)^{-\gamma_n} \log^m(\lambda x) + (\lambda x)^{-s_{ij}} V^{ij}(\lambda x)$ with $D^{ij}_{nm} := K^{ij}_{nm}$ for $Re(\gamma_n) \leqslant r_2$. If $d^{ij}_\mu(\lambda) := \int_b^\infty (\lambda x)^{-s_2} O_{ij}(\lambda x)\, x^{\alpha_i} \log^j(\lambda x)\, e^{(i\lambda-\mu)x}\, dx$, two cases occur. If $Re(\alpha_i)+1 \leqslant r_2 < s_2$, then $\lim_{\mu\to 0^+} d^{ij}_\mu(\lambda) = O(\lambda^{-s_2}) = o(\lambda^{-r_2})$ and if $s_{ij} > 1+Re(\alpha_i) > r_2$, one may write

$$d^{ij}_\mu(\lambda) = \lambda^{-s_{ij}} \int_b^\infty x^{\alpha_i - s_{ij}} \log^j(x)\, V^{ij}(\lambda x)\, e^{(i\lambda-\mu)x}\, dx$$

$$+ \sum_{m,\, r_2 < Re(\gamma)\leqslant 1+Re(\alpha_i)} \sum_{l=0}^{m} C^l_m D^{ij}_{nm} J^{nl}_\mu(\lambda)\, \lambda^{-\gamma_n} \log^{m-l}\lambda \tag{5·11}$$

with $J^{nl}_\mu(\lambda) := \int_b^\infty x^{\alpha_i-\gamma_n} \log^{j+l}(x)\, e^{(i\lambda-\mu)x}\, dx$. If $Re(\alpha_i)+1 > Re(\gamma_n)$, $J^{nl}_\mu(\lambda)$ is bounded and if $Re(\alpha_i)+1 = Re(\gamma_n)$ one integration by part leads to $\lim_{\mu\to 0^+} J^{nl}_\mu(\lambda) = o(1)$. Hence, for $0 < b \leqslant +\infty$, $R^1(\lambda) := \lim_{\mu-0^+} R^1_\mu(\lambda) = o(\lambda^{-r_2}) = o(\lambda^{-r})$.

As far as $R^2_\mu(\lambda)$ is concerned, if $g_\mu(\lambda, x) := x^{s_1}(\lambda x)^{-s_2} W_{r_1,r_2}(x, \lambda x)\, e^{(i\lambda-\mu)x}$ three terms are introduced: $C^1_\mu(\lambda) := fp\int_0^{A/\lambda} g_\mu(\lambda, x)\, dx$, $C^2_\mu(\lambda) := fp\int_{A/\lambda}^{\eta} g_\mu(\lambda, x)\, dx$ and also $C^3_\mu(\lambda) := fp\int_\eta^b g_\mu(\lambda, x)\, dx$.

(*a*) For λ large enough, $A/\lambda \leqslant \eta$ and combination of expansions (4·2), (4·7) and of definition of $x^{s_1} u^{-s_2} W_{r_1,r_2}(x, u)$ yields $C^1_\mu(\lambda) = C^{1'}_\mu(\lambda) - \sum_{m,\, Re(\gamma)\leqslant r_2} l^{nm}_\mu(\lambda)$ with $C^{1'}_\mu(\lambda) = \int_0^{A/\lambda} x^{s_1} H_{r_1}(x, \lambda x)\, e^{(i\lambda-\mu)x} dx$, $l^{nm}_\mu(\lambda) = fp\int_0^{A/\lambda} x^{s_1} L_{nm}(x)\, (\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x} dx$. Due to assumption (4·3), observe that

$$|C^{1'}_\mu(\lambda)| \leqslant \int_0^{A/\lambda} x^{s_1} |H_{r_1}(x, \lambda x)|\, dx = \lambda^{-(s_1+1)} \int_0^A u^{s_1} |H_{r_1}(u/\lambda, u)|\, du.$$

Thereafter, if $C^{1'}(\lambda) := \lim_{\mu\to 0^+} C^{1'}_\mu(\lambda)$, then $C^{1'}(\lambda) = O(\lambda^{-(s_1+1)}] = o(\lambda^{-r})$. Recall that the definition of integer V_n shows that in any case $V_n \geqslant r_2 \geqslant Re(\gamma_n)$ and if $s_1 > V_n - 1 \geqslant Re(\gamma_n) - 1$ then $|l^{nm}_\mu(\lambda)| \leqslant \lambda^{-(s_1+1)} \int_0^A |u^{s_1-\gamma_n} L_{nm}(u/\lambda) \log^m(u)|\, du =$

$o[\lambda^{-(r_1+1)}] = o(\lambda^{-r})$, else use of expansion (4·6) for $s_1 \leqslant V_n - 1$ ensures that

$$l_\mu^{nm}(\lambda) = \sum_{j,\, s_1 \leqslant Re(\alpha) \leqslant V_n - 1} \sum_{l=0}^{j} C_j^l (-1)^l K_{nm}^{ij} A_\mu^{ijl}(\lambda)\, \lambda^{-(\alpha_i+1)} \log^l \lambda + e_\mu^{nm}(\lambda), \qquad (5·12)$$

where $A_\mu^{ijl}(\lambda) = fp \int_0^{A/\lambda} (\lambda x)^{\alpha_i - \gamma_n} \log^{m+j-l}(\lambda x)\, e^{(i\lambda-\mu)x} d(\lambda x)$ and also complex $e_\mu^{nm}(\lambda)$ obeys $e_\mu^{nm}(\lambda) = \int_0^{A/\lambda} h_{nm}(x)(\lambda x)^{-\gamma_n} \log^m(\lambda x)\, e^{(i\lambda-\mu)x} dx$. Observing that $h_{nm}(x) = x^t H_{nm}(x)$ with $t > V_n - 1 \geqslant \operatorname{Max}[Re(\gamma_n) - 1, s_1]$ and H_{nm} bounded near zero on the right, it follows that

$$\lim_{\mu \to 0^+} [e_\mu^{nm}(\lambda)] = \lambda^{-(t+1)} \int_0^{A/\lambda} (\lambda x)^{t-\gamma_n} \log^m(\lambda x)\, H_{nm}(x)\, e^{i\lambda x} d(\lambda x)$$

$$= \lambda^{-(t+1)} \int_0^{A} u^{t-\gamma_n} \log^m(u)\, H_{nm}(u/\lambda)\, e^{iu} du = o(\lambda^{-r}). \qquad (5.13)$$

Change of scale $u = \lambda x$ is applied to each term $A^{ijl}(\lambda) := \lim_{\mu\to 0^+}[A_\mu^{ijl}(\lambda)]$ and leads (see Lemma 3) to $A^{ijl}(\lambda) = fp^* \int_0^A u^{\alpha_i - \gamma_n} \log^{m+j-l}(u)\, e^{iu} du + C \log^{m+j-l+1}(\lambda)$, where the latter contribution is the potential corrective term. Thus, $\lim_{\mu\to 0^+}[l_\mu^{nm}(\lambda)] = o[\lambda^{-(r_1+1)}] = o(\lambda^{-r})$.

(*b*) Since $|W_{r_1, r_2}(x, u)| \leqslant D$ for $(x, u) \in]0, \eta] \times [A, +\infty[$, $|C_\mu^2(\lambda)| \leqslant D\lambda^{-s_2} \int_{A/\lambda}^{\eta} x^{s_1 - s_2} dx$, i.e. $\lim_{\mu\to 0^+} C_\mu^2(\lambda) = o(\lambda^{-r})$ for $r \leqslant \operatorname{Max}(r_1 + 1, r_2)$.

(*c*) To conclude, use of behaviours (4·1) and (4·5) for $u = \lambda x \geqslant \eta$ gives

$$C_\mu^3(\lambda) = \lambda^{-s_2} \int_\eta^b x^{-s_2} G_{r_2}(x, \lambda x)\, e^{(i\lambda-\mu)x} dx$$

$$- \sum_{j,\, Re(\alpha) \leqslant r_1} \int_\eta^b (\lambda x)^{-s_2} O_{ij}(\lambda x)\, x^{\alpha_i} \log^j x\, e^{(i\lambda-\mu)x} dx. \qquad (5·14)$$

Assumption (4·3) and a treatment similar to the one employed for $R_\mu^1(\lambda)$ if $b < +\infty$ easily show that $\lim_{\mu\to 0^+} C_\mu^3(\lambda) = o(\lambda^{-r})$.

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