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Hadamard's finite part concept in dimension $n \geqslant 2$; definition and change of variables, associated Fubini's theorem, derivation

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Abstract

Some usual and important operations: change of variables, application of Fubini's theorem and derivation with respect to the isolated singularity (in the present work with respect to the origin of the spherical coordinates (r, θ)) are studied for the following singular integral

$$I_{\alpha,j}(a) := fp \int_{\Omega,U} a(\theta)\, r^{\alpha} \log^{j} r \, dx,$$

where $\alpha \in C$, $Re(\alpha) \leqslant -n$, $j \in \mathbb{N}$, $a \in L^1(\Sigma_n, C)$ and the symbol $fp \int_{\Omega,U}$ means an integration on the set Ω in the finite part sense of Hadamard with respect to the domain configuration U. Moreover, applications to integral operators are outlined.

1. *Introduction*

Consider Ω an open, bounded and simply connected subset of $\mathbb{R}^n (n \geqslant 2)$, O a point belonging to Ω and $d(O, \partial\Omega)$ the distance from O to its boundary $\partial\Omega$. Throughout this paper $M := x$ designates an arbitrary point of $\mathbb{R}^n$, dx stands for the associated Lebesgue measure and $(r, \theta_1, \ldots, \theta_{n-1})$ is a set of spherical coordinates of origin O with $r := OM$ and $\theta := (\theta_1, \ldots, \theta_{n-1}) := x/r$. Recall that $dx = r^{n-1}\, dr\, d\sigma_n$ where $d\sigma_n$ is Lebesgue measure on $\Sigma_n := \{M \in \mathbb{R}^n, r = 1\}$. The zero function on Σ_n is designated by Θ_{Σ_n}. If C denotes the set of complex numbers, one often encounters (see Section 4) the integral

$$I_{\alpha,j}(a) := fp \int_{\Omega,U} a(\theta)\, r^{\alpha} \log^{j} r \, dx, \tag{1·1}$$

where $\alpha \in C$, $Re(\alpha) \leqslant -n$, $j \in \mathbb{N}$, $a \neq \Theta_{\Sigma_n}$, $a \in L^1(\Sigma_n, C)$ and the symbol $fp \int_{\Omega,U}$ means an integration on the set Ω in the finite part of Hadamard with respect to the domain configuration U (see next section for further explanations). Since Ω is bounded, $Re(\alpha) \leqslant -n$ and $a \neq \Theta_{\Sigma_n}$, the complex function $s^a_{\alpha,j}$, defined for $M \neq O$ by $s^a_{\alpha,j}(M) := a(x/r)\, r^{\alpha} \log^j r$, is singular at origin O and the integral $\mathscr{I}_{\alpha,j}(a) := \int_\Omega s^a_{\alpha,j}(x)\, dx$ is divergent. At this stage, many regularization methods are available to deal with this divergent integral $\mathscr{I}_{\alpha,j}(a)$ (see, for instance, Estrada and Kanwal[**2**]). The one here employed is Hadamard's finite part concept which leads to the quantity $I_{\alpha,j}(a)$ and require a great care as soon as one is eager to perform a change of variables or to apply usual Fubini's theorem. Such operations may generate corrective terms and the aim of this work is to derive those extra terms for the integral $I_{\alpha,j}(a)$.

This paper is divided into three sections. In Section 2, the meaning of the notation $fp\int_{\Omega,U}$ is introduced and the question of a change of variables is studied. An associated 'Fubini's' theorem is derived in Section 3. Finally, Section 4 is devoted to the derivations of the integral $I_{\alpha,j}(a)$ with respect to the isolated singularity O and to application to integral operators.

2. *The change of variables*

First, the integration in the finite part sense of Hadamard is presented.

Definition 2·1. For a real $\beta \succ 0$, a complex function f defined on the open set $]0,\beta[$ is of the first kind if and only if there exist a real $0 \prec \eta \prec \beta$, a family of positive integers $(M(n))$, two complex families (γ_n), (f_{nm}) and a complex function F such that

$$\left.\begin{aligned} &f(\epsilon) = \sum_{n=0}^{N}\sum_{m=0}^{M(n)} f_{nm}\,\epsilon^{\gamma_n}\log^m \epsilon + F(\epsilon);\ \text{in }]0,\eta[,\\ &Re(\gamma_N) \leqslant Re(\gamma_{N-1}) \leqslant \dots \leqslant Re(\gamma_1) \leqslant Re(\gamma_0) := 0,\\ &\lim_{\epsilon\to 0} F(\epsilon)\in C \quad \text{and} \quad f_{00}=0 \quad \text{if} \quad \gamma_0 = 0.\end{aligned}\right\} \tag{2·1}$$

If there exists (n,m) with $f_{nm} \neq 0$, then $f(\epsilon)-F(\epsilon)$ is a sum of a finite number of diverging terms as ϵ tends to zero. Following Hadamard[**4**] or Schwartz[**8**] the complex $\lim_{\epsilon\to 0}F(\epsilon)$ is called the finite part in the Hadamard sense of $f(\epsilon)$ and noted $fp[f(\epsilon)]$.

A closed surface $U(O)$ is a domain configuration with respect to the origin O if there exists a strictly positive and continuous function r_U on Σ_n such that $U(O) = \{P\in\mathbb{R}^n, OP = r_U(\theta), \theta\in\Sigma_n\}$. Observe that this function r_U only needs to be continuous. This property allows us to set $d_U := \sup\{r_U(\theta), \theta\in\Sigma_n\} > 0$. For a real $a>0$, $B(a) := \{M\in\mathbb{R}^n, OM < a\}$, $\Sigma(a) = \partial B(a) := \{M\in\mathbb{R}^n,\ OM = a\}$ and $DU(a) := \{M\in\mathbb{R}^n, OM < ar_U(\theta), \theta := x/r\}$. Moreover for h belonging to $L^1_{\text{loc}}(\Omega\backslash\{O\}, C)$, and ϵ such that $0<\epsilon<d(O,\partial\Omega)/d_U$ the complex function $f^U_h(\epsilon)$ is introduced as

$$f^U_h(\epsilon) := \int_{\Omega\backslash DU(\epsilon)} h(x)\,dx := \int_{\Omega,\,U(\epsilon)} h(x)\,dx. \tag{2·2}$$

The set $\mathscr{S}^U_O(\Omega)$ and the transformation $fp\int_{\Omega,U}$ on it are now defined.

Definition 2·2. Given $U(O)$ a domain configuration relatively to O, $\mathscr{S}^U_O(\Omega)$ is the set of complex functions h belonging to $L^1_{\text{loc}}(\Omega\backslash\{O\}, C)$ and such that $f^U_h(\epsilon)$ is of the first kind (for $0<\epsilon<d(O,\partial\Omega)/d_U$). Moreover, for $h\in\mathscr{S}^U_O(\Omega)$

$$fp\int_{\Omega,\,U} h(x)\,dx := fp[f^U_h(\epsilon)] = fp\left[\int_{\Omega\backslash DU(\epsilon)} h(x)\,dx\right]. \tag{2·3}$$

Proposition 2·3. *For reals c and d with $0<c<d$, $j\in\mathbb{N}$ and $\alpha\in C$*

$$\int_c^d x^\alpha \log^j x\,dx = P^j_\alpha(d) - P^j_\alpha(c), \tag{2·4}$$

with $$P^j_{-1}(t) := \frac{\log^{j+1}(t)}{j+1}, \quad \textit{else} \quad P^j_\alpha(t) := t^{\alpha+1}\sum_{k=0}^{j}\frac{(-1)^{j-k}j!}{k!(\alpha+1)^{1+j-k}}\log^k(t). \tag{2·5}$$

Consider the set of specific pseudofunctions $\mathscr{E}_O := \{s^a_{\alpha,j};\ \alpha \in C, Re(\alpha) \leqslant -n, j \in \mathbb{N}$, and $a \in L^1(\Sigma_n, C)\}$. Observe that $\mathscr{E}_O$ contains the zero function on Ω. Proposition 2·3 easily leads to the following theorem.

THEOREM 2·4. *For $U(O)$ a domain configuration with respect to O, if $s^a_{\alpha,j} \in \mathscr{E}_O$ then $s^a_{\alpha,j} \in \mathscr{S}^U_O(\Omega)$ and for any real $0 < \eta < d(O, \partial\Omega)/d_U$*

$$fp \int_{\Omega, U} s^a_{\alpha,j}(x)\,dx := fp\left[\int_{\Omega, U(\epsilon)} s^a_{\alpha,j}(x)\,dx\right]$$

$$= \int_{\Omega\setminus DU(\eta)} s^a_{\alpha,j}(x)\,dx + \int_{\Sigma_n} a(\theta)\left[P^j_{\alpha+n-1}(\eta r_U(\theta)) - s_\alpha \frac{\log^{j+1}(r_U(\theta))}{j+1}\right] d\sigma_n \tag{2·6}$$

with $s_\alpha := 1$ if $\alpha = -n$; else $s_\alpha := 0$.

Proof. Clearly, for $0 < \epsilon < d(O, \partial\Omega)/d_U$, $f(\epsilon) := \int_{\Omega\setminus DU(\epsilon)} s^a_{\alpha,j}(x)\,dx$ and consequently $fp\int_{\Omega, U} s^a_{\alpha,j}(x)\,dx$ no depend on η. We choose reals ϵ and η with

$$0 < \epsilon < \eta < d(O, \partial\Omega)/d_U.$$

Then $f(\epsilon)$ may be rewritten as

$$f(\epsilon) = \int_{\Omega\setminus DU(\eta)} s^a_{\alpha,j}(x)\,dx + \int_{DU(\eta)\setminus DU(\epsilon)} s^a_{\alpha,j}(x)\,dx. \tag{2·7}$$

Use of Proposition 2·3 ensures for the last integral on the right-hand side of (2·7)

$$g(\epsilon) := \int_{DU(\eta)\setminus DU(\epsilon)} s^a_{\alpha,j}(x)\,dx = \int_{\Sigma_n} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\eta r_U(\theta)} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n$$

$$= \int_{\Sigma_n} a(\theta)\,[P^j_{\alpha+n-1}(\eta r_U(\theta)) - P^j_{\alpha+n-1}(\epsilon r_U(\theta))]\,d\sigma_n. \tag{2·8}$$

From Definition 2·1 and relations (2·5) it follows that the complex functions $g(\epsilon)$ and $P^j_{\alpha+n-1}[\epsilon r_U(\theta)]$ are of the first kind. If $\alpha = -n, P^j_{\alpha+n-1}[\epsilon r_U(\theta)] = \log^{j+1}[\epsilon r_U(\theta)/(j+1)$ and by means of the usual Newton binomial formula one gets either

$$fp[P^j_{\alpha+n-1}(\epsilon r_U(\theta))] = \log^{j+1}[r_U(\theta)]/(j+1)$$

or else one obtains $fp[P^j_{\alpha+n-1}(\epsilon r_U(\theta))] = 0$.

Usually a symmetric neighbourhood of O is removed from Ω, i.e. one chooses $DU(\epsilon) := B(\epsilon)$ which corresponds to $r_U(\theta) = 1$ in the above theorem. Regarding Theorem 2·4 it is also worth outlining the influence of the domain configuration U.

PROPOSITION 2·5. *If $(U(O), V(O))$ is a pair of domains configuration with respect to O and $s^a_{\alpha,j} \in \mathscr{E}_O$, then*

$$fp \int_{\Omega, U} s^a_{\alpha,j}(x)\,dx = fp\int_{\Omega, V} s^a_{\alpha,j}(x)\,dx + s_\alpha \int_{\Sigma_n} a(\theta)\left[\int_{r_U(\theta)}^{r_V(\theta)} r^{-1}\log^j r\,dr\right] d\sigma_n. \tag{2·9}$$

Proof. For given domain configurations with respect to O, $U(O)$ and $V(O)$, if

$$0 < \epsilon < \mathrm{Min}\,\{d(O, \partial\Omega)/d_U, d(O, \partial\Omega)/d_v\}$$

we consider the function

$$h(\epsilon) := \int_{\Omega\setminus DU(\epsilon)} s^a_{\alpha,j}(x)\,dx - \int_{\Omega\setminus DV(\epsilon)} s^a_{\alpha,j}(x)\,dx.$$

Use of Newton's binomial formula with notation $C_j^k := j!/[k!(j-k)!]$ for j, k positive integers ($k \leqslant j$) and change of variable $r = \epsilon t$ makes it possible to put $h(\epsilon)$ into the following form

$$h(\epsilon) := \int_{DV(\epsilon)\backslash DU(\epsilon)} s^{a}_{\alpha,j}(x)\,dx = \int_{\Sigma_n} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\epsilon r_V(\theta)} r^{\alpha+n-1}\log^j r\,dr\right]d\sigma_n$$

$$= \sum_{k=0}^{j} C_j^k\left\{\int_{\Sigma_n} a(\theta)\left[\int_{r_U(\theta)}^{r_V(\theta)} t^{\alpha+n-1}\log^{j-k} t\,dt\right]d\sigma_n\right\}\epsilon^{\alpha+n}\log^k\epsilon. \quad (2\cdot10)$$

Thus, $h(\epsilon)$ is of the first kind and $fp[h(\epsilon)] = 0$ unless $\alpha = -n$ and $k = 0$ which circumstances provide the additional term in (2·9).

We now consider the question of the change of variables $x = T(x')$, where the transformation T is both differentiable and one-to-one with $x' = T^{-1}(x)$, $O' := T^{-1}(O)$. Since differentiable T is also continuous on Ω and consequently $\Omega' = T^{-1}(\Omega)$ is an open subset of $\mathbb{R}^n$. Recall that if $J_T(x')$ denotes the Jacobian at point x' of the transformation T, $dx = |J_T(x')|\,dx'$. Moreover (for instance see Rudin[7], for an application of the usual change of variables), the transformation T^{-1} is assumed to be continuous on Ω'. If $U(O)$ is a domain configuration with respect to O and $h \in \mathscr{S}_O^U(\Omega)$, the problem consists in finding the link between the two integrals $I := fp\int_{\Omega,U} h(x)\,dx$ and $I' := fp\int_{T^{-1}(\Omega),\,U(O')} h\circ T(x')\,|J_T(x')|\,dx'$, where $U(O')$ means the domain configuration with respect to O' defined by

$$U(O') := \{P \in \mathbb{R}^n, O'P := r_U(\theta), \theta := [x_P - x_{O'}]/O'P\}.$$

The answer is not trivial and deeply depends not only on functions h and T but also on the domain configuration $U(O)$. One may consult for partial results Di Pasquantonio and Lavoine[1] or Schwartz[8] (those authors dealing only with the case $U = \Sigma_n$) and Jones[5].

Clearly, if T reduces to a shift, $I' = I$. Consequently the study is by now restricted to transformations T belonging to the set

$$\mathscr{P}(O,\Omega) := \{T; T \text{ is one-to-one and differentiable on } \Omega \text{ with } T(O) = O, T^{-1} \text{ is continuous on } \Omega' := T^{-1}(\Omega)\}.$$

Since T^{-1} is one to one, for subsets A and B of $\mathbb{R}^n$, if $A \subset B$ then

$$T^{-1}(B\backslash A) = T^{-1}(B)\backslash T^{-1}(A).$$

Thus, for $0 < \epsilon < d(O,\partial\Omega)/d_U$, $T \in \mathscr{P}(O,\Omega)$ and $h \in \mathscr{S}_O^U(\Omega)$, application of the usual change of variables (Rudin[7]) ensures that if $H(x') := h\circ T(x')\,|J_T(x')|$ then $H \in L^1_{\text{loc}}(T^{-1}(\Omega)\backslash\{O\}, C)$. Moreover, it makes it possible to write

$$f_h^U(\epsilon) = \int_{\Omega\backslash DU(\epsilon)} h(x)\,dx = \int_{T^{-1}(\Omega)\backslash DU(\epsilon)} h\circ T(x')\,|J_T(x')|\,dx' + \int_{T[DU(\epsilon)]\backslash DU(\epsilon)} h(x)\,dx. \quad (2\cdot11)$$

As soon as f_h^U is of the first kind, equality (2·11) shows that the function H belongs to $\mathscr{S}_O^U(T^{-1}(\Omega))$ if and only if the new function $g^U_{h,T}(\epsilon) := \int_{T[DU(\epsilon)]\backslash DU(\epsilon)} h(x)\,dx$ is also of the first kind. If $H \in \mathscr{S}_O^U(T^{-1}(\Omega))$ the corrective term is thereafter given by

$$fp\int_{\Omega,U} h(x)\,dx - fp\int_{T^{-1}(\Omega),U} h\circ T(x')\,|J_T(x')|\,dx' = fp[g^U_{h,T}(\epsilon)]. \quad (2\cdot12)$$

Hence the work reduces to the study of the function $g^U_{h,T}$. In this paper, this is achieved for $h \in \mathscr{E}_O$ and for specific changes of variables T belonging to $\mathscr{P}(O,\Omega)$. For $h \in \mathscr{E}_O$ and T a given element of $\mathscr{P}(O,\Omega)$, observe that if there exists $U(O)$ a domain configuration with respect to O such that $H \in \mathscr{S}^U_O(T^{-1}(\Omega))$ then application of Proposition 2·5 and equality (2·12) yield for any domain configuration $V(O)$ with respect to O

$$fp\int_{\Omega,V} s^a_{\alpha,j}(x)\,dx = fp\int_{T^{-1}(\Omega),U} s^a_{\alpha,j}\circ T(x')\,|J_T(x')|\,dx' + fp[g^U_{s,T}(\epsilon)] + s_\alpha\int_{\Sigma_N} a(\theta)\left[\int_{r_V(\theta)}^{r_U(\theta)} \frac{\log^j r}{r}\,dr\right]d\sigma_n.$$

THEOREM 2·6. *Given $U(O)$ a domain configuration with respect to O, if T is a linear and one-to-one transformation with $T(O)=O$ then $T\in\mathscr{P}(O,\Omega)$ and for $s^a_{\alpha,j}\in\mathscr{E}_O$*

$$fp\int_{\Omega,U} s^a_{\alpha,j}(x)\,dx = fp\int_{\Omega',U} s^a_{\alpha,j}\circ T(x')\,|J_T(x')|\,dx' + s_\alpha\int_{\Sigma_n} a(\theta)\left[\int_{r_U(\theta)}^{r_V(\theta)} \frac{\log^j r}{r}\,dr\right]d\sigma_n, \tag{2·13}$$

where $V(O)$ is the domain configuration defined by $V(O) := T[U(O)]$.

Proof. Such a change of variables belongs to $\mathscr{P}(O,\Omega)$ and satisfies: $|J_T(x')|$ is constant. Moreover, if $x\in\mathbb{R}^n$, $T(\epsilon x)=\epsilon T(x)$ so that if

$$V(O) := T[U(O)],\ T[DU(\epsilon)] = DV(\epsilon) \text{ for } 0<\epsilon<d(O,\partial\Omega)/d_U.$$

Thus, the function $g^U_{s,T}$ becomes

$$g^U_{s,T}(\epsilon) = \int_{DV(\epsilon)\backslash DU(\epsilon)} s^a_{\alpha,j}(x)\,dx = \int_{\Sigma_n} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\epsilon r_V(\theta)} r^{\alpha+n-1}\log^j r\,dr\right]d\sigma_n. \tag{2·14}$$

As a consequence, $g^U_{s,T}$ is of the first kind and the value of $fp[g^U_{s,T}(\epsilon)]$ (which one is already known, see Proposition 2·5) ensures the result.

A useful application is the change of scale $x=\lambda x'$ with λ real and non-zero. If $\lambda>0$, $r_V(\theta)=\lambda r_U(\theta)$, and for $\Omega' := \{\lambda^{-1}x;\, x\in\Omega\}$, Theorem 2·6 provides the following result for $I_{\alpha,j}(a) := fp\int_{\Omega,U} s^a_{\alpha,j}(x)\,dx$

$$I_{\alpha,j}(a) = fp\int_{\Omega',U} s^a_{\alpha,j}(\lambda x')\,\lambda^n\,dx' + s_\alpha\int_{\Sigma_n} a(\theta)\left[\int_{r_U(\theta)}^{\lambda r_U(\theta)} r^{-1}\log^j r\,dr\right]d\sigma_n$$

$$= \sum_{k=0}^{j} C^k_j \log^{j-k}\lambda\left\{\lambda^{\alpha+n} fp\int_{\Omega',U} s^a_{\alpha,k}(x')\,dx' + s_\alpha\frac{\log\lambda}{j+1-k}\int_{\Sigma_n} a(\theta)\log^k r_U(\theta)\,d\sigma_n\right\}.$$

THEOREM 2·7. *Consider $U(O)$ a domain configuration with respect to O and $s^a_{\alpha,j}\in\mathscr{E}_O$. If $[\![a]\!]$ denotes the integer part of the real a, it is possible to define for $i\in\mathbb{N}$, the sets of functions*

$$\mathscr{D}^\alpha_i := \{\phi(r,\theta);\ \phi \text{ is a real, positive, one-to-one and differentiable function on } [0,d(O,\partial\Omega)]\times\Sigma_n,\ a_k(\theta) := [\partial^k\phi(r,\theta)/\partial r^k](0,\theta) \text{ exists for } 0\leqslant k\leqslant 2+i+[\![-(1+i)(Re(\alpha)+n)]\!] \text{ with } a_k(\theta)=0 \text{ for } 0\leqslant k\leqslant i \text{ and } a_{i+1}(\theta)>0\}.$$

Assume that $T \in \mathscr{P}(O,\Omega)$ *is a change of variables such that* $x \in (r,\theta) = T(x') = T(r',\theta')$ *and that there exists* $\phi \in \mathscr{D}_i^\alpha$ *with* $r = \phi(r',\theta)$, *then*

$$fp\int_{\Omega,U} s^a_{\alpha,j}(x)\,dx = fp\int_{T^{-1}(\Omega),U} s^a_{\alpha,j}\circ T(x')\,|J_T(x')|\,dx' + E_i^U(s^a_{\alpha,j}), \tag{2·15}$$

where $E_i^U(s^a_{\alpha,j}) = 0$ *except if there exists a positive integer* l *with* $\alpha = -n-l/(i+1)$, *and in these circumstances*

$$E_i^U(s^a_{\alpha,j}) = \sum_{Faa,l}\frac{1}{m_1!\ldots m_q!}\int_{\Sigma_n} a\frac{d^p}{dX^p}\left[H^U_{\alpha,j}\left(\frac{a_{i+1}(\theta)}{(i+1)!}\right)\right]\left[\frac{a_{i+2}(\theta)}{(i+2)!}\right]^{m_1}\cdots\left[\frac{a_{q+i+1}(\theta)}{(q+i+1)!}\right]^{m_q} d\sigma_n,$$

$$H^U_{\alpha,j}(X) = \sum_{m=0}^{j} C_j^m \sum_{k=0}^{m+1} p^m_{\alpha+n-1,k}\sum_{q=0}^{k} C_k^q\, i^{k-q}\log^{j-m+k-q}[r_U(\theta)]\,X^{\alpha+n}\log^q X,$$

where the sum $\sum_{Faa,l}$ *and coefficients* $p^m_{\alpha+n-1,k}$ *are defined below.*

Proof. First, a useful lemma is recalled.

LEMMA 2·7. *For functions f and g admitting derivatives up to order n respectively in a neighbourhood V of x and a neighbourhood* $W := f(V)$ *of* $f(x)$, *then* $g\circ f$ *admits derivatives up to order n in V and*

$$\frac{d^n}{dx^n}[g\circ f(x)] = \sum_{Faa,m}\frac{n!}{m_1!\ldots m_q!}g^{(p)}[f(x)]\left[\frac{f^{(1)}(x)}{1!}\right]^{m_1}\cdots\left[\frac{f^{(q)}(x)}{q!}\right]^{m_q}, \tag{2·16}$$

where $f^{(i)} := d^if/(dx^i)$ *for* $i \in \mathbb{N}$ *and the sum* $\sum_{Faa,n}$ *means a sum over all families* $(m_i)_{i\in\{1,\ldots,q\}}$ *of positive integers such that* $m_1+2m_2+\ldots+qm_q = n$ *and* $p := m_1+m_1+\ldots+m_q$.

Result (2·16) is known as the Faa de Bruno formula and the reader is referred to Gradshteyn and Ryzhik [4]. For $g_\theta(t) := \phi(t,\theta)/t$, if $t>0$, the use both of change of variables $r = \epsilon r_U(\theta)v$ and of the Newton binomial formula ensures for $g^U_{s,T}(\epsilon)$ the following form

$$g^U_{s,T}(\epsilon) = \sum_{m=0}^{j} C_j^m\int_{\Sigma_n} a(\theta)\,\epsilon^{\alpha+n}r_U(\theta)^{\alpha+n}\left[\int_1^{g_\theta(\epsilon r_U(\theta))} v^{\alpha+n-1}\log^m v\,dv\right]\log^{j-m}(\epsilon r_U(\theta))\,d\sigma_n. \tag{2·17}$$

For $t = \epsilon r_U(\theta)$, Proposition 2·3 allows us to write

$$F^m_\theta(t) := t^{\alpha+n}\int_1^{g_\theta(t)} v^{\alpha+n-1}\log^m v\,dv = t^{\alpha+n}P^m_{\alpha+n-1}[g_\theta(t)] - t^{\alpha+n}P^m_{\alpha+n-1}[1], \tag{2·18}$$

where it is recalled (see equalities (2·5)) that $P^m_\beta(t) := \sum_{k=0}^{m+1} p^m_{\beta,k}t^{\beta+1}\log^k t$ with $p^m_{-1,k} := 0$ for $0 \leqslant k \leqslant m$; $p^m_{-1,m+1} := 1/(m+1)$ and if $\beta \neq -1$; $p^m_{\beta,m+1} := 0$ and for $0 \leqslant k \leqslant m$; $p^m_{\beta,k} := (-1)^{m-k}m!/[k!(\beta+1)^{1+m-k}]$.

Clearly, if $d^m_\alpha(\epsilon) := P^m_{\alpha+n-1}[1]\int_{\Sigma_n} a(\theta)\,\epsilon^{\alpha+n}r_U^{\alpha+n}(\theta)\log^{j-m}[\epsilon r_U(\theta)]\,d\sigma_n$, the function d^m_α is of the first kind and for α with $Re(\alpha)+n \leqslant 0$, $fp[d^m_\alpha(\epsilon)] = 0$ (since $P^m_{-1}[1] = 0$)). The first contribution on the right-hand side of (2·18) associated to $\log^{j-m}(t)$ (see 2·17) is treated by introducing the function $W_\theta(t) := \sum_{m=0}^{j} C_j^m P^m_{\alpha+n-1}[g_\theta(t)]\,t^{\alpha+n}\log^{j-m}(t)$. Consider $l_{\alpha,i} := 2+i+[\![-(1-i)(Re(\alpha)+n)]\!]$. The assumptions on function ϕ ensure in a neighbourhood of zero (on the right) the form below for $g_\theta(t)$

$$g_\theta(t) = t^i\left\{\sum_{k=0}^{l_{\alpha,i}-i-1} a_{k+i+1}(\theta)\,t^k/(k+i+1)! + t^{l_{\alpha,i}-i-1}o(t)\right\} := t^ih_\theta(t), \tag{2·19}$$

with $h_\theta > 0$. Thus, definition of $P^m_{\alpha+n-1}(t)$ allows us to cast $W_\theta(t)$ into the form:

$$W_\theta(t) = \sum_{m=0}^{j} C_j^m \sum_{k=0}^{m+1} p^m_{\alpha+n-1,k} \sum_{q=0}^{k} C_k^q\, i^{k-q} t^{(\alpha+n)(1+i)} \log^{j-m+k-q}(t)\,[h_\theta(t)]^{\alpha+n} \log^q[h_\theta(t)]. \tag{2·20}$$

Consider, for $X > 0$, the function $A(X) := X^{\alpha+n}\log^q X$. An expansion of $A[h_\theta(t)]$ is sought for small values of t up to order $l_{\alpha,i}-i-1$. Since $h_\theta(t) = X(t)+t^{l_{\alpha,i}-i-1}o(t)$ with $X(t) := \sum_{k=0}^{l_{\alpha,i}-i-1} a_{k+i+1}(\theta)\, t^k/(k+i+1)!$, this development is supplied by the expansion up to this order $l_{\alpha,i}-i-1$ of the function $A[X(t)]$. Use of the Faa de Bruno formula gives the derivatives $d^pA[X(0)]/dt^p$ and ensures

$$A[h_\theta(t)] = \sum_{p=0}^{l_{\alpha,i}-i-1} \frac{d^p}{dt^p}[A(X(0))]\frac{t^p}{p!} + t^{l_{\alpha,i}-i-1}o(t). \tag{2·21}$$

Consequently, $W_\theta(\epsilon r_U(\theta))$ is a linear combination terms such as

$$d_A(\epsilon) := \sum_{p=0}^{l_{\alpha,i}-i-1} \frac{d^p}{dt^p}[A(X(0))]\frac{[\epsilon r_U(\theta)]^{(\alpha+n)(i+1)+p}}{p!}\log^{j-m+k-q}[\epsilon r_U(\theta)] + [\epsilon r_U(\theta)]^{(\alpha+n)(i+1)+l_{\alpha,i}-i-1}\, o[\epsilon r_U(\theta)]. \tag{2·22}$$

Since $l_{\alpha,i}-i-1+(1+i)[Re(\alpha)+n] > 0$, d_A is of the first kind and $fp[d_A(\epsilon)] = 0$, unless there exists a positive integer l with $0 \leqslant l \leqslant l_{\alpha,i}-i-2 = [\![-(1+i)(Re(\alpha)+n)]\!]$ such that $(\alpha+n)(1+i)+l = 0$, i.e. $\alpha = -n-l/(1+i)$. In such a case, Definition 2·1 yields

$$fp[d_A(\epsilon)] = \frac{\log^{j-m+k-q}[r_U(\theta)]}{l!}\frac{d^l}{dt^l}[A(X(0))]. \tag{2·23}$$

Gathering the contributions associated to different values of positive integers m, k and q for $W_\theta(\epsilon r_U(\theta))$, it is useful to introduce for $X > 0$ the function

$$H^U_{\alpha,j}(X) := \sum_{m=0}^{j} C_j^m \sum_{k=0}^{m+1} p^m_{\alpha+n-1,k} \sum_{q=0}^{k} C_k^q\, i^{k-q}\log^{j-m+k-q}[r_U(\theta)]X^{\alpha+n}\log^q X. \tag{2·24}$$

Application of the Faa de Bruno formula with this notation leads for $\alpha = -n-l/(1+i)$, $l \in \mathbb{N}^*$ to the additional term

$$E_i^U(s^a_{\alpha,j}) = \sum_{Faa,l} \frac{1}{m_1!\dots m_q!}\int_{\Sigma_n} a(\theta)\frac{d^p}{dX^p}\left[H^U_{\alpha,j}\left(\frac{a_{i+1}(\theta)}{(i+1)!}\right)\right]\left[\frac{a_{i+2}(\theta)}{(i+2)!}\right]^{m_1}\dots\left[\frac{a_{q+i+1}(\theta)}{(q+i+1)!}\right]^{m_q} d\sigma_n. \tag{2·25}$$

Some remarks on Theorem 2·7 are in order.

The result depends on the domain configuration $U(O)$ and this dependence is contained in the function $H^U_{\alpha,j}$. When $U(O) := \Sigma(1) := \{P \in \mathbb{R}^n, OP = 1\}$, $H^U_{\alpha,j} = P^j_{\alpha+n-1}$ (obtained with $m = j$, and $q = k$ in equality (2·24)).

If $\alpha = -n$, then $l = 0$ and the extra term is

$$E_i^U(s^a_{-n,j}) = \sum_{m=0}^{j}\sum_{q=0}^{m+1} \frac{i^{m+1-q}j!}{q!(j-m)!(m+1-q)!}\int_{\Sigma_n} a(\theta)\log^{j+1-q}[r_U(\theta)]\log^q\left[\frac{a_{i+1}(\theta)}{(i+1)!}\right]d\sigma_n. \tag{2·26}$$

Usually $i = 0$, i.e. the particular α are $\alpha = -n-l$, $l \in \mathbb{N}$. In this case

$$H^U_{\alpha,j}(X) = \textstyle\sum_{m=0}^{j} C_j^m \log^{j-m}[r_U(\theta)]\,P^m_{-l-1}(X)$$

and

$$E_0^U(s^a_{\alpha,j}) = \sum_{Faa,l} \frac{1}{m_1!\dots m_q!}\int_{\Sigma_n} a(\theta)\frac{d^p}{dX^p}[H^U_{\alpha,j}(a_1(\theta))]\left[\frac{a_2(\theta)}{2!}\right]^{m_1}\dots\left[\frac{a_{q+1}(\theta)}{(q+1)!}\right]^{m_q} d\sigma_n. \tag{2·27}$$

For instance, if $x = \lambda x'$ with $\lambda > 0$, then $\phi(r', \theta) = \lambda r'$ with $a_1(\theta) = \lambda > 0$ and $a_k(\theta) = 0$ for $k \geqslant 2$. Thus, equality (2·27) leads to $E_0^U(s^a_{\alpha,j}) = 0$ for $\alpha = -n-l, l \geqslant 1$. If $\alpha = -n$, (2·26) is rewritten with $i = 0$ as

$$E_0^U(s^a_{-n,j}) = \sum_{m=0}^{j} C_j^m \frac{\log^{m+1}\lambda}{m+1}\left[\int_{\Sigma_n} a(\theta)\log^{j-m} r_U(\theta)\, d\sigma_n\right], \tag{2·28}$$

which is relevant to previous result.

Consider $T \in \mathscr{P}(O,\Omega)$ which is characterized by two functions R and Θ with $T(x') = T(r',\theta') = x = (r,\theta) = (R(r',\theta'), \Theta(r',\theta'))$. Whenever it is possible to find $i \in \mathbb{N}$, $\alpha \in C$, $Re(\alpha) \leqslant -n$ and $\phi \in \mathscr{D}^\alpha_i$ such that $r = R(r',\theta') = \phi(r', \Theta(r',\theta'))$, Theorem 2·7 applies to change of variables T for the pseudofunction $s^a_{\alpha,j}$. Note that ϕ is assumed to possess derivatives at $r = 0^+$ up to a specific order. The proposition below is concerned with an example of function ϕ belonging to none of the sets $\mathscr{D}^\alpha_i$.

PROPOSITION 2·9. *For $U(O)$ a domain configuration with respect to O and $s^a_{\alpha,j} \in \mathscr{E}_O$ if $T \in \mathscr{P}(O,\Omega)$ is such that there exist a real $\gamma > 0$ and a real and differentiable function $f(\theta) > 0$ with $T(x') = T(r',\theta') = x = (r,\theta) = (r'^{\gamma}f(\theta),\theta)$ then*

$$fp\int_{\Omega,U} s^a_{\alpha,j}(x)\,dx = fp\int_{\Omega',U} s^a_{\alpha,j}[T(x')]\,|J_T(x')|\,dx' + s_\alpha\int_{\Sigma_n} a\left[\int_{r_U(\theta)}^{r^\gamma_U(\theta)f(\theta)} \frac{\log^j r}{r}\,dr\right] d\sigma_n. \tag{2·29}$$

Proof. Here the function $g^U_{s,T}$ is written as

$$g^U_{s,T}(\epsilon) := \int_{T[DU(\epsilon)]\backslash DU(\epsilon)} s^a_{\alpha,j}(x)\,dx = \int_{\Sigma_n} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\epsilon^\gamma r^\gamma_U(\theta)f(\theta)} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n. \tag{2·30}$$

Hence, $g^U_{s,T} = \int_{\Sigma_n} a(\theta)\{P^j_{\alpha+n-1}[\epsilon^\gamma r^\gamma_U(\theta)f(\theta)] - P^j_{\alpha+n-1}[\epsilon r_U(\theta)]\}\,d\sigma_n$. Definition of the function $P^j_{\alpha+n-1}$ shows that $g^U_{s,T}$ is of the first kind and leads to the announced extra term.

3. *Associated Fubini's theorem*

For Ω given, Π_Ω denotes its characteristic function and, for any real $p > 1$, the set $\mathscr{E}^p_O$ is defined as $\mathscr{E}^p_O := \{s^a_{\alpha,j}; \alpha \in C, Re(\alpha) \leqslant -n, j \in \mathbb{N}, a \in L^1(\Sigma_n, C) \cap L^p(\Sigma_n, C)\}$. Since $s^a_{\alpha,j}$ is not measurable on Ω, application of the usual Fubini's theorem to the special integral $I_{\alpha,j}(a)$ (see (1·1)) is not legitimate. Hence, it is necessary to derive an adequate 'Fubini's theorem' and this section studies the extra terms occurring.

First, some useful notation is introduced. For $x = (x_1, \dots, x_n) = (r,\theta)$ an arbitrary point of $\mathbb{R}^n$, the set of angular coordinates $\theta := (\theta_1, \dots, \theta_{n-1})$ is such that $\theta_k \in [0,\pi]$ for $1 \leqslant k \leqslant n-2$, $\theta_{n-1} \in [0, 2\pi]$ (if $n = 2$, this reduces to $\theta_{n-1} = \theta_1 \in [0,2\pi]$),

$$x_n = r\pi^{n-1}_{k=1}\sin\theta_k,$$

and for $1 \leqslant j \leqslant n-1$, $x_j = r[\pi^{j-1}_{k=1}\sin\theta_k]\cos\theta_j$. Moreover

$$d\sigma_n = \Pi^{n-1}_{j=1}[\sin\theta_j^{n-(j+1)}\,d\theta_j].$$

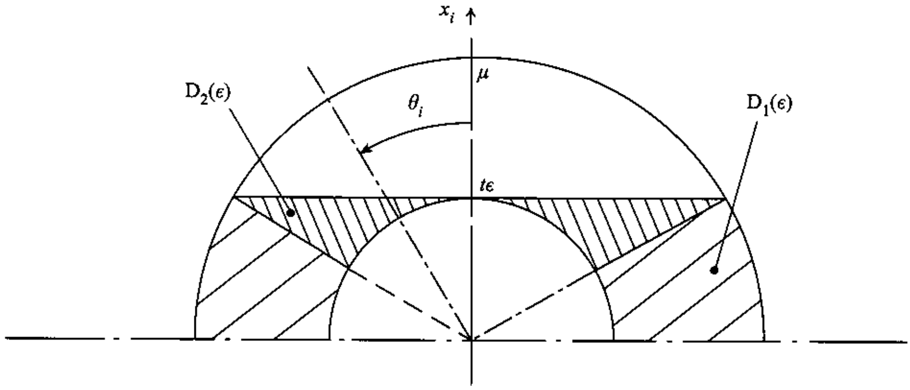

Fig. 1

By now, we set $y_j(\theta) := x_j/r$ and the point $y := (y_1, \dots, y_n)$ belongs to Σ_n. For

$$i \in \{1, \dots, n\},$$

the subsets $\Sigma^+_{n,i}$ and $\Sigma^-_{n,i}$ of Σ_n obey the definition

$$\Sigma^+_{n,i} := \{y = (y_1, \dots, y_n) \in \Sigma_n ; y_i \geqslant 0\}$$

and $\Sigma^-_{n,i} := \{y = (y_1, \dots, y_n) \in \Sigma_n ; -y_i \geqslant 0\}$.

An important lemma is stated below.

LEMMA 3·1. *For $i \in \{1, \dots, n\}$ and reals p, μ, t with $p > 1$, $t > 0$, $\mu > 0$ if $s^a_{\alpha,j} \in \mathscr{E}^p_0$ and for $0 < \epsilon < \mu/t$; $\Delta^\pm_i(\mu, t, \epsilon) := \{M \in \mathbb{R}^n, t\epsilon \leqslant r \leqslant \mu$ and $0 \leqslant \pm x_i \leqslant t\epsilon\}$ then*

$$fp\left[\int_{\Delta^\pm_i(\mu,t,\epsilon)} s^a_{\alpha,j}(x)\,dx\right] = s_\alpha \sum_{k=0}^{j} \frac{C^k_j(-1)^{k+1}\log^{j-k} t}{k+1} \int_{\Sigma^\pm_{n,i}} a(\theta)\log^{k+1}|y_i(\theta)|\,d\sigma_n. \quad (3·1)$$

Proof. Observe that the result does not depend on μ. For the given integer $i \in \{1, \dots, n\}$, a new set of angular coordinates $\theta := (\theta_1, \theta_2, \dots, \theta_{n-1})$ is chosen such that if $x = (x_1, \dots, x_n)$ then $x_i = r\cos\theta_i$ with $\theta_i \in [0, (1+\delta_{n2})\pi]$ (where δ denotes the Kroneker delta) and $d\sigma_n = [\sin\theta_i^{n-2}]\,d\theta_i\,d\sigma_{n-1}$ with $d\sigma_{n-1} = F(\theta_2, \dots, \theta_{n-1})\prod_{k=2}^{n-1} d\theta_k$ for $n > 2$; else $d\sigma_1 := 1$. The derivation is achieved for the set $\Delta^+_i(\mu, t, \epsilon) = \Delta_i(\epsilon)$ (the case of $\Delta^-_i(\mu, t, \epsilon)$ is left to the reader). For $0 < \epsilon < \mu/t$, the question consists both in showing that the function $g_i(\epsilon) := \int_{\Delta_i(\epsilon)} s^a_{\alpha,j}(x)\,dx$ is of the first kind and in calculating $fp[g_i(\epsilon)]$. As illustrated by Fig. 1 above, the domain $\Delta_i(\epsilon)$ is shared into the two subsets $\mathscr{D}_1(\epsilon) := \{M \in \mathbb{R}^n ; 0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu$ and $t\epsilon \leqslant r \leqslant \mu\}$ and $\mathscr{D}_2(\epsilon) := \{M \in \mathbb{R}^n ; t\epsilon/\mu \leqslant \cos\theta_i \leqslant 1$ and $t\epsilon \leqslant r \leqslant t\epsilon/\cos\theta_i\}$.

This division yields

$$\begin{aligned} g_i(\epsilon) &= \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta)\left[\int_{t\epsilon}^{\mu} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n \\ &\quad + \int_{\cos\theta_i \geqslant t\epsilon/\mu} a(\theta)\left[\int_{t\epsilon}^{t\epsilon/\cos\theta_i} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n \\ &= \int_{0 \leqslant \cos\theta_i \leqslant 1} a(\theta)\left[\int_{t\epsilon}^{t\epsilon/\cos\theta_i} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n \\ &\quad + \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta)\left[\int_{t\epsilon/\cos\theta_i}^{\mu} r^{\alpha+n-1}\log^j r\,dr\right] d\sigma_n. \end{aligned} \quad (3·2)$$

Note $A(\epsilon)$, the first contribution on the right-hand side of equality (3·2). Taking into account successively a change of variable $r = \epsilon t u$, Proposition 2·3 and the definition of functions P^j_β, one obtains

$$A(\epsilon) = \sum_{k=0}^{j} C_j^k [t\epsilon]^{\alpha+n} \log^{j-k}[t\epsilon] \left\{ -P^k_{\alpha+n-1}[1] \int_{\Sigma^+_{n,i}} a(\theta)\, d\sigma_n \right.$$

$$\left. + \sum_{m=0}^{k+1} p^k_{\alpha+n-1,m} \int_{0 \leqslant \cos\theta_i \leqslant 1} a(\theta) [\cos\theta_i]^{-(\alpha+n)} \log^m[1/\cos\theta_i]\, d\sigma_n \right\}. \quad (3·3)$$

As for $x > 0$ and $a \in C, |x^a| = x^{Re(a)}$ and $d\sigma_n = \sin\theta_i^{n-2}\, d\theta_i\, d\sigma_{n-1}$, for any real $q > 0$ if

$$D^q_m := \int_{0 \leqslant \cos\theta_i \leqslant 1} |[\cos\theta_i]^{-(\alpha+n)} \log^m[1/\cos\theta_i]|^q\, d\sigma_n$$

then

$$D^q_m \leqslant \left[\int_0^{\pi/2} [\cos\theta_i]^{-q(Re(\alpha)+n)} |\log[1/\cos\theta_i]|^{mq} |\sin\theta_i|^{n-2}\, d\theta_i \right] \left[\int_{\Sigma_{n-1}} d\sigma_{n-1} \right]. \quad (3·4)$$

Since $q > 0$ and $Re(\alpha)+n \leqslant 0$, $-q(Re(\alpha)+n)+1 > 0$ and it is easy to show that D^q_m is bounded. Assumption $a \in L^p(\Sigma_n, C)$, choice of q such that $1/p+1/q = 1$ and application of Hölder's inequality prove that

$$\int_{0 \leqslant \cos\theta_i \leqslant 1} a(\theta) [\cos\theta_i]^{-(\alpha+n)} \log^m[1/\cos\theta_i]\, d\sigma_n$$

exists. Hence equality (3·3) ensures that the function A is of the first kind and application of Definition 2·1 easily leads (since $P^k_{-1}[1] = 0$) to

$$fp[A(\epsilon)] = s_\alpha \sum_{k=0}^{j} \frac{C_j^k \log^{j-k} t}{k+1} \int_{\Sigma^+_{n,i}} a(\theta) \log^{k+1}[1/\cos\theta_i]\, d\sigma_n. \quad (3·5)$$

The second integral on the right-hand side of (3·2) is

$$h_i(\epsilon) = P^j_{\alpha+n-1}[\mu] \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta)\, d\sigma_n - \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta) P^j_{\alpha+n-1}[t\epsilon/\cos\theta_i]\, d\sigma_n. \quad (3·6)$$

The first contribution tends to zero with ϵ. The second one is

$$w_i(\epsilon) = -\sum_{k=0}^{j+1} p^j_{\alpha+n-1,k} \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta) \left[\frac{t\epsilon}{\cos\theta_i} \right]^{\alpha+n} \log^k[t\epsilon/\cos\theta_i]\, d\sigma_n. \quad (3·7)$$

For $q > 0$, and $E^q_k := \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} |[t\epsilon/\cos\theta_i]^{\alpha+n} \log^k[t\epsilon/\cos\theta_i]|^q\, d\sigma_n$, above arguments yield

$$E^q_k \leqslant \left[\int_{\arccos[t\epsilon/\mu]}^{\pi/2} \left[\frac{t\epsilon}{\cos\theta_i} \right]^{q(Re(\alpha)+n)} |\log[t\epsilon/\cos\theta_i]|^{kq} |\sin\theta_i|^{n-2}\, d\theta_i \right] \left[\int_{\Sigma_{n-1}} d\sigma_{n-1} \right]. \quad (3·8)$$

Change of variable $\cos\theta_i = t\epsilon v$ in the integration bearing on θ_i in the last inequality leads to

$$E^q_k \leqslant \left[t\epsilon \int_0^{1/\mu} v^{-q(Re(\alpha)+n)} |\log v|^{kq} E_n(\epsilon)\, dv \right] \left[\int_{\Sigma_{n-1}} d\sigma_{n-1} \right], \quad (3·9)$$

with $E_n := 1$, if $n \geqslant 3$; else $E_2(\epsilon) := [1-\epsilon^2 t^2/\mu^2]^{-1/2}$. Moreover $-q(Re(\alpha)+n)+1 > 0$. Consequently, there exists a real $R_k > 0$ such that for ϵ small enough $E_k^q \leqslant t\epsilon R_k$. Application of Hölder's inequality for q such that $1/p+1/q = 1$ shows that

$$\left| \int_{0 \leqslant \cos\theta_i \leqslant t\epsilon/\mu} a(\theta) \left[\frac{t\epsilon}{\cos\theta_i}\right]^{\alpha+n} \log^k[t\epsilon/\cos\theta_i]\, d\sigma_n \right| \leqslant \epsilon^{1/q}[tR_k]^{1/q} \left[\int_{\Sigma_n} |a(\theta)|^p\, d\sigma_n\right]^{1/p}. \tag{3·10}$$

Since $q > 1$, (3·7) ensures that $\lim_{\epsilon\to 0} w_i(\epsilon) = 0$. Thus the function g_i is of the first kind and $fp[g_i(\epsilon)] = fp[A(\epsilon)]$ which is given by (3·5).

Consider two reals b and c with $b < 0 < c$ and a function $h \in L^1_{\text{loc}}(]b,c[\backslash\{0\}, C)$. It is also convenient for $V := (v_-, v_+)$ with $v_- > 0$, $v_+ > 0$ to introduce the integration in the finite part sense of Hadamard of the function h on the open set $]b,c[$ with respect to V. More precisely, if the function $f_h^V(\epsilon) := [\int_b^{-\epsilon v_-} + \int_{\epsilon v_+}^c] h(x)\, dx$ for

$$0 < \epsilon < \text{Min}\,(c/v_+, -b/v_-))$$

is of the first kind then $fp \int_{]b,c[,V} h(x)\, dx := fp[f_h^V(\epsilon)]$. Observe that whenever $h \in L^1(]b,c[, C)$, $fp\int_{]b,c[,V} h(x)\, dx = \int_{]b,c[,V} h(x)\, dx$, for any V.

THEOREM 3·2. *Given $U(O)$ a domain configuration with respect to O, $V := (v_-, v_+)$ and $i \in \{1, \ldots, n\}$, if $s^a_{\alpha,j} \in \mathscr{E}^p_O$ (with $p > 1$) then*

$$fp\int_{\Omega,U} s^a_{\alpha,j}(x)\, dx = fp\int_{\mathbb{R}\backslash[-\epsilon v_-,\epsilon v_+]} \left[\int_{\mathbb{R}^{n-1}} \Pi_\Omega(x)\, s^a_{\alpha,j}(x) \prod_{k\neq i} dx_k\right] dx_i + F_i^{U,V}(s^a_{\alpha,j}), \tag{3·11}$$

where the additional 'Fubini' term $F_i^{U,V}(s^a_{\alpha,j})$ is

$$\begin{aligned} F_i^{U,V}(s^a_{\alpha,j}) = s_\alpha \Bigg\{ & \int_{\Sigma^+_{n,i}} a(\theta) \left[\int_{r_V(\theta)}^{v_+} \frac{\log^j r}{r} dr\right] d\sigma_n + \int_{\Sigma^-_{n,i}} a(\theta) \left[\int_{r_U(\theta)}^{v_-} \frac{\log^j r}{r} dr\right] d\sigma_n \\ & + \sum_{k=0}^{j} \frac{C_j^k(-1)^{k+1}}{k+1} \left[\log^{j-k}(v_+) \int_{\Sigma^+_{n,i}} a(\theta) \log^{k+1}|y_i(\theta)|\, d\sigma_n \right. \\ & \left. + \log^{j-k}(v_-) \int_{\Sigma^-_{n,i}} a(\theta) \log^{k+1}|y_i(\theta)|\, d\sigma_n \right]\Bigg\}. \end{aligned} \tag{3·12}$$

Proof. If $V = (v_-, v_+)$ we set $d_V := \text{Min}\,(v_-^{-1}, v_+^{-1})$. For given integer $i \in \{1, \ldots, n\}$ and $0 < \epsilon < d_V\, d(O, \partial\Omega)$ consider $\Omega_i(\epsilon) := \{x = (x_1, \ldots, x_n) \in \Omega; x_i > \epsilon v_+ \text{ or } x_i < -\epsilon v_-\}$ and $\mathbb{R}_\epsilon := \mathbb{R}\backslash[-\epsilon v_-, \epsilon v_+]$. Observe that the origin O does not belong to $\Omega_i(\epsilon)$ and that $\Pi_{\Omega_i(\epsilon)}(x) = \Pi_\Omega(x)\, \Pi_{\mathbb{R}_\epsilon}(x_i)$. Since $s^a_{\alpha,j} \in L^1_{\text{loc}}(\Omega\backslash\{0\}, C)$, the integral

$$f_i^V(\epsilon) := \int_{\Omega_i(\epsilon)} s^a_{\alpha,j}(x)\, dx = \int_{\mathbb{R}^n} \Pi_{\Omega_i(\epsilon)}(x)\, s^a_{\alpha,j}(x)\, dx$$

exists and consequently the usual Fubini's theorem applies to it. Hence the function

$$\begin{aligned} f_i^V(\epsilon) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} \Pi_{\Omega_i(\epsilon)}(x)\, s^a_{\alpha,j}(x)\, dx\right] \\ &= \int_{\mathbb{R}\backslash[-\epsilon v_-, \epsilon_+]} \left[\int_{\mathbb{R}^{n-1}} \Pi_\Omega(x)\, s^a_{\alpha,j}(x) \prod_{k\neq i} dx_k\right] dx_i, \end{aligned} \tag{3·13}$$

possesses a sense.

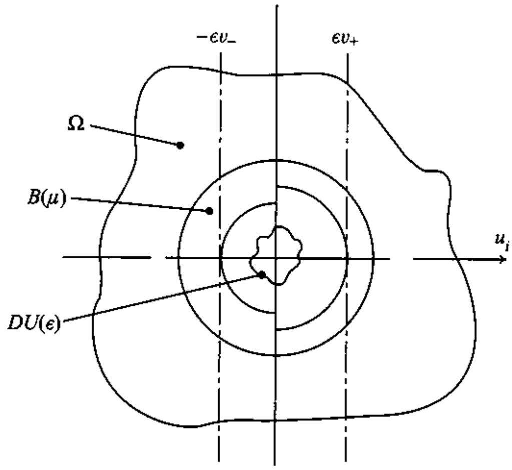

Fig. 2

Given $U(O)$ a domain configuration relative to O, reals μ and ϵ are chosen such that $0 < \mu < d(O, \partial\Omega)$ and $0 < \epsilon < \mathrm{Min}(\mu/d_U, \mu d_V)$. For $B(\mu) := \{M \in \mathbb{R}^n; OM = r < \mu\}$, $B(\mu) \subset \Omega$ and $[\Omega\backslash B(\mu)](\epsilon) := \{x = (x_1, \ldots, x_n) \in \Omega\backslash B(\mu), x_i > \epsilon v_+ \text{ or } x_i < -\epsilon v\}$. This notation and Fig. 2 above yield

$$f_i^V(\epsilon) := \int_{\Omega_i(\epsilon)} s^a_{\alpha,j}(x)\,dx = \int_{\Omega_i(\epsilon)\cap B(\mu)} s^a_{\alpha,j}(x)\,dx + \int_{[\Omega\backslash B(\mu)](\epsilon)} s^a_{\alpha,j}(x)\,dx, \tag{3·14}$$

$$f_s^U(\epsilon) := \int_{\Omega\backslash DU(\epsilon)} s^a_{\alpha,j}(x)\,dx = \int_{B(\mu)\backslash DU(\epsilon)} s^a_{\alpha,j}(x)\,dx + \int_{\Omega\backslash B(\mu)} s^a_{\alpha,j}(x)\,dx. \tag{3·15}$$

Since $s^a_{\alpha,j} \in L^1_{\mathrm{loc}}(\Omega\backslash\{O\}, C)$, as ϵ tends to zero

$$F(\epsilon) := \int_{[\Omega\backslash B(\mu)](\epsilon)} s^a_{\alpha,j}(x)\,dx - \int_{\Omega\backslash B(\mu)} s^a_{\alpha,j}(x)\,dx$$

tends to zero. Moreover, inspection of Fig. 2 leads to

$$\begin{aligned}K(\epsilon) &:= \int_{B(\mu)\backslash DU(\epsilon)} s^a_{\alpha,j}(x)\,dx - \int_{\Omega_i(\epsilon)\cap B(\mu)} s^a_{\alpha,j}(x)\,dx \\ &= \int_{\Sigma^+_{m,i}} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\epsilon v_+} r^{\alpha+n-1}\log^j r\,dr\right]d\sigma_n + \int_{\Delta_i^+(\mu, v_+, \epsilon)} s^a_{\alpha,j}(x)\,dx \\ &\quad + \int_{\Sigma^-_{n,i}} a(\theta)\left[\int_{\epsilon r_U(\theta)}^{\epsilon v_-} r^{\alpha+n-1}\log^j r\,dr\right]d\sigma_n + \int_{\Delta_i^-(\mu, v_+, \epsilon)} s^a_{\alpha,j}(x)\,dx.\end{aligned} \tag{3·16}$$

Consequently use of Theorem 2·4, Lemma 3·2 and equalities (3·14), (3·15) ensures that $K(\epsilon) = f_s^U(\epsilon) - f_i^V(\epsilon) + F(\epsilon) = I(\epsilon)$ with $\lim_{\epsilon\to 0} F(\epsilon) = 0$ and I function of the first kind. Thus, $f_s^U - f_i^V$ is of the first kind and previous results provide the formula (3·12).

Usually $v_- = v_+ = v > 0$. In such a case

$$F_i^{U,V}(s^a_{\alpha,j}) = s_\alpha \int_{\Sigma_n} a(\theta)\left[\int_{r_U(\theta)}^{v} \frac{\log^j r}{r}\,dr + \sum_{k=0}^{j} \frac{C_j^k(-1)^{k+1}}{k+1}\log^{j-k}(v)\log^{k+1}|y_i(\theta)|\right]d\sigma_n. \tag{3·17}$$

Moreover, if $v = 1$, this term reduces to

$$F_i^{U,V}(s_{\alpha,j}^a) = -\frac{s_\alpha}{j+1}\int_{\Sigma_n} a(\theta)\,[\log^{j+1} r_U(\theta) + (-1)^j \log^{j+1}|y_i(\theta)|]\,d\sigma_n. \tag{3·18}$$

Application of Theorem 3·2 leads to:

PROPOSITION 3·3. *Consider integers i and l belonging to $\{1,\ldots,n\}$ with $i \neq l$ and $(v_-^i, v_+^i, v_-^l, v_+^l) \in \mathbb{R}_+^{*4}$. Then, if $s_{\alpha,j}^a \in \mathscr{E}_0^p$ (with $p > 1$)*

$$fp\int_{\mathbb{R}\setminus[-\epsilon v_-^i,\,\epsilon v_+^i]}\left[\int_{\mathbb{R}^{n-1}} \Pi_\Omega(x)\, s_{\alpha,j}^a(x) \prod_{k\neq i} dx_k\right] dx_i = D_{i,l}^{v_-^i, v_+^i, v_-^l, v_+^l}(s_{\alpha,j}^a)$$
$$+ fp\int_{\mathbb{R}\setminus[-\epsilon v_-^l,\,\epsilon v_+^l]}\left[\int_{\mathbb{R}^{n-1}} \Pi_\Omega(x)\, s_{\alpha,j}^a(x) \prod_{k\neq l} dx_k\right] dx_l, \tag{3·19}$$

with

$$D_{i,l}^{v_-^i, v_+^i, v_-^l, v_+^l} = -s_\alpha\Bigg\{\int_{\Sigma_{n,i}^+}\frac{a(\theta)}{j+1}\log^{j+1}(v_+^i)\,d\sigma_n + \int_{\Sigma_{n,i}^-}\frac{a(\theta)}{j+1}\log^{j+1}(v_-^i)\,d\sigma_n$$
$$-\int_{\Sigma_{n,l}^+}\frac{a(\theta)}{j+1}\log^{j+1}(v_+^l)\,d\sigma_n - \int_{\Sigma_{n,l}^-}\frac{a(\theta)}{j+1}\log^{j+1}(v_-^l)\,d\sigma_n + \sum_{k=0}^{j}\frac{C_j^k(-1)^{k+1}}{k+1}$$
$$\times\Bigg[\log^{j-k}(v_+^i)\int_{\Sigma_{n,i}^+} a(\theta)\log^{k+1}|y_i(\theta)|\,d\sigma_n + \log^{j-k}(v_-^i)$$
$$\times\int_{\Sigma_{n,i}^-} a(\theta)\log^{k+1}|y_i(\theta)|\,d\sigma_n - \log^{j-k}(v_+^l)\int_{\Sigma_{n,l}^+} a(\theta)\log^{k+1}$$
$$\times|y_l(\theta)|\,d\sigma_n - \log^{j-k}(v_-^l)\int_{\Sigma_{n,l}^-} a(\theta)\log^{k+1}|y_l(\theta)|\,d\sigma_n\Bigg]\Bigg\}. \tag{3·20}$$

The case $(v_-^i, v_+^i, v_+^l, v_+^l) = (1,1,1,1)$ is often encountered. (3·19) leads to

$$D_{i,l}^{1,1,1,1}(s_{\alpha,j}^a) = \frac{(-1)^j s_\alpha}{j+1}\left\{\int_{\Sigma_n} a(\theta)\,[\log^{j+1}|y_i(\theta)| - \log^{j+1}|y_l(\theta)|]\,d\sigma_n\right\}. \tag{3·21}$$

4. *Derivation with respect to the isolated singularity; applications*

Consider $y = (y_1,\ldots,y_n)$, a point belonging to the open set Ω. If $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, it is understood that $x \pm y := (x_1 \pm y_1, \ldots, x_n \pm y_n)$. By now, U_y designates a domain configuration with respect to y and defined by the function r_U, i.e.

$$U_y := \{x \in \mathbb{R}^n;\, x - y = (r_U(\theta), \theta) \quad \text{where} \quad \theta := (x-y)/\|x-y\|\}.$$

Obviously for $\epsilon > 0$,

$$DU_y(\epsilon) := \{x \in \mathbb{R}^n;\, x - y = (r, \theta), \quad \text{and} \quad r < \epsilon r_U(\theta)\}.$$

Moreover for any integer $i \in \{1,\ldots,n\}$, $\mathbf{e}_i$ denotes the usual unit vector along the direction i and $\partial_i g(x) := \partial g(x)/\partial x_i$.

PROPOSITION 4·1. *Consider $y = (y_1,\ldots,y_n) \in \Omega$, U_y a domain configuration with respect to y, an integer $i \in \{1,\ldots,n\}$ and $s_{\alpha,j}^a(X) := a(\theta)\, r^\alpha \log^j r$ (for $X := (r,\theta)$) with*

$\alpha \in C$, $j \in \mathbb{N}$ and such that $\partial_i s^a_{\alpha,j}(X)$ exists for $X \neq O$. The function $I(y)$ is defined by $I(y) := fp\int_{\Omega,U_y} s^a_{\alpha,j}(x-y)\,dx$. Then if $a \neq 1-n$

$$\frac{\partial I(y)}{\partial y_i} = \frac{\partial}{\partial y_i}\left[fp\int_{\Omega,U_y} s^a_{\alpha,j}(x-y)\,dx\right] = fp\int_{\Omega,U_y}\frac{\partial}{\partial y_i}[s^a_{\alpha,j}(x-y)]\,dx; \tag{4·1}$$

else

$$\frac{\partial I(y)}{\partial y_i} = fp\int_{\Omega,B_y}\frac{\partial}{\partial y_i}[s^a_{1-n,j}(x-y)]\,dx - \delta_{j0}\left[\int_{\Sigma_n} a(\theta)\frac{x_i-y_i}{r}\,d\sigma_n\right], \tag{4·2}$$

where δ denotes Kronecker Delta, $x-y=(r,\theta)$ and $B_y := \{x \in \mathbb{R}^n : x-y=(1,\theta)\}$.

Proof. Note that if $Re(\alpha) > -n, I(y)$ turns out to be, for any U_y, a usual integral $I(y) = \int_\Omega s^a_{\alpha,j}(x-y)\,dx$. For $i \in \{1,\ldots,n\}$ and real h such that $0 < h < d(y,\partial\Omega)$, the set $\Omega'_h := \{x - h\mathbf{e}_i := (x_1,\ldots,x_{i-1},x_i-h,x_{i+1},\ldots,x_n);\, x=(x_1,\ldots,x_n)\in\Omega\}$ is introduced. Observe that Ω'_h is open and choice of h ensure $y \in \Omega'_h$. Change of variable $x = x' + h\mathbf{e}_i$ (always valid without any corrective term since it reduces to a shift) allows us to write

$$\begin{aligned} I(y+h\mathbf{e}_i) &= fp\int_{\Omega,U_{y+h\mathbf{e}_i}} s^a_{\alpha,j}(x-y-h\mathbf{e}_i)\,dx = fp\int_{\Omega'_h,U_y} s^a_{\alpha,j}(x'-y)\,dx' \\ &= fp\int_{\Omega,U_y} s^a_{\alpha,j}(x'-y)\,dx' + \int_{\Omega'_h\setminus(\Omega'_h\cap\Omega)} s^a_{\alpha,j}(x'-y)\,dx' - \int_{\Omega\setminus(\Omega'_h\cap\Omega)} s^a_{\alpha,j}(x'-y)\,dx'. \end{aligned} \tag{4·3}$$

In fact the two last integrals on the right-hand side of equality (4·3) are usual integrations (since $y \in \Omega'_y$, it belongs neither to $\Omega'_h\setminus(\Omega'_h\cap\Omega)$, nor to $\Omega\setminus(\Omega'_h\cap\Omega)$). The first integral on the right-hand side of (4·3) is $I(y) = fp\int_{\Omega,U_y} s^a_{\alpha,j}(x'-y)\,dx'$. The assumption of the existence of $\partial_i(s^a_{\alpha,j})$ implies that the function $s^a_{\alpha,j}(x)$ is continuous with respect to x_i. Application of the usual mean theorem yields

$$\frac{\partial I(y)}{\partial y_i} := \lim_{h\to0}\frac{I(y+h\mathbf{e}_i)-I(y)}{h} = -\int_{\partial\Omega} s^a_{\alpha,j}(x-y)\,\mathbf{n}.\mathbf{e}_i\,dS, \tag{4·4}$$

where $\mathbf{n}$ is the unit vector on $\partial\Omega$ directed outwards from Ω. Note that the result depends on U_y. For $\epsilon > 0$, consider $B_y(\epsilon) := \{x \in \mathbb{R}^n;$ if $x-y=(r,\theta)$, then $r<\epsilon\}$. As $\partial_i[s^a_{\alpha,j}(x-y)]$ exists in $\Omega\setminus B_y(\epsilon)$, application of Green's theorem ensures

$$\frac{\partial I(y)}{\partial y_i} = -\int_{\Omega\setminus B_y(\epsilon)}\partial_i[s^a_{\alpha,j}(x-y)]\,dx - \int_{\partial B_y(\epsilon)} s^a_{\alpha,j}(x-y)\,\mathbf{n}.\mathbf{e}_i\,dS, \tag{4·5}$$

with $\mathbf{n}$ directed outwards from $B_y(\epsilon)$, i.e. $\mathbf{n} = \mathbf{e}_r$. The boundary integral occurring in equality (4·5) is easy to transform. Indeed, use of spherical polar coordinates (r,θ) of origin y (that is to say $x-y=(r,\theta)$) leads to

$$d(\epsilon) := \int_{\partial B_y(\epsilon)} s^a_{\alpha,j}(x-y)\,\mathbf{n}.\mathbf{e}\,dS = \epsilon^{\alpha+n-1}\log^j\epsilon\left[\int_{\Sigma_n} a(\theta)\frac{x_i-y_i}{r}\,d\sigma_n\right]. \tag{4·6}$$

Thereafter d is of the first kind and $fp[d(\epsilon)] = 0$, unless $\alpha = 1-n$ and $j = 0$. Moreover, combination of equalities (4·4), (4·5) and (4·6) proves that the function g such that $g(\epsilon) := \int_{\Omega\setminus B_y(\epsilon)}\partial_i[s^a_{\alpha,j}(x-y)]\,dx$ is of the first kind too. Consequently

$$\frac{\partial I(y)}{\partial y_i} = fp\int_{\Omega,B_y}\frac{\partial}{\partial y_i}[s^a_{\alpha,j}(x-y)]\,dx - fp[\epsilon^{\alpha+n-1}\log^j\epsilon]\left[\int_{\Sigma_n} a(\theta)\frac{x_i-y_i}{r}\,d\sigma_n\right]. \tag{4·7}$$

Here $\partial[s^a_{\alpha,j}(x-y)]/\partial y_i = -\partial[s^a_{\alpha,j}(X)]/\partial X_i$, if $X := x-y = (r,\theta)$. The choice of a set of spherical polar coordinates (r,θ_j) such that $X_i = r\cos\theta_i$ for the given integer i, yields $\partial[s^a_{\alpha,j}(X)]/\partial X_i = \cos\theta_i \times \partial s^a_{\alpha,j}/\partial r - r^{-1}\sin\theta_i \times \partial s^a_{\alpha,j}/\partial\theta_i$, i.e.

$$\partial_i[s^a_{\alpha,0}] = r^{\alpha-1}\left[\alpha a(\theta)\cos\theta_i - \sin\theta_i \frac{\partial a}{\partial\theta_i}\right], \tag{4·8}$$

$$\partial_i[s^a_{\alpha,0}] = r^{\alpha-1}\log^j r\left[\alpha a(\theta)\cos\theta_i - \sin\theta_i\frac{\partial a}{\partial\theta_i}\right] + ja(\theta)\cos\theta_i r^{\alpha-1}\log^{j-1} r, \quad \text{for} \quad j \geqslant 1. \tag{4·9}$$

Hence, if $\alpha \neq 1-n$, Proposition 2·5 gives

$$fp\int_{\Omega,B_y}\frac{\partial}{\partial y_i}[s^a_{\alpha,j}(x-y)]\,dx = fp\int_{\Omega,U_y}\frac{\partial}{\partial y_i}[s^a_{\alpha,j}(x-y)]\,dx. \tag{4·10}$$

When $\alpha = 1-n$, (4·8), (4·9) and Proposition 2·5 allow us to give the link between the two integrals above.

To conclude, the range of applications of this paper is outlined and some examples are given. At first some definitions and usual notation are introduced. For $x = (x_1,\ldots,x_n)\in\mathbb{R}^n$ and $\beta = (\beta_1,\ldots,\beta_n)\in\mathbb{N}^n$, it is understood that

$$|\beta| := \beta_1+\ldots+\beta_n,\ \beta! := \beta_1!\ldots\beta_n!;\ x^\beta = x_1^{\beta_1}\ldots x_n^{\beta_n}$$

and $D^\beta := \partial^{\beta_1}/\partial x_1^{\beta_1}\ldots\partial^{\beta_n}/x_n^{\beta_n} = D^\beta/\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}$. If Ω is an open and bounded subset of $\mathbb{R}^n$, for $N\in\mathbb{N}$ a complex function ϕ belongs to the set $\mathscr{D}^N(\Omega)$ if and only if ϕ admits derivatives up to order $N+1$ in Ω with those of order $N+1$ bounded in $\Omega\cup\partial\Omega$ (which for instance is satisfied as soon as those derivatives of order $N+1$ are continuous on the compact $\Omega\cup\partial\Omega$). Moreover, $Q^N_{\phi,y}$ and $R^N_{\phi,y}$ respectively designate its Taylor polynomial expansion of order N at point y of Ω and the associated remainder. These functions obey the well-known relations

$$Q^N_{\phi,y}(x) := \sum_{|\beta|=0}^{N}\frac{1}{\beta!}D^\beta\phi(y)(x-y)^\beta, \tag{4·11}$$

$$\begin{aligned} R^N_{\phi,y}(x) &:= \phi(x) - Q^N_{\phi,y}(x) \\ &= (N+1)\sum_{|\beta|=N+1}\frac{(x-y)^\beta}{\beta!}\int_0^1(1-t)^N D^\beta\phi[y+t(x-y)]\,dt. \end{aligned} \tag{4·12}$$

Observe that if $h\in\mathscr{D}^N(\Omega)$ then $R^N_{\phi,y}(x)/\|x-y\|^{N+1}$ is bounded on Ω. For $\mathscr{D}$ an open subset of $\mathbb{R}^n$ containing y, the set of pseudofunctions $\mathscr{A}_y(\mathscr{D})$ is defined below.

Definition 4·2. A complex function h belongs to the set $\mathscr{A}_y(\mathscr{D})$ if and only if there exist $\Omega\subset\mathscr{D}$ an open, bounded and simply connected neighbourhood of y such that $h\in L^1(\mathscr{D}\backslash\Omega, C)$ and also a positive integer I, a family of positive integers $(J(i))_{\{i,\ldots,I\}}$, a complex family (α_i), a family of complex functions $(a^h_{ij}))$, a complex function $R_h(x)$ such that in spherical polar coordinates (r,θ) of origin y

$$\left.\begin{aligned} &h(M) = h(r,\theta) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} a^h_{ij}(\theta)\,r^{\alpha_i}\log^j r + R_h(x) \quad \text{a.e. in } \Omega, \\ &R_h\in L^1(\Omega,C);\quad \forall(i,j)\,a^h_{ij}\in L^1(\Sigma_n,C), \\ &Re(\alpha_I)\leqslant Re(\alpha_{I-1})\leqslant\ldots\leqslant Re(\alpha_1)\leqslant Re(\alpha_0) := -n \quad \text{and} \quad \exists(m,k), a^h_{mk}\neq\Theta_{\Sigma_n}, \end{aligned}\right\} \tag{4·13}$$

where Θ_{Σ_n} is the zero function on Σ_n.

It is understood that $\Omega, I, (J(i)), (\alpha_i)$ depend on the function h which is singular at point y (since there exists (m,k), $a^h_{mk} \neq \Theta_{\Sigma_n}$ and $Re(\alpha_m) \leqslant -n$). Definition 4·2 ensures for any element h of $\mathscr{A}_y(\mathscr{D})$ and for any domain configuration U_y with respect to y

$$fp\int_{\mathscr{D},U_y} h(x)\,dx := \sum_{i=0}^{I}\sum_{j=0}^{J(i)} fp\int_{\Omega,U_y} s^{a^h_{ij}}_{\alpha_i,j}(x)\,dx + \int_\Omega R_h(x)\,dx + \int_{\mathscr{D}\backslash\Omega} h(x)\,dx. \quad (4·14)$$

Equality (4·14) defines the quantity $I = fp\int_{\mathscr{D},U_y} h(x)\,dx$ and results (as soon as the required assumptions bearing on functions a^h_{ij} are satisfied) of Sections 2 and 3 provide the eventual corrective terms occurring when attempting on I the studied operations: change of domain configuration U_y, change of variables, application of 'Fubini's theorem'.

Actually, one often deals with integral operators written as

$$T(y) := fp\int_{\mathscr{D},U_y} K(x-y)\,\phi(x)\,dx, \quad (4·15)$$

where $\mathscr{D}$ is an open subset of $\mathbb{R}^n$ containing y, U_y is a domain configuration with respect to $y, K(X)$ is a kernel singular at $X = O$, ϕ a smooth enough and unknown density and there exists Ω an open, bounded and simply connected neighbourhood of y such that $h(x) := K(x-y)\,\phi(x) \in L^1(\mathscr{D}\backslash\Omega, C)$. When there exist $(\alpha, j, N) \in C \times \mathbb{N}^2$ with $Re(\alpha) \leqslant -n, N := [\![-Re(\alpha)-n]\!]$ and $a \in L^1(\Sigma_n, C)$ such that $K(X) := s^a_{\alpha,j}(X)$ and $\phi \in \mathscr{D}^N(\Omega)$, then the above-mentioned complex function h belongs to $\mathscr{A}_y(\mathscr{D})$. More precisely, if $x = y + (r,\theta) \in \Omega_y$ and $A_\beta(\theta) := (x-y)^\beta / r^{|\beta|}$ then

$$h(x) = s^a_{\alpha,j}(x-y)\,\phi(x) = \sum_{|\beta|=0}^{[\![-Re(\alpha)-n]\!]} \frac{D^\beta\phi(y)}{\beta!}\, a(\theta)\, A_\beta(\theta)\, r^{\alpha+|\beta|} \log^j r + R^N_{h,y}(x), \quad (4·16)$$

where the function $R^N_{h,y}$ belonging to $L^1(\Omega, C)$ obeys (with $N := [\![-Re(\alpha)-n]\!]$) the relation

$$R^N_{h,y}(x) := (N+1) \sum_{|\beta|=N+1} \left[\frac{a(\theta)\,A_\beta(\theta)}{\beta!} \int_0^1 (1-t)^N D^\beta\phi(y+t(x-y))\,dt\right] r^{\alpha+|\beta|} \log^j r. \quad (4·17)$$

Moreover, if for $i \in \{1, \dots, n\}$ the function $\partial s^a_{\alpha,j}(X)/\partial X_i$ exists at $X \neq 0$ (and is given by equalities (4·8), (4·9)) and if $H(x) := \phi(x)\,\partial[s^a_{\alpha,j}(x-y)]/\partial y_i \in L^1(\mathscr{D}\backslash\Omega, C)$ and for $N := [\![-Re(\alpha)-n]\!]$, $\phi \in \mathscr{D}^{N+2}(\Omega)$ then

$$T(y) = \sum_{|\beta|=0}^{[\![-Re(\alpha)-n]\!]+1} \frac{D^\beta\phi(y)}{\beta!} fp\int_{\Omega,U_y} a(\theta)\,A_\beta(\theta)\, r^{\alpha+|\beta|}\log^j r\,dx + \int_\Omega R^{N+1}_{h,y}(x)\,dx + \int_{\mathscr{D}\backslash\Omega} s^a_{\alpha,j}(x-y)\,\phi(x)\,dx, \quad (4·18)$$

with $h(x) := s^a_{\alpha,j}(x-y)\,\phi(x)$, $R^{N+1}_{h,y} \in L^1(\Omega, C)$, $\phi \in \mathscr{D}^{N+2}(\Omega)$ and equality (4·17) involving that $\partial[R^{N+1}_{h,y}]/\partial y_i \in L^1(\Omega, C)$. Derivation of (4·18) with respect to y_i with use of Proposition 4·1 shows that $\partial T(y)/\partial y_i = fp\int_{\mathscr{D},U_y} \partial[s^a_{\alpha,j}(x-y)]/\partial y_i\,\phi(x)\,dx$, except if there exists $k \in \mathbb{N}$, $\alpha = 1-n-k$. In such a case (see Proposition 4·1)

$$\frac{\partial T(y)}{\partial y_i} = fp\int_{\mathscr{D},B_y} \frac{\partial s^a_{\alpha,j}(x-y)}{\partial y_i}\phi(x)\,dx - \delta_{j0}\sum_{|\beta|=k}\frac{D^\beta\phi(y)}{\beta!}\left[\int_{\Sigma_n} a(\theta)\,A_\beta(\theta)\frac{x_i-y_i}{r}\,d\sigma_n\right]. \quad (4·19)$$

Examples. For $i\in\{1,\dots,n\}$, if ϕ is a function such that ϕ is Hölder continuous at y, i.e. there exists $(\eta,B,\gamma)\in\mathbb{R}_+^{*3}$ with $|\phi(x)-\phi(y)|<B|x-y|^\gamma$ for $|x-y|<\eta$ and moreover if $\phi\in L^1(\mathscr{D}\backslash B_y(\eta),C)$ we consider

$$T(y)=vp\int_{\mathscr{D},B_y}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\,dx:=\lim_{\epsilon\to0}\left[\int_{\mathscr{D}\backslash B_y(\epsilon)}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\,dx\right]$$

$$=\phi(y)\,vp\int_{\mathscr{D},B_y}\frac{\cos(\theta_i)}{r^n}dx+\int_{\mathscr{D}}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}[\phi(x)-\phi(y)]\,dx, \tag{4·20}$$

where $x=y+(r,\theta)$ and $x_i=y_i+r\cos\theta_i$ with an appropriated choice of angular coordinates $(\theta_1,\dots,\theta_{n-1})$. The concept of principal value of an integral ($vp\int$) is encountered when dealing with integral equations (the reader is, for instance, referred to Mikhlin and Prössdorf[6]). The following results are obtained

(i) The change of variables $x=T(x')$ such that $\forall k\in\{1,\dots,n\};x_k=\lambda_k x'_k$ with $\prod_{k=1}^n\lambda_k\neq0$ is treated by applying Theorem 2·6 to the first integral on the right-hand side of (4·20) and $V_y:=T[B_y]$ is an ellipsoid domain defined by $r_E(\theta)$. (Its equation is $x_1^2/\lambda_1^2+\dots+x_n^2/\lambda_n^2=1$.) Here $a(\theta):=\cos\theta_i\in L^p(\Sigma_n,C)$ for $p\geqslant1$, $\alpha=-n, j=0$ and the additional term is $-\int_{\Sigma_n}\cos\theta_i\log r_E(\theta)\,d\sigma_n$. For symmetry reasons this term is zero.

(ii) For $l\in\{1,\dots,n\}$, application of Theorem 3·2 to $T(y)$ with $v_-=v_+=1$ gives no additional term since $\int_{\Sigma_n}\cos\theta_i\log|y_l(\theta)|\,d\sigma_n=0$, and leads to

$$vp\int_{\mathscr{D},B_y}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\,dx=fp\int_{\mathbb{R}\backslash[-\epsilon,\epsilon]}\left[\int_{\mathbb{R}^{n-1}}\Pi_\Omega(x)\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\prod_{k\neq l}dx_k\right]dx_l.$$

(iii) For $l\in\{1,\dots,n\}$, if there exists $\Omega\in\mathscr{D}$ an open, bounded and simply connected neighbourhood of y such that $\phi\in\mathscr{D}^2(\Omega)$, and

$$H(x):=\phi(x)\,\partial[s^a_{-n,0}(x-y)]/\partial y_l\in L^1(\mathscr{D}\backslash\Omega,C)$$

$$T(y)=\phi(y)\,vp\int_{\mathscr{D},B_y}\frac{\cos\theta_i}{r^n}dx+\sum_{k=1}^n\frac{\partial\phi(y)}{\partial x_k}\int_\Omega\frac{\cos\theta_i}{r^{n-1}}A_k(\theta)\,dx$$
$$+\int_{\Omega_y}R^1_{h,y}(x)\,dx+\int_{\mathscr{D}\backslash\Omega}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\,dx, \tag{4·21}$$

where $h(x):=\cos\theta_i\,\phi(x)/r^n$, and $R^1_{h,y}$ is given by (4·17). Application of (4·19) leads to

$$\frac{\partial}{\partial y_l}\left[vp\int_{\mathscr{D},B_y}\frac{(\mathbf{x}-\mathbf{y}).\mathbf{e}_i}{|x-y|^{n+1}}\phi(x)\,dx\right]=fp\int_{\mathscr{D},B_y}\frac{\partial}{\partial y_l}\left[\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{e}_i}{|x-y|^{n+1}}\right]\phi(x)\,dx$$
$$-\sum_{k=1}^n\frac{\partial\phi(y)}{\partial x_k}\int_{\Sigma_n}\frac{x_i-y_i}{r}\frac{x_k-y_k}{r}\frac{x_l-y_l}{r}d\sigma_n. \tag{4·22}$$

For symmetry reasons, the last sum on the right-hand side of (4·22) is zero and consequently there is no corrective term. If

$$\phi\in\mathscr{D}^2(\Omega_y),H(x):=\phi(x)\,\partial[|x-y|^{-n}]/\partial y_l\in L^1(\mathscr{D}\backslash\Omega,C)$$

and U_y is a domain configuration with respect to y, one obtains (with $\int_{\Sigma_n} y_l^2\, d\sigma_n = \Gamma(1/2)^n/\Gamma(1+n/2)$)

$$\frac{\partial}{\partial y_l}\left[fp\int_{\mathscr{D},\,U_y}\frac{\phi(x)}{|x-y|^n}dx\right]=fp\int_{\mathscr{D},\,B_y}\frac{\partial}{\partial y_l}\left[\frac{\phi(x)}{|x-y|^n}\right]dx-\frac{\Gamma(1/2)^n}{\Gamma(1+n/2)}\frac{\partial\phi(y)}{\partial x_l}. \quad (4{\cdot}23)$$

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