# Asymptotic expansion of a general integral

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For $b \in \mathbb{R}_+^*$, a real function $h(x)$ and also complex pseudo-functions $f(x)$ and $K(x, u)$ obeying weak assumptions, an asymptotic expansion of the general integral

$$I(\lambda) := \boldsymbol{fp} \int_0^b f(x) K[h(x), \lambda x] \, \mathrm{d}x,$$

is proposed, with respect to the real and large parameter $\lambda$, where the notation $\boldsymbol{fp}$ designates an integration in the finite part sense of Hadamard.

## 1. Introduction

For $\epsilon > 0, -\infty < a < 0 < b < +\infty$ and also a smooth enough complex function $f$ and a real function $h$ such that $h(0) \neq 0$ we consider the following integral:

$$W_h(\epsilon) = \int_a^b f(x) \, \mathrm{d}x / \sqrt{x^2 + \epsilon^2 h^2(x)}. \tag{1.1}$$

Such an integral naturally arises in the well-known slender-body theory, in the case of an axially symmetric flow, both for $h = 1$ (see Tuck 1992) and also for $h$ smooth enough and related to the area of the cross section (consult Handelsman & Keller (1967)). Actually the key step of this theory consists in expanding the quantity $W_h(\epsilon)$ with respect to the positive and small slenderness parameter $\epsilon$. In this specific case, Handelsman & Keller (1967) built up the expansion inductively by applying a tedious method. Instead of $W_h(\epsilon)$, one may also encounter the more general integral,

$$M_h(\epsilon) = \boldsymbol{fp} \int_a^b f(x) K[x, \epsilon h(x)] \, \mathrm{d}x, \tag{1.2}$$

where the symbol $\boldsymbol{fp}$ means integration in the finite part sense of Hadamard (see § 2). The new kernel $K$ is a '$Q$ pseudo-homogeneous' pseudo-function with $Q$ an integer (positive or negative), i.e. it obeys the following pseudo-homogenous property of order $Q$:

$$K(tx, ty) = t^Q S(t) K(x, y), \quad \text{for} \quad (t, x, y) \in \mathbb{R}^{*3}, \tag{1.3}$$

where $S(t) := 1$ for $t \in \mathbb{R}^*$ or $S(t) := \mathrm{sgn}(t) = t/|t|$ for $t \in \mathbb{R}^*$. For instance, the previous integral $W_h(\epsilon)$ is associated with $K(x, y) = [x^2 + y^2]^{-1/2}$, where $Q = -1, S(t) = \mathrm{sgn}(t)$. The aim of this paper is in part to present a systematic formula for the asymptotic expansion of $M_h(\epsilon)$ with respect to the small and strictly positive parameter $\epsilon$, when weak assumptions are made concerning the behaviour of the complex pseudo-functions $f, K$ and the real function $h$ in the neighbourhood of zero and at infinity. Guermond (1987) has dealt with the case $h = 1$ for smooth enough

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functions $f$ and $K$. The most general case of an arbitrary function $h$ seems difficult to tackle; the asymptotic expansion clearly involves the behaviour of $f$ and $h$ near zero on the right ($b>0$) and on the left ($a<0$). Thus, here we study a class of integrals, defined by the following integral:

$$J_h(\epsilon) := \boldsymbol{fp}\int_0^b f(x)L[x,\epsilon h(x)]\,\mathrm{d}x, \tag{1.4}$$

where $b$ is positive real and, this time, $L$ is a complex and '$\mu$ homogeneous' kernel, i.e. it satisfies, for $\mu$ real, the homogeneous property of order $\mu$:

$$L(tx,ty)=t^{\mu}L(x,y), \quad \text{for} \quad t\in\mathbb{R}_+^*, \quad x\in\mathbb{R}_+^*, \quad y\in\mathbb{R}^*. \tag{1.5}$$

As a result of equality (1.5),

$$J_h(\epsilon)=\epsilon^{\mu}\boldsymbol{fp}\int_0^b f(x)L[x/\epsilon,h(x)]\,\mathrm{d}x=\epsilon^{\mu}I_h(\lambda),$$

with $\lambda:=\epsilon^{-1}\to+\infty$ and for $K(x,u):=L(u,x)$

$$I_h(\lambda):=\boldsymbol{fp}\int_0^b f(x)K[h(x),\lambda x]\,\mathrm{d}x. \tag{1.6}$$

The objective of the analysis is to obtain the asymptotic expansion of $I_h(\lambda)$ for large values of the real parameter $\lambda$. This will be achieved for the more general kernel $K$ which is not necessarily a '$\mu$ homogeneous' pseudo-function.

If $h(x)=1$, by setting $H(x):=K(1,x)$, this problem reduces to the study of $I_1(\lambda)=\boldsymbol{fp}\int_0^b f(x)H(\lambda x)\,\mathrm{d}x$. For such an integral, different points of view may be adopted in order to deal with more or less restrictive assumptions on the behaviour of both complex pseudo-functions $f$ and $H$ near zero and at infinity. The reader is for instance referred to Bleistein & Handelsman (1975) and Wong (1989) for an approach based on integration by parts or the notion of the Mellin transform, to Estrada & Kanwal (1990, 1994) for an elegant distributional approach and to Sellier (1994, theorem 3) for an alternative method based on the concept of integration in the finite part sense of Hadamard.

Observe that the choice of a new pseudo-function $K_h$ such that $K_h(x,u):=K[h(x),u]$ allows us to write

$$I_h(\lambda)=\boldsymbol{fp}\int_0^b f(x)K_h(x,\lambda x)\,\mathrm{d}x. \tag{1.7}$$

When both pseudo-functions $f(x)$ and $K_h(x,u)$ satisfy specific assumptions, it seems possible to treat this last form of $I_h(\lambda)$ by applying the results of Sellier (1994, theorem 1). It is clearly essential to relate the asymptotic expansion of $I_h(\lambda)$ to the particular behaviour of $h$ and $K$. However, the properties of the pseudo-function $K_h(x,u)$ are not so easily deduced from those of the function $h$ and of the pseudo-function $K(x,u)$ despite it is clearly essential to relate the asymptotic expansion of $I_h(\lambda)$ to the specific behaviours of $h$ and of $K$. This study therefore deals with the following general integral

$$I(\lambda):=\boldsymbol{fp}\int_0^b f(x)K[h(x),\lambda x]\,\mathrm{d}x. \tag{1.8}$$

An asymptotic expansion of $I(\lambda)$ for large $\lambda$ is obtained when the pseudo-functions

$h, f$ and $K$ fulfil adequate assumptions. This integral is said to be general in the sense that it includes the following usual cases:

$$I_1(\lambda) = \boldsymbol{fp}\int_0^b g(x)H(\lambda x)\,\mathrm{d}x, \quad \text{for} \quad f = g,\ h = 1,\ H(u) = K[1,u];$$

$$I_2(\lambda) = \boldsymbol{fp}\int_0^b g[h(x)]H(\lambda x)\,\mathrm{d}x, \quad \text{for} \quad f = 1,\ K(x,u) = g(x)H(u);$$

$$I_3(\lambda) = \boldsymbol{fp}\int_0^b f(x)K(x,\lambda x)\,\mathrm{d}x, \quad \text{for} \quad h = x.$$

The paper is organized in the following manner. In § 2, useful mathematical tools are presented. A general theorem for the expansion of $I(\lambda)$ is derived in § 3. The cases of a few '$\mu$ homogeneous' kernels and also of a more general kernel are investigated in § 4. Many examples are included, indicating the wide range of applications of the results.

## 2. Mathematical concepts

Before stating the basic theorem yielding the asymptotic expansion of integral $I(\lambda)$, several useful mathematical tools and results are introduced. More precisely, the basic concept of integration in the finite part of Hadamard is presented by definitions 9 and 10. This concept not only allows us to give a sense, for real value $0 < b \leqslant +\infty$ and the specific and complex pseudo-function $f$, to the integral

$$\boldsymbol{fp}\int_0^b f(x)\,\mathrm{d}x,$$

but also plays a central role in expanding a class of integrals with respect to a large parameter $\lambda$ (see Sellier 1994). The other aim of this section is mainly to exhibit sufficient assumptions for the real function $h$ and complex pseudo-functions $F$ and $g$ which yield a good behaviour near zero (on the right) for the complex pseudo-function $f(x) := F(x)g[h(x)]$. Consequently, many symbols are introduced in this section to allow us to present a self-contained study and also to clearly define the range of applications of the main theorems established in § 3 and § 4. The symbol $C$ designates the set of complex numbers and $a$ is a real number.

*Definition 1.* A complex pseudo-function $f$ belongs to the set $D_+(a, C)$ if and only if there exist a complex $\alpha_f$, real values $\eta^+ > 0$ and $s^+ > \mathrm{Re}(\alpha_f)$, a complex function $F^+$ bounded in $[0, \eta^+]$, a positive integer $J^+$ and also a non-zero and complex family $(f_j^+)$ such that

$$f(x) = |x-a|^{\alpha_f}\left\{\sum_{j=0}^{J^+} f_j^+ \log^j |x-a|\right\} + |x-a|^{s^+}F^+(|x-a|), \quad \text{a.e. in} \quad ]a, a+\eta^+]. \tag{2.1}$$

To each element $f$ of $D_+(a, C)$ is associated the real $S_a^+(f) := \mathrm{Re}(\alpha_f)$.

*Definition 2.* For any real $r$, $D_+^r(a, C)$ is the set of complex pseudo-functions $f$ such that there exist real values $\eta^+ > 0$ and $s^+ > r$, a complex function $F_r^+$ bounded in $[0, \eta^+]$, a positive integer $I^+$, a family of positive integers $(J^+(i))$ and also two complex families $(\alpha_i^+), (f_{ij}^+)$ such that

$$\left.\begin{array}{c} f(x) = \displaystyle\sum_{i=0}^{I^+}\sum_{j=0}^{J^+(i)} f_{ij}^+|x-a|^{\alpha_i^+}\log^j|x-a| + |x-a|^{s^+}F_r^+(|x-a|), \\ \text{a.e. in} \quad ]a, a+\eta^+], \\ \mathrm{Re}(\alpha_0^+) < \mathrm{Re}(\alpha_1^+) < \cdots < \mathrm{Re}(\alpha_{I^+}^+) \leqslant r, \\ \forall (i,j)\ f_{ij}^+ = 0 \quad \text{or} \quad \forall i \in \{0,\ldots,I^+\} \quad \text{then} \quad (f_{ij}) \neq (0). \end{array}\right\} \tag{2.2}$$

For instance, if $f \in D_+(a,C)$ and $r < S_a^+(f)$ then $f_{ij}^+ = 0$ for each $(i,j)$. If $\forall(i,j)\ f_{ij}^+ \neq 0$, then $f \in D_+(a,C)$ with $S_a^+(f) := \mathrm{Re}(\alpha_0^+) < \mathrm{Re}(\alpha_1^+) < \cdots < \mathrm{Re}(\alpha_{I^+}^+) \leqslant r$. For $\epsilon > 0$, the notation

$$A(\epsilon) = \sum_{j,\mathrm{Re}(\alpha_i)\leqslant r} a_{ij}\epsilon^{\alpha_i}\log^j \epsilon$$

means that there exist a positive integer $I$, a family of positive integers $(J(i))$ and two complex families $(\alpha_i), (a_{ij})$ such that $\mathrm{Re}(\alpha_0) < \mathrm{Re}(\alpha_1) < \cdots < \mathrm{Re}(\alpha_I) \leqslant r$ and

$$A(\epsilon) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} a_{ij}\epsilon^{\alpha_i}\log^j \epsilon.$$

For $r' \leqslant r$, the notation

$$B(\epsilon) = \sum_{j,r'\leqslant\mathrm{Re}(\alpha_i)\leqslant r} a_{ij}\epsilon^{\alpha_i}\log^j \epsilon$$

is also introduced.

*Definition 3.* For $k \in \mathbb{N}$, $\mathcal{C}_+^k(a,C)$ is the set of complex functions $f$ such that there exists $\eta^+ > 0$ with $f$ admitting up to order $k$ continuous derivatives on $[a, a+\eta^+]$.

This definition means that, for $0 \leqslant i \leqslant k$, $f^{(i)}(x)$ exists for $x \in ]a, a+\eta^+[$ and that the complex number $f^{(i)}(a)$ and $f^{(i)}(a+\eta^+)$ respectively designating derivatives of order $i$ at $a$ on the right and at $a+\eta^+$ on the left exist. If $k \in \mathbb{N}^*$, the usual Taylor formula applies to $f$ and ensures (if $f^0 := f$) that the following useful relation holds for $t \in [a, a+\eta^+]$:

$$f(t) = \sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}(t-a)^i + \frac{(t-a)^k}{(k-1)!}\int_0^1 (1-u)^{k-1} f^{(k)}[a+u(t-a)]\,du. \tag{2.3}$$

Since it is continuous, $f^{(k)}$ is bounded on compact $[a, a+\eta^+]$. Consequently, the last term on the right-hand side of (2.3) may be rewritten as $T_k(t) = (t-a)^k P_k(t-a)$ with $P_k$ bounded on $[a, a+\eta^+]$. If $k = 0$, $T_0(t) := f(t) - f(a)$ is also bounded on $[a, a+\eta^+]$. Hence, for real $r < k \in \mathbb{N}$, $\mathcal{C}_+^k(a,C) \subset D_+^r(a,C)$.

Observe that, if $(f,g) \in D_+^r(a,C) \times D_+^r(a,C)$ and $(\beta,\gamma) \in C^2$, then $\beta f + \gamma g \in D_+^r(a,C)$ too. The short proposition below gives conditions regarding the complex pseudo-functions $f, g$ which ensure that $fg \in D_+^r(a,C)$.

**Proposition 4.** *Consider $(f,g) \in D_+(a,C) \times D_+(a,C)$, $r$ a given real and $r_1 := r - S_a^+(g)$, $r_2 := r - S_a^+(f)$. If $f \in D_+^{r_1}(a,C)$ and $g \in D_+^{r_2}(a,C)$ then $fg \in D_+^r(a,C)$.*

*Proof.* Two cases arise.

*Case 1:* $r \geqslant S_a^+(f)+S_a^+(g)$. In such circumstances, $r_1 \geqslant S_a^+(f), r_2 \geqslant S_a^+(g)$ and, according to definition 2, it is possible to find real values $\eta > 0, s_1 > r_1, s_2 > r_2$ and complex functions $F_{r_1}^+$ and $G_{r_2}^+$ bounded in $[0,\eta]$ such that, for almost any $x \in ]a, a+\eta]$,

$$f(x) = \sum_{j,S_a^+(f)\leqslant \mathrm{Re}(\alpha_i^+)\leqslant r_1} f_{ij}^+|x-a|^{\alpha_i^+}\log^j|x-a| + |x-a|^{s_1}F_{r_1}^+(|x-a|),$$

$$g(x) = \sum_{l,S_a^+(g)\leqslant \mathrm{Re}(\beta_k^+)\leqslant r_2} g_{kl}^+|x-a|^{\beta_k^+}\log^l|x-a| + |x-a|^{s_2}G_{r_2}^+(|x-a|).$$

Hence, the function $fg$ admits for almost any $x$ in $]a, a+\eta]$ the decomposition

$$f(x)g(x) = \sum_{j,l,\mathrm{Re}(\alpha_i^+ + \beta_k^+)\leqslant r} f_{ij}^+ g_{kl}^+ |x-a|^{\alpha_i^+ + \beta_k^+}\log^{j+l}|x-a| + Q(|x-a|)$$

with

$$\begin{aligned} Q(|x-a|) = &\sum_{j,l,r<\mathrm{Re}(\alpha_i^+ + \beta_k^+)\leqslant r_1+r_2} f_{ij}^+ g_{kl}^+|x-a|^{\alpha_i^+ + \beta_k^+}\log^{j+l}|x-a| \\ &+|x-a|^{s_1+s_2}F_{r_1}^+(|x-a|)G_{r_2}^+(|x-a|) \\ &+\sum_{j,S_a^+(f)\leqslant\mathrm{Re}(\alpha_i^+)\leqslant r_1} f_{ij}^+|x-a|^{\alpha_i^+ + s_2}\log^j|x-a|G_{r_2}^+(|x-a|) \\ &+\sum_{l,S_a^+(g)\leqslant\mathrm{Re}(\beta_k^+)\leqslant r_2} g_{kl}^+|x-a|^{\beta_k^+ + s_1}\log^l|x-a|F_{r_1}^+(|x-a|). \end{aligned} \tag{2.4}$$

Here, $s_1+\mathrm{Re}(\beta_k^+) > r_1 + S_a^+(g) = r$, $s_2 + \mathrm{Re}(\alpha_i^+) > r_2 + S_a^+(f) = r$ and $s_1+s_2 > r_1+s_2 \geqslant S_a^+(f)+s_2 > r$. Consequently, it is possible to find $s > r$ and $P$ bounded in $[0,\eta]$ such that $Q(|x-a|) = |x-a|^s P(|x-a|)$.

*Case 2:* $r < S_a^+(f)+S_a^+(g)$. According to definition 1 there exist $\eta > 0$, $v^+ > 0$, $w^+ > 0$, positive integers $J^+$, $K^+$ and complex functions $F^+$, $G^+$ bounded in $[0,\eta]$ such that $f(x)g(x) = |x-a|^r P(|x-a|)$, where the function

$$P(|x-a|) := |x-a|^{S_a^+(f)+S_a^+(g)-r}\left\{\sum_{j=0}^{J^+} f_j^+ \log^j|x-a| + |x-a|^{v^+}F^+(|x-a|)\right\}$$
$$\times\left\{\sum_{k=0}^{K^+} g_k^+\log^k|x-a| + |x-a|^{w^+}G^+(|x-a|)\right\}$$

is bounded in $[0,\eta]$. ■

This derivation also shows that, if $f \in D_+(a,C)$ and the expansion of $g$ near $x=a^+$ reduces to $g(x) = |x-a|^{s_2}G_{r_2}(|x-a|)$, $s_2 > r_2 := r - S_a^+(f)$, then $fg \in D_+^r(a,C)$. In these circumstances $g$ belongs to $D_+^{r_2}(a,C)\setminus D_+(a,C)$, i.e. the real $S_a^+(g)$ may not exist.

At this stage it is both convenient and straightforward to define the sets $D_-(a,C)$ and $D_-^r(a,C)$ by replacing in the associated definitions 1 and 2 concerning $D_+(a,C)$ and $D_+^r(a,C)$ each superscript + by superscript − and also the condition a.e. in $]a, a+\eta^+]$ by a.e. in $[a-\eta^-, a[$. Note that for $f \in D_+^r(a,C)\cap D_-^r(a,C)$, definition 2 may introduce different behaviours (see (2.2)) of the function $f$ near point $a$ and respectively for $x > a$ and $x < a$, i.e. different sets $(I^+, (J^+(i)), (\alpha_i^+), (f_{ij}^+), \eta^+)$ and $(I^-, (J^-(i)), (\alpha_i^-), (f_{ij}^-), \eta^-)$. For $k \in \mathbb{N}$ the definition of $\mathcal{C}_-^k(a, C)$ is obvious and $\mathcal{C}^k(a, C)$ is the set of complex functions admitting up to order $k$ continuous derivatives on an open set containing a compact neighbourhood of $a$. Of course, proposition 4 also applies to

$$f \in D_-(a, C) \cap D_-^{r-S_a^-(g)}(a, C), \qquad g \in D_-(a, C) \cap D_-^{r-S_a^-(f)}(a, C)$$

and then $fg \in D_-^r(a, C)$.

*Definition 5.* For a given real value $r$, $(h, g)$ belongs either to $\Pi_+^{r,1}(a, C)$ or $\Pi_+^{r,2}(a, C)$ if and only if the following properties are fulfilled:

1. $h$ is a real function such that there exist $c \in \mathbb{R}\gamma \in \mathbb{R}_+^*$, $\eta > 0$ and a function $h_a$ with $h(x) - c =_-^+ (x-a)^\gamma h_a(x)$, $h_a(x) > 0$ and

$$h_a(x) = \sum_{m=0}^{M} a_m \log^m(x-a) + o(x-a) \quad \text{in} \quad ]a, a+\eta],$$

where $M \in \mathbb{N}$ and $a_M \neq 0$.

2. For $r' := \gamma^{-1} r$, if $h(x) = c + (x-a)^\gamma h_a(x)$ then $g \in D_+^{r'}(c, C)$, else if $h(x) = c - (x-a)^\gamma h_a(x)$ then $g \in D_-^{r'}(c, C)$. More precisely, there exist real values $\eta > 0$ and $s' > r'$, a positive integer $I$, a complex function $G_{r'}$ bounded in $[0, \eta]$, a family of positive integers $(J(i))$, two complex families $(\alpha_i), (g_{ij})$ such that $\mathrm{Re}(\alpha_0) < \cdots < \mathrm{Re}(\alpha_I) \leqslant r'$ and in the adequate neighbourhood of $c$

$$g(X) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} g_{ij} |X-c|^{\alpha_i} \log^j |X-c| + |X-c|^{s'} G_{r'}(|X-c|). \qquad (2.5)$$

3. $(h, g)$ belongs to $\Pi_+^{r,1}(a, C)$ if the following assumptions 3.1 and 3.2 are true

3.1. The above decomposition (2.5) reduces to

$$g(X) = \sum_{i=0}^{I} g_i (X-c)^i + |X-c|^{s'} G_{r'}(|X-c|), \qquad (2.6)$$

with $I := \max(0, [\![r']\!])$ if $[\![d]\!]$ designates the integer part of real $d$.

3.2. For any $i \in \{0, \ldots, I\}$ such that $g_i \neq 0$, and $h_i(x) := [h(x) - c]^i$ then $h_i \in D_+^r(a, C)$ and for almost any $x \in ]a, a+\eta_i]$

$$[h(x) - c]^i = \sum_{p_i=0}^{P_i} \sum_{q_i=0}^{Q_i(p_i)} \Delta_{p_i q_i}^i (x-a)^{\tau_{p_i}^i} \log^{q_i}(x-a) + (x-a)^{s_i} \Delta_r^i(x-a), \qquad (2.7)$$

where $\eta_i > 0$, $i\gamma \leqslant \mathrm{Re}(\tau_{p_i}^i) \leqslant r$, $s_i > r$ and the complex function $\Delta_r^i$ is bounded in $[0, \eta_i]$.

4. $(h, g) \in \Pi_+^{r,2}(a, C)$ when decomposition (2.6) is not satisfied but (2.5) holds with $S_c(g) := \mathrm{Re}(\alpha_0), \forall i \in \{0, \ldots, I\}$ then $(g_{ij}) \neq (0)$ and moreover $h_a \in \mathcal{C}_+^p(a, C)$ with $p := [\![r - \gamma S_c(g)]\!] + 1$.

Some properties stated in definition 5 require a few remarks.

The case of a function $h$ constant in a neighbourhood on the right of $a$ is excluded by assumption 1.

According to the assumptions $s' > r' := \gamma^{-1} r$ and

$$h_a(x) = \sum_{m=0}^{M} a_m \log^m(x-a) + o(x-a),$$

it is possible to find $R > r$ and a function $H_r$ bounded in a neighbourhood on the right of zero and such that $H(x) := |h(x)-c|^{s'} G_{r'}(|h(x)-c|) = (x-a)^R H_r(x-a)$. First, observe that $\lim_{x\to a^+}(x-a)^\delta h_a(x) = 0$ for $\delta \in \mathbb{R}_+^*$. Moreover, if $t := (s'-r')/2 > 0$ and $x \in ]a, a+\eta]$ then

$$\begin{aligned} H(x) &= [(x-a)^\gamma h_a(x)]^{s'} G_{r'}[(x-a)^\gamma h_a(x)] \\ &= (x-a)^{\gamma(r'+t)}[(x-a)^{\gamma t} h_a(x)^{s'}] G_{r'}[(x-a)^\gamma h_a(x)]. \end{aligned}$$

The form near $x = a$ of $h_a$ shows that $(x-a)^{\gamma t} h_a(x)^{s'}$ is bounded near $a$ for $\gamma t > 0$ and $s' \in \mathbb{R}$. Consequently, the choice of $R := \gamma(r'+t) = r + \gamma t > r$ and of $H_r(x-a) := (x-a)^{\gamma t} h_a(x)^{s'} G_{r'}[(x-a)^\gamma h_a(x)]$ is approriate.

Expansion (2.6) includes the case $g(X) = |X-c|^{s'} G_{r'}(|X-c|)$ by choosing $(g_i) = (0)$, i.e. $S_c(g) > r'$. In such circumstances, the previous remark ensures that $g[h(x)] = (x-a)^R H_r(x-a)$ with $R > r$ and $H_r$ bounded in a neighbourhood on the right of zero, i.e. that $g \circ h \in D_+^r(a, C)$. This also explains,why, for $(h,g) \in \Pi_+^{r,2}(a, C)$, decomposition (2.5) is written with $S_c(g) = \mathrm{Re}(\alpha_0) < \cdots < \mathrm{Re}(\alpha_I) \leqslant r'$ under the assumption $(g_{ij}) \neq (0), \forall i \in \{0, \ldots, I\}$. In fact, $(h,g) \in \Pi_+^{r,2}(a,C)$ implies that $r' := \gamma^{-1} r \geqslant S_c(g)$.

According to definition 5 note that, for $m \in \{1,2\}$, $r' > r$ then $(h,g) \in \Pi_+^{r',m}(a,C)$ implies that $(h,g) \in \Pi_+^{r,m}(a,C)$.

For further applications, it is worth exhibiting a class of functions $h$ obeying relations (2.7) for certain values of the positive integer $i$.

**Proposition 6.** *Consider $f \in D_+(a,C)$ with $\gamma := S_a^+(f) > 0$ and $r \in \mathbb{R}$, $L \in \mathbb{N}$ such that $r \geqslant L\gamma$. If $t := r - \gamma(L-1)$ and $f \in D_+^t(a,C)$ then $f^i \in D_+^r(a,C)$ for integers $i \geqslant L$.*

*Proof.* Under these assumptions, it is possible to find $\eta > 0$ such that for $x \in ]a, a+\eta]$, $f(x) = (x-a)^\gamma g(x)$ with $g(x) = B(x) + C(x)$ where

$$B(x) := \sum_{m, 0 \leqslant \mathrm{Re}(\tau_n) \leqslant r - \gamma L} a_{nm}(x-a)^{\tau_n} \log^m(x-a),$$

$\mathrm{Re}(\tau_0) = 0$, $(a_{0m}) \neq (0)$ with $C(x) := (x-a)^v G(x-a)$ with $v > r - \gamma L$ and $G$ bounded in $[0, \eta]$. Use of the well-known multinomial formula

$$(b_1 + \ldots + b_j)^k = \sum_{\substack{p_1, \ldots, p_j \geqslant 0 \\ p_1 + \ldots + p_j = k}} \frac{k!}{p_1! \ldots p_j!} b_1^{p_1} \ldots b_j^{p_j}, \tag{2.8}$$

and of assumption $(a_{0m}) \neq (0)$ shows that $B^k(x) \in D_+^w(a,C) \cap D_+(a,C)$ for $k \in \mathbb{N}$, $w \in \mathbb{R}$ with $S_a^+[B^k(x)] = 0$. Now, Newton binomial formula ensures that $g^i(x) = \sum_{l=0}^{i} C_i^l [B(x)]^{i-l}[C(x)]^l$ with $C_i^l := i!/[l!(i-l)!]$. Since $S_a^+[B^k(x)] = 0$ and $v > r - \gamma L$, one clearly gets $g^i(x) \in D_+^{r-\gamma L}(a,C)$. Consequently $\gamma > 0$ and $i \geqslant L$ lead to $i\gamma + r - L\gamma \geqslant r$ and thereby $f^i(x) \in D_+^r(a,C)$. ■

For definition 5, if $\mathcal{S} := \{i, 0 \leqslant i \leqslant I \text{ and } g_i \neq 0\}$ two cases may arise. If $\mathcal{S}$ is the empty set, then the above remarks show that assumption

$$h(x) - c =^{+}_{-} (x-a)^{\gamma}\left\{\sum_{m=0}^{m} a_m \log^M (x-a) + o(x-a)\right\}$$

is adequate. Else, if $L := \min(\mathcal{S})$, then $S_c(g) = L \leqslant r' = \gamma^{-1} r$, i.e. $r \geqslant \gamma L$ with $\gamma = S_a^+(h-c)$ and proposition 6 applies to $f = h-c$. Thus, each function $h$ such that $h-c \in D_+^t(a,C) \cap D_+(a,C)$ with $t := r - S_a^+(h-c)[S_c(g)-1]$ satisfies decompositions (2.7). Among these functions are the real functions $h = c + (x-a)^{\gamma}\phi(x)$, where $\phi(a) \neq 0$, and there exists a positive integer $k$ such that $k > r - \gamma S_c(g) > 0$ and $\phi \in \mathcal{C}_+^k(a,C)$.

**Proposition 7.** *For a real value $r$, if $(h,g) \in \Pi_+^{r,1}(a,C)$ or $(h,g) \in \Pi_+^{r,2}(a,C)$ and $w(x) := g[h(x)]$ then $w \in D_+^r(a,C)$. More precisely, there exist $s > r$, $s' > r' := \gamma^{-1} r$, complex families $(\Delta_{p_i q_i}^{ij}), (\tau_{p_i}^i)$ and $\eta > 0$ such that for almost any $x \in ]a, a+\eta]$*

$$w(x) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)}\sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_{ij}(p_i)} g_{ij}\Delta_{p_i q_i}^{ij}(x-a)^{\tau_{p_i}^i}\log^{q_i}(x-a)$$
$$+\sum_{i=0}^{I}\sum_{j=0}^{J(i)} g_{ij}(x-a)^s R^{ij}(x) + |h(x)-c|^{s'} G_{r'}(|h(x)-c|), \quad (2.9)$$

*where each quantity keeps the meaning introduced in definition 5, the complex functions $R^{ij}$ are bounded in an adequate neighbourhood of $a$ and $\mathrm{Re}(\tau_{p_i}^i) \leqslant r$.*

*Proof.* First recall that according to the second remark concerning definition 5, the last term on the right-hand side of (2.9) may be rewritten as $H(x) := |h(x) - c|^{s'} G_{r'}(|h(x)-c|) = (x-a)^R H_r(x-a)$ with $R > r$ and $H_r$ bounded in a neighbourhood on the right of zero.

*Case 1.* The case of $(h,g) \in \Pi_+^{r,1}(a,C)$. If $(g_i) = (0)$, then $w(x) = H(x) \in D_+^r(a,C)$. If there exist $0 \leqslant i \leqslant I$ with $g_i \neq 0$, the combination of decomposition (2.6) and expansion (2.7) immediately yields

$$w(x) - H(x) = \sum_{i=0}^{I}\sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_i(p_i)} g_i \Delta_{p_i q_i}^i (x-a)^{\tau_{p_i}^i}\log^{q_i}(x-a)$$
$$+\sum_{i=0}^{I} g_i (x-a)^{s_i}\Delta_r^i(x-a). \quad (2.10)$$

Thus, equality (2.9) holds with $J(i) := 0, Q_{i0}(p_i) := Q_i(p_i)$, $\Delta_{p_i q_i}^{i0} := \Delta_{p_i q_i}^i$, $s := \min\{s_i$, if $i$ is such that $g_i \neq 0\}$, $R^{i0}(x) := (x-a)^{s_i - s}\Delta_r^i(x-a)$.

*Case 2.* The case of $(h,g) \in \Pi_+^{r,2}(a,C)$. Then $\gamma^{-1} r \geqslant S_c(g)$ and $S_c(g) = \mathrm{Re}(\alpha_0) < \cdots < \mathrm{Re}(\alpha_I) \leqslant \gamma^{-1} r$. First, a useful lemma is recalled.

**Lemma 8.** *For complex functions $f$ and $g$ admitting derivatives up to order integer $n$ respectively in $V$ an open neighbourhood of $x$ and $W := f(V)$ a neighbourhood of $f(x)$, then $g \circ f$ admits derivatives up to order $n$ in $V$ and*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[g \circ f](x) = \sum_{Faa,n} \frac{n!}{m_1! \ldots m_q!} g^{(p)}[f(x)]\left[\frac{f^{(1)}(x)}{1!}\right]^{m_1} \cdots \left[\frac{f^{(q)}(x)}{q!}\right]^{m_q}, \quad (2.11)$$

*where the sum $\sum_{Faa,n}$ means a sum over all families $(m_i)_{i\in\{1,\ldots,q\}}$ of positive integers such that $m_1+2m_2+\ldots+qm_q=n$ and $p:=m_1+m_2+\ldots+m_q$.*

Such a result (2.11) is known as the Faa de Bruno formula. Note that if $f\in\mathcal{C}_+^n(a,C)$ and $g\in\mathcal{C}^n(f(a),C)$ then this formula (2.11) also applies (by continuity of derivatives of the function $f$ at point $a$ on the right) at point $a$ and ensures that $g\circ f\in\mathcal{C}_+^n(a,C)$.

According to decomposition (2.5), in a neighbourhood on the right of point $a$ then

$$w(x):=g[h(x)]=\sum_{i=0}^{I}\sum_{j=0}^{J(i)}g_{ij}\sum_{l=0}^{j}C_j^l\gamma^{j-l}W_{ij}^l(x)+H(x)$$

with functions $W_{ij}^l$ defined by $W_{ij}^l:=(x-a)^{\gamma\alpha_i}\log^{j-l}(x-a)F_l^i[h_a(x)]$ if $F_l^i(X):=X^{\alpha_i}\log^l(X)$ for $X>0$. For each $i\in\{0,\ldots,I\}$, if $k_i:=[\![r-\gamma\operatorname{Re}(\alpha_i)]\!]+1$, then $k_i\in\mathbb{N}^*$ and $k_i\leqslant p:=[\![r-\gamma\operatorname{Re}(\alpha_0)]\!]+1$. Thus, the assumption $h_a\in\mathcal{C}_+^p(a,C)$ (see definition 5, property 4) ensures that $h_a\in\mathcal{C}_+^{k_i}(a,C)$ and previous lemma 8 (since $F_l^i$ is smooth at any $X>0$ and $h_a(x)>0$) shows that $F_l^i[h_a(x)]\in\mathcal{C}_+^{k_i}(a,C)$. Use of the Taylor expansion (2.3) for this latter function leads to the form

$$W_{ij}^l(x)=\sum_{p_i=0}^{k_i-1}\frac{D_l^i(p_i)}{p_i!}(x-a)^{\gamma\alpha_i+p_i}\log^{j-l}(x-a)+(x-a)^{\gamma\alpha_i+k_i}\log^{j-l}(x-a)P_{k_i}(x),\tag{2.12}$$

where

$$P_{k_i}(x):=[(k_i-1)!]^{-1}\int_0^1(1-u)^{k_i-1}\frac{\mathrm{d}^{k_i}F_l^i}{\mathrm{d}y^{k_i}}[a+u(x-a)]\,\mathrm{d}u$$

is bounded in a neighbourhood on the right of zero and the complex

$$D_l^i(p_i):=\frac{\mathrm{d}^{p_i}[F_l^i\circ h_a]}{\mathrm{d}y^{p_i}(a)}$$

are obtained by applying the Faa de Bruno formula (2.11). Consequently, if $q:=j-l$,

$$\begin{aligned}w(x)-H(x)=&\sum_{i=0}^{I}\sum_{j=0}^{J(i)}\sum_{p_i=0}^{k_i-1}\sum_{q=0}^{j}\frac{g_{ij}C_j^q\gamma^qD_{j-q}^i(p_i)}{p_i!}(x-a)^{\gamma\alpha_i+p_i}\log^q(x-a)\\&+\sum_{i=0}^{I}\sum_{j=0}^{J(i)}g_{ij}(x-a)^{\gamma\alpha_i+k_i}\left\{\sum_{q=0}^{j}C_j^q\gamma^q\log^q(x-a)P_{k_i}(x)\right\}.\end{aligned}\tag{2.13}$$

This expansion agrees with (2.9) if one chooses $P_i:=k_i-1=[\![r-\gamma\operatorname{Re}(\alpha_i)]\!]$, $q_i:=q$, $Q_{ij}(p_i):=j$, $\tau_{p_i}^i:=\gamma\alpha_i+p_i$, $\Delta_{p_iq_i}^{ij}:=C_j^{q_i}\gamma^{q_i}D_{j-q_i}^i(p_i)/p_i!$, $s:=\min\{\gamma\operatorname{Re}(\alpha_i)+k_i\}>r$ and

$$R^{ij}(x):=\sum_{q=0}^{j}C_j^q\gamma^q(x-a)^{\gamma\alpha_i+k_i-s}\log^q(x-a)P_{k_i}(x-a).$$

Observe that the leading term arising on the right-hand side of (2.13) (obtained by setting $i=p_i=0$) is

$$T_1(x)=\sum_{j=0}^{J(0)}g_{0j}\sum_{q=0}^{j}C_j^q\gamma^qD_{j-q}^i(0)(x-a)^{\gamma\alpha_0}\log^q(x-a).$$

The definition of $D^i_{j-q}(0)$ leads to

$$T_1(x) = h_a(x)^{\alpha_0}(x-a)^{\gamma\alpha_0}\sum_{j=0}^{J(0)} g_{0j}\left\{\sum_{q=0}^{j} C_j^q \gamma^q \log^{j-q}[h_a(a)]\log^q(x-a)\right\},$$

i.e. to

$$T_1(x) = h_a(x)^{\alpha_0}(x-a)^{\gamma\alpha_0}\left[\sum_{j=0}^{J(0)} g_{0j} d^j(x-a)\right]$$

if $d(x-a) := \log[h_a(a)(x-a)^\gamma]$. Because $(g_{0j}) \neq (0)$, it is easy to prove that $T_1(x)$ is not the zero function in a neighbourhood on the right of $a$. Consequently, if $(h,g) \in \Pi_+^{r,2}(a,C)$ then $S_a^+(g \circ h) = \gamma\,\mathrm{Re}(\alpha_0)$. ∎

The reader may easily introduce the definitions of sets $\Pi_-^{r,1}(a,C)$, $\Pi_-^{r,2}(a,C)$ and show that $(h,g) \in \Pi_-^{r,m}(a,C)$ for $m \in \{1,2\}$ implies $w(x) := g[h(x)] \in D_-^r(a,C)$.

*Definition 9.* For $r > 0$, the complex function $h$ is of the second kind on the set $]0,r[$ if and only if there exist a complex function $H$, a family of positive integers $(M(n))$, and two complex families $(\beta_n)$ and $(h_{nm})$ such that

$$\forall \epsilon \in ]0,r[, \qquad h(\epsilon) = \sum_{n=0}^{N}\sum_{m=K(n)}^{M(n)} h_{nm}\epsilon^{\beta_n}\log^m(\epsilon) + H(\epsilon), \tag{2.14}$$

$$\left.\begin{array}{l}\mathrm{Re}(\beta_N) < \mathrm{Re}(\beta_{N-1}) < \ldots < \mathrm{Re}(\beta_1) < \mathrm{Re}(\beta_0) := 0,\\ \lim_{\epsilon\to 0} H(\epsilon) \in C \quad \text{and} \quad h_{00} := 0 \quad \text{for} \quad \beta_0 = 0.\end{array}\right\} \tag{2.15}$$

The above notation $\mathrm{Re}(\beta_0) := 0$ occurring in equation (2.14) means that complex number $\beta_0$ is such that $\mathrm{Re}(\beta_0) = 0$. Naturally the associated coefficients $h_{0m}$ may be zero.

According to Hadamard's concept (see Hadamard 1932; Gel'fand & Shilov 1964; Schwartz 1966) the finite part in the Hadamard sense of the quantity $h(\epsilon)$, noted $\boldsymbol{fp}[h(\epsilon)]$, is the complex $\lim_{\epsilon\to 0} H(\epsilon)$.

*Definition 10.* A complex function $f$ belongs to the set $\mathcal{P}(]0,+\infty[,C)$ if and only if $f \in L^1_{\mathrm{loc}}(]0,+\infty[,C)$ and there exist positive reals $\eta_f$ and $A_f$, two functions $F^0 \in L^1([0,\eta_f],C)$ and $F^\infty \in L^1([A_f,+\infty[,C)$, two families of positive integers $(J(i)),(K(l))$ and complex families $(\alpha_i),(f^0_{ij}),(\gamma_l)$ and $(f^\infty_{lk})$ such that

$$f(x) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^0_{ij}x^{\alpha_i}\log^j x + F^0(x), \quad \text{a.e. in} \quad ]0,\eta_f],$$

$$\mathrm{Re}(\alpha_I) < \mathrm{Re}(\alpha_{I-1}) < \ldots < \mathrm{Re}(\alpha_1) < \mathrm{Re}(\alpha_0) := -1;$$

$$f(x) = \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f^\infty_{lk}x^{-\gamma_l}\log^k x + F^\infty(x), \quad \text{a.e. in} \quad [A_f,+\infty[,$$

$$\mathrm{Re}(\gamma_L) < \mathrm{Re}(\gamma_{L-1}) < \ldots < \mathrm{Re}(\gamma_1) < \mathrm{Re}(\gamma_0) := 1.$$

Moreover, to each element $f$ of $\mathcal{P}(]0,+\infty[,C)$ is associated its integral in the finite part sense of Hadamard

$$\begin{aligned}\boldsymbol{fp}\int_0^{\infty} f(x)\,\mathrm{d}x := \boldsymbol{fp}\left[\int_{\epsilon}^{1/\epsilon} f(x)\,\mathrm{d}x\right] &= \int_{\delta}^{B} f(x)\,\mathrm{d}x + \int_0^{\delta} F^0(x)\,\mathrm{d}x + \int_B^{\infty} F^{\infty}(x)\,\mathrm{d}x \\ &\quad + \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f_{ij}^0 P_{\alpha_i}^j(\delta) - \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^{\infty} P_{-\gamma_l}^k(B), \end{aligned} \tag{2.16}$$

where $(\delta, B)$ is any pair such that $0 < \delta \leqslant \eta_f$, $A_f \leqslant B < +\infty$ and also

$$P_{-1}^j(t) := \frac{\log^{j+1}(t)}{j+1}, \quad \text{else} \quad P_{\alpha}^j(t) := t^{\alpha+1}\sum_{k=0}^{j} \frac{(-1)^{j-k} j!}{k!(\alpha+1)^{1+j-k}} \log^k(t). \tag{2.17}$$

For $0 < a < c$, $j \in \mathbb{N}$ and $\alpha \in C$, use of the equality

$$\int_a^c x^{\alpha} \log^j x\,\mathrm{d}x = P_{\alpha}^j(c) - P_{\alpha}^j(a),$$

where the functions $P_{\alpha}^j$ are given by above definition (2.17), indeed ensures that the function $h_f$ defined by

$$h_f(\epsilon) := \int_{\epsilon}^{\epsilon^{-1}} f(x)\,\mathrm{d}x$$

is of the second kind as soon as $f \in \mathcal{P}(]0, +\infty[, C)$. Relation (2.16) offers no difficulties and the proof is left to the reader. By the way, if $f \in L^1_{\text{loc}}(\mathbb{R}, C)$ then

$$\boldsymbol{fp}\int_0^{\infty} f(x)\,\mathrm{d}x$$

reduces to the Lebesgue integration

$$\int_0^{\infty} f(x)\,\mathrm{d}x.$$

For $0 < b < +\infty$, $\mathcal{P}(]0, b[, C)$ is the set of complex functions $f$ such that if $\mathcal{F}(x) := H_e(b - x)f(x)$ where $H_e$ designates the Heaviside function then $\mathcal{F} \in \mathcal{P}(]0, +\infty[, C)$. For instance, $(D_+^r(0, C) \cap L^1_{\text{loc}}(]0, b[, C)) \subset \mathcal{P}(]0, b[, C)$ for $0 < b < +\infty$ and $r \geqslant -1$.

The two equalities,

$$\lim_{x \to 0} f(x) = \sum_{j, \mathrm{Re}(\alpha_i) \leqslant r} a_{ij} x^{\alpha_i} \log^j x, \quad \lim_{x \to +\infty} f(x) = \sum_{m, \mathrm{Re}(\gamma_n) \leqslant r} a_{nm} [x^{-1}]^{\gamma_n} \log^m x, \tag{2.18}$$

mean that there exist a real $s > r$, a complex function $F_r$ bounded in a neighbourhood respectively on the right of zero and on the left of infinity in which

$$f(x) = \sum_{j, \mathrm{Re}(\alpha_i) \leqslant r} a_{ij} x^{\alpha_i} \log^j x + x^s F_r(x)$$

or

$$f(x) = \sum_{m, \mathrm{Re}(\gamma_n) \leqslant r} a_{nm} [x^{-1}]^{\gamma_n} \log^m x + x^{-s} F_r(x).$$

Finally for real values $r_1$ and $r_2, \mathcal{E}_{r_1}^{r_2}(]0,b[,C)$ is the set of complex functions such that

$$\lim_{x\to 0} f(x) = \sum_{j,\mathrm{Re}(\alpha_i)\leqslant r_1} a_{ij}x^{\alpha_i}\log^j x$$

and if $0<b<+\infty, f\in L^1_{\mathrm{loc}}(]0,b],C)$; else $f\in L^1_{\mathrm{loc}}(]0,b[,C)$ and also

$$\lim_{x\to+\infty} f(x) = \sum_{m,\mathrm{Re}(\gamma_n)\leqslant r_2} a_{nm}[x^{-1}]^{\gamma_n}\log^m x.$$

Note that $\mathcal{E}_{r_1}^{r_2}(]0,b[,C)\subset D_+^{r_1}(0,C)$.

For detailed explanations regarding the use of the concept of Hadamard's finite part in obtaining asymptotic expansion of a class of integrals the reader is referred to Sellier (1994).

## 3. A general theorem

This section presents, for $0<b<+\infty$, the asymptotic expansion of

$$I(\lambda) := \boldsymbol{fp}\int_0^b f(x)K[h(x),\lambda x]\,\mathrm{d}x, \tag{3.1}$$

with respect to the large real parameter $\lambda$ when $(f,h,K)$ belongs to a specific set defined below.

*Definition 11.* For real values $0<b<+\infty, r_1$ and $r_2$ the triplet $(f,h,K)$ belongs to the set $\mathcal{L}_{r_1}^{r_2}(]0,b[,C)$ if and only if the following properties are satisfied by the real function $h$ and complex pseudo-functions $f$ and $K(x,u)$ for $\lambda$ large enough

1. $f\in D_+(0,C)$ with $S_0(f):=S_0^+(f)$ and if $g_\lambda(x):=f(x)K[h(x),\lambda x]$ then $g_\lambda\in\mathcal{P}(]0,b[,C)$ for $\lambda$ large enough.

2. If $E:=h(]0,b[)$ then there exist a positive integer $N$, a complex family $(\gamma_n)$ with $\mathrm{Re}(\gamma_0)<\cdots<\mathrm{Re}(\gamma_N):=r_2$, families of positive integers $(M(n))$ and of complex pseudo-functions $(K_{nm}(X))$, a real $s_2>r_2$, a complex function $V_{r_2}(X,u)$, a real $B\geqslant 0$ and a real $\eta>0$ such that for any $(X,u)\in E\times[\eta,+\infty[$

(*a*)

$$K(X,u) = \sum_{n=0}^{N}\sum_{m=0}^{M(n)} K_{nm}(X)u^{-\gamma_n}\log^m u + u^{-s_2}V_{r_2}(X,u), \tag{3.2}$$

(*b*)

$$\left|\int_\eta^b f(x)x^{-s_2}V_{r_2}[h(x),\lambda x]\,\mathrm{d}x\right| \leqslant B < +\infty, \tag{3.3}$$

(*c*) $\forall n\in\{0,\ldots,N\},\forall m\in\{0,\ldots,M(n)\}$ if $g_{nm}(x):=f(x)K_{nm}[h(x)]$ then $g_{nm}\in L^1_{\mathrm{loc}}(]0,b],C)$ and for $t_n:=\max[r_1-S_0(f),\mathrm{Re}(\gamma_n)-1-S_0(f)]$ then $(h,K_{nm})\in(\Pi_+^{t_n,1}(0,C)\cup\Pi_+^{t_n,2}(0,C))$ with relation $h(x)-c =^{+}_{-} x^\gamma h_0(x)$ (see definition 5). Moreover, there exist a positive integer $I$, a complex family $(\alpha_i)$ with $\mathrm{Re}(\alpha_0)<\cdots<\mathrm{Re}(\alpha_I)=t_1:=\gamma^{-1}[r_1-S_0(f)]$, a family of positive integers $(J(i))$ and a complex family $(K_{nm}^{ij})$ such that $\forall n\in\{0,\ldots,N\},\forall m\in\{0,\ldots,M(n)\}$ decomposition (2.5) on the appropriate neighbourhood of $c$ for $K_{nm}$ takes the following form

$$K_{nm}(X) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K_{nm}^{ij} |X - c|^{\alpha_i} \log^j |X - c| + |X - c|^{t_1'} V_{nm}(|X - c|), \tag{3.4}$$

with $t_1' > t_1$ and the complex functions $V_{nm}$ bounded in a neighbourhood on the right of zero.

3. There exist real values $A \geqslant \eta > 0$, $B \geqslant 0$, a family of complex pseudo-functions $(a^{ij}(u))$, a complex function $H_{t_1}(x, u)$ such that

(*a*) for $u > 0$

$$K(x, u) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} a^{ij}(u) |X - c|^{\alpha_i} \log^j |X - c| + |X - c|^{t_1'} H_{t_1}(X - c, u) \tag{3.5}$$

in an adequate neighbourhood of $c$, i.e. on the right or on the left if respectively $h(x) - c = x^\gamma h_0(x)$ or $h(x) - c = -x^\gamma h_0(x)$ near zero;

(*b*)

$$\int_0^{A/\lambda} f(x) |h(x) - c|^{t_1'} H_{t_1}[h(x) - c, \lambda x] \, \mathrm{d}x = o[\lambda^{-(r_1+1)}]; \tag{3.6}$$

(*c*) if $(a^{ij}) = (0)$ then we set $S = \gamma^{-1}[r_1 - S_0(f)]$. When $(a^{ij}) \neq (0)$ it is understood that there exists $j \in \{0, \ldots, I(0)\}$ such that $a^{0j} \neq 0$ and this time $S = \mathrm{Re}(\alpha_0) \leqslant \gamma^{-1}[r_1 - S_0(f)]$ with the new assumption: $f \in D_+^{r_1 - \gamma S}(0, C)$;

(*d*) $\forall i \in \{0, \ldots, I\}$, $\forall j \in \{0, \ldots, J(i)\}$, $a^{ij} \in D_+(0, C)$ with $S_0(a^{ij}) = S_0^+(a^{ij}) \geqslant -r_1 - 1$ and also $a^{ij} \in \mathcal{E}_R^{R'}(]0, +\infty[, C)$ where $R := \max[-1 - r_1, -1 - \gamma S - S_0(f)]$ and $R' := \max(r_1 + 1, r_2)$. It is understood that

$$\lim_{u \to 0} a^{ij}(u) = \sum_{q, \mathrm{Re}(\beta_p) \leqslant R} A_{pq}^{ij} u^{\beta_p} \log^q u$$

and that there exists a complex function $O_{ij}$ bounded in the neighbourhood of infinity $[A, +\infty[$ in which

$$a^{ij}(u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}^{ij} u^{-\gamma_n} \log^m u + u^{-s_2} O_{ij}(u). \tag{3.7}$$

4. The complex function $w_{t_1', s_2}$ defined by

$$u^{-s_2} w_{t_1', s_2}[X - c, u] := H_{t_1}(X - c, u) - \sum_{n=0}^{N} \sum_{m=0}^{M(n)} V_{nm}(|X - c|) u^{-\gamma_n} \log^m u$$

is bounded for $(X, u) \in h([0, \eta]) \times [A, +\infty[$.

Assumption 2(*d*) means that there exist a real $t > r_1 - \gamma S$ and a complex function $F$ bounded in a neighbourhood on the right of zero in which

$$f(x) = \sum_{l=0}^{L} \sum_{k=0}^{K(l)} f_{lk}^0 x^{\delta_l} \log^k x + x^t F(x), \tag{3.8}$$

with $\mathrm{Re}(\delta_0) < \cdots < \mathrm{Re}(\delta_L) \leqslant r_1 - \gamma S$. Observe that $r_1 - \gamma S \geqslant S_0(f)$. For given

$(i,j)$ observe that according to expansion (3.9) if $a^{ij}=0$ then $(K^{ij}_{nm})=(0)$ and $(K^{ij}_{nm})\neq(0)$ implies that $a^{ij}\neq 0$.

According to the above definition, the next basic theorem holds.

**Theorem 12.** *For real values $r$ and $0<b<+\infty$, if there exist $r_1\geqslant r-1$ and $r_2\geqslant r$ such that $(f,h,K)\in\mathcal{L}^{r_2}_{r_1}(]0,b[,C)$ then the integral $I(\lambda)$ admits the following expansion with respect to the large and real parameter $\lambda$*

$$
\begin{aligned}
I(\lambda) &= \boldsymbol{fp}\int_0^b f(x)K[h(x),\lambda x]\,\mathrm{d}x \\
&= \sum_{m,\mathrm{Re}(\gamma_n)\leqslant r}\sum_{e=0}^{m} C^e_m\left[\boldsymbol{fp}\int_0^b f(x)K_{nm}[h(x)]x^{-\gamma_n}\log^{m-e}(x)\,\mathrm{d}x\right]\lambda^{-\gamma_n}\log^e\lambda \\
&\quad+\sum_{\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant r-1}^{l,k,i,j,p_i,q_i} f^0_{lk}\Delta^{ij}_{p_iq_i}\sum_{v=0}^{q_i+k}C^v_{q_i+k}(-1)^v \\
&\quad\times\left\{\boldsymbol{fp}\int_0^\infty a^{ij}(u)u^{\tau^i_{p_i}+\delta_l}\log^{q_i+k-v}(u)\,\mathrm{d}u\right. \\
&\qquad-\sum_{\{p;\beta_p=-\tau^i_{p_i}-\delta_l-1\}}\sum_{q=0}^{Q(p)}\frac{A^{ij}_{pq}}{1+q_i+k+q-v}\log^{1+q_i+k+q-v}\lambda \\
&\qquad\left.+\sum_{\{n;\gamma_n=\tau^i_{p_i}+\delta_l+1\}}\sum_{m=0}^{M(n)}\frac{K^{ij}_{nm}}{1+q_i+k+m-v}\log^{1+q_i+k+m-v}\lambda\right\} \\
&\quad\times\lambda^{-(\tau^i_{p_i}+\delta_l+1)}\log^v\lambda+o(\lambda^{-r}),
\end{aligned}
\tag{3.9}
$$

*where the meaning of each quantity is given by proposition 7 or definition 11, the notation*

$$\sum_{\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant r-1}^{l,k,i,j,p_i,q_i}$$

*is explained below and $\sum_{\{p;p=t\}}A(p)$ means $A(t)$ if there exists a positive integer $p$ such that $p=t$, else it equals zero.*

For real values $t_1,t_2,\epsilon>0$ and two functions $A(l,k,i,j,p_i,q_i)$ and $B(l,q_i)$ the complex

$$S_{t_2}(\epsilon)=\sum_{\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant t_2}^{l,k,i,j,p_i,q_i}A(l,k,i,j,p_i,q_i)\epsilon^{\tau^i_{p_i}+\delta_l}\log^{B(l,q_i)}\epsilon$$

is the sum of terms of

$$S=\sum_{l=0}^{L}\sum_{k=0}^{K(l)}\sum_{i=0}^{I}\sum_{j=0}^{J(i)}\sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_{ij}(p_i)}A(l,k,i,j,p_i,q_i)\epsilon^{\tau^i_{p_i}+\delta_l}\log^{B(l,q_i)}\epsilon$$

which satisfy $\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant t_2$. Naturally, if $\mathrm{Re}(\tau^i_{p_i}+\delta_l)>t_2$ for any $(i,p_i,l)$ then

$S_{t_2}(\epsilon)=0$. Morover for $t_1<t_2$,

$$\sum_{t_1<\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant t_2}^{l,k,i,j,p_i,q_i} A(l,k,i,j,p_i,q_i)\epsilon^{\tau^i_{p_i}+\delta_l}\log^{B(l,q_i)}\epsilon := S_{t_2}(\epsilon)-S_{t_1}(\epsilon).$$

Note that expansion (3.9) involves coefficients of the sequence $\lambda^\delta \log^k \lambda$ which are integrals in the finite part sense of Hadamard even if quantity $I(\lambda)$ reduces to a Lebesgue integration.

*Proof.* Assume that $(f,h,K)\in\mathcal{L}^{r_2}_{r_1}(]0,b[,C)$ and also that each notation introduced in definition 11 keeps its meaning. A new complex pseudo-function $\overline{K}(x,u)$ is defined as $\overline{K}(x,u):=f(x)K[h(x),u]$ for $x\in]0,b[$ and $u\in\mathbb{R}^*_+$. Under the assumptions bearing on the real function $h$ and on the complex pseudo-functions $f$ and $K(x,u)$ then $g_\lambda(x)\overline{K}(x,\lambda x)\in\mathcal{P}(]0,b[,C)$ for $\lambda$ large enough so that $I(\lambda)$ exists and the following properties are fulfilled.

1. Since if $x\in]0,b[$ then $X:=h(x)\in E$, expansion (3.2) yields for $(x,u)\in ]0,b[\times]\eta,+\infty[$

(*a*)

$$\overline{K}(x,u)=\sum_{n=0}^{N}\sum_{m=0}^{M(n)}\overline{K}_{nm}(x)u^{-\gamma_n}\log^m u+u^{-s_2}G_{r_2}(x,u), \tag{3.10}$$

with $\overline{K}_{nm}(x):=f(x)K_{nm}[h(x)]=g_{nm}(x)\in L^1_{\mathrm{loc}}(]0,b],C)$ and also $G_{r_2}(x,u):=f(x)V_{r_2}[h(x),u]$. Hence, inequality (3.3) may be rewritten as

(*b*)

$$\left|\int_\eta^b x^{-s_2}G_{r_2}(x,\lambda x)\,\mathrm{d}x\right|\leqslant B<+\infty. \tag{3.11}$$

(*c*) $\forall n\in\{0,\dots,N\}$, $\forall m\in\{0,\dots,M(n)\}$ the fact that $(h,K_{nm})$ belongs to $\Pi^{r_1-S_0(f),1}_+(0,C)\cup\Pi^{r_1-S_0(f),2}_+(0,C)$ authorizes us to apply proposition 7. Consequently, $K_{nm}[h(x)]\in D^{r_1-S_0(f)}_+(0,C)$ with the following expansion on the right of zero:

$$K_{nm}[h(x)]=\sum_{i=0}^{I}\sum_{j=0}^{J(i)}K^{ij}_{nm}\left\{\sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_{ij}(p_i)}\Delta^{ij}_{p_iq_i}x^{\tau^i_{p_i}}\log^{q_i}x+x^sR^{ij}(x)\right\}$$
$$+[x^\gamma h_0(x)]^{t'_1}V_{nm}[x^\gamma h_0(x)], \tag{3.12}$$

where the positive integers $P_i,Q_{ij}(p_i)$, the complex families $(\Delta^{ij}_{p_iq_i}),(\tau^i_{p_i})$ such that $\mathrm{Re}(\tau^i_{p_i})\leqslant r_1-S_0(f)$, the complex functions $R^{ij}$ bounded on the right of zero and the real $s>r_1-S_0(f)$ have previously been defined. Recall that $t'_1>t_1:=\gamma^{-1}[r_1-S_0(f)]$. After observing that both real $s$ and functions $R^{ij}$ do not depend on $(n,m)$ two cases are considered.

If $(K^{ij}_{nm})=(0)$ (i.e. $\forall(i,j,n,m)K^{ij}_{nm}=0$), then on the right of zero

$$\overline{K}_{nm}(x):=f(x)K_{nm}[h(x)]=x^{\gamma t'_1}f(x)h_0(x)^{t'_1}V_{nm}[x^\gamma h_0(x)]=x^{r_1+t}P(x),$$

where $t:=[\gamma t'_1-r_1+S_0(f)]/2>0$ and the function

$$P(t):=[x^{t-S_0(f)}f(x)]h_0(x)^{t'_1}V_{nm}[x^\gamma h_0(x)]$$

turns out to be bounded on the right of zero.

If $(K^{ij}_{nm}) \neq (0)$ then $(a^{ij}) \neq (0), S = \mathrm{Re}(\alpha_0) \leqslant \gamma^{-1}[r_1 - S_0(f)]$ and also (see proposition 7) $\mathrm{Re}(\tau^0_0) = \gamma S$. Moreover, the combination of decomposition (3.8) for the function $f$ and equality (3.12) yields the following expansion near zero on the right

$$\overline{K}_{nm}(x) = \sum_{\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant r_1}^{l,k,i,j,p_i,q_i} K^{ij}_{nm} f^0_{lk} \Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}+\delta_l} \log^{q_i+k} x + x^{s_1} L_{nm}(x), \tag{3.13}$$

where the last term $x^{s_1}L_{nm}(x)$ may be rewritten as $x^{s_1}L_{nm}(x) = T_{nm}(x) + T'_{nm}(x) + T''_{nm}(x) + T'''_{nm}(x)$ with

$$T_{nm}(x) := \sum_{r_1<\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant 2r_1 - S_0(f)-\gamma S}^{l,k,i,j,p_i,q_i} K^{ij}_{nm} f^0_{lk} \Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}+\delta_l} \log^{q_i+k} x \tag{3.14}$$

and also

$$\left.\begin{aligned} T'_{nm}(x) &:= x^t F(x) \sum_{i=0}^{I} \sum_{j=0}^{J(i)} \sum_{p_i=0}^{P_i} \sum_{q_i=0}^{Q_i(p_i)} K^{ij}_{nm} \Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}} \log^{q_i} x, \\ T''_{nm}(x) &:= \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K^{ij}_{nm} x^s f(x) R^{ij}(x); \\ T'''_{nm}(x) &:= x^{\gamma t'_1} f(x) h_0(x)^{t'_1} V_{nm}[x^\gamma h_0(x)], \end{aligned}\right\} \tag{3.15}$$

where $t > r_1 - \gamma S$ and (see proposition 7 and definition of $S$) $\mathrm{Re}(\tau^i_{p_i}) \geqslant \gamma S$.

For $x$ given, observe that $T_{nm}(x), T'_{nm}(x)$ and $T''_{nm}(x)$ depend linearly on way of family $(K^{ij}_{nm})$. Therefore it is possible to define three operators $T[x, (F^{ij})]$, $T'[x, (F^{ij})]$ and $T''[x, (F^{ij})]$ which are linear with respect to the complex family $(F^{ij})$, $i \in \{0, \ldots, I\}$, $j \in \{0, \ldots, J(i)\}$ and satisfy

$$T[x, (K^{ij}_{nm})] = T_{nm}(x), \quad T'[x, (K^{ij}_{nm})] = T'_{nm}(x), \quad T''[x, (K^{ij}_{nm})] = T''_{nm}(x). \tag{3.16}$$

Note that $w > r_1 - S_0(f)$ ensures $x^w f(x) = x^{w'} V(x)$ with $w' > r_1$ and $V$ bounded on the right of zero. Moreover, $t + \mathrm{Re}(\tau^i_{p_i}) > r_1 - \gamma S + \gamma S = r_1$ so that $T'_{nm}(x) = x^{t'} V'(x)$ with $t' > r_1$ and $V'$ bounded near zero. Thus, it is legitimate to set $T_{nm}(x) + T'_{nm}(x) + T''_{nm}(x) + T'''_{nm}(x) = x^{s_1} L_{nm}(x)$ with $s_1 > r_1$ and $L_{nm}$ bounded on the right of zero.

Hence, in any case, $\overline{K}_{nm} \in D^{r_1}_{+}(0, C)$. The same arguments also show that

$$\overline{K}_{nm} \in D^{t_n + S_0(f)}_{+}(0, C) \quad \text{for} \quad t_n := \max[r_1 - S_0(f), \mathrm{Re}(\gamma_n) - 1 - S_0(f)].$$

According to the assumption

$$g_{nm} = \overline{K}_{nm} \in L^1_{\mathrm{loc}}(]0, b], C) \quad \text{with} \quad 0 < b < +\infty,$$

it follows that

$$\overline{K}_{nm} \in \mathcal{E}^{1-\mathrm{Re}(\gamma_n)}_{v_n}(]0, b[, C) \quad \text{if} \quad v_n := \max[r_1, \mathrm{Re}(\gamma_n) - 1].$$

This justifies the existence of each integral

$$\boldsymbol{fp} \int_0^b f(x) K_{nm}[h(x)] x^{-\gamma_n} \log^{m-e}(x)\, \mathrm{d}x$$

on the right-hand side of result (3.9).

2. Application of decomposition (3.5) makes it possible to write for $u > 0$ and $0 < x < W \leqslant b$

(*a*)

$$\overline{K}(x,u) = \sum_{\operatorname{Re}(\tau^i_{p_i}+\delta_l)\leqslant r_1}^{l,k,i,j,p_i,q_i} a^{ij}(u) f^0_{lk} \Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}+\delta_l} \log^{q_i+k} x + x^{s_1}\mathcal{H}_{t_1}(x,u), \tag{3.17}$$

with $\overline{K}(x,u) - x^{s_1}\mathcal{H}_{t_1}(x,u) = 0$ if $(a^{ij}) = (0)$, else $f \in D^{r_1-\gamma S}_{+}(0,C)$ with this time $x^{s_1}\mathcal{H}_{t_1}(x,u) = S(x,u) + S'(x,u) + S''(x,u) + S'''(x,u)$ where

$$\begin{aligned} S(x,u) &:= \sum_{r_1<\operatorname{Re}(\tau^i_{p_i}+\delta_l)\leqslant 2r_1 - S_0(f)-\gamma S}^{l,k,i,j,p_i,q_i} a^{ij}(u) f^0_{lk} \Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}+\delta_l} \log^{q_i+l} x \\ &= T[x, (h^{ij}(u))] \end{aligned} \tag{3.18}$$

and also

$$S'(x,u) = x^t F(x) \sum_{i=0}^{I}\sum_{j=0}^{J(i)}\sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_i(p_i)} a^{ij}(u)\Delta^{ij}_{p_i q_i} x^{\tau^i_{p_i}} \log^{q_i} x = T'[x, (a^{ij}(u))], \tag{3.19}$$

$$S''(x,u) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} a^{ij}(u) x^s f(x) R^{ij}(x) = T''[x, (a^{ij}(u))], \tag{3.20}$$

$$S'''(x,u) = f(x)|h(x)-c|^{t'_1} H_{t_1}[h(x)-c, u]. \tag{3.21}$$

(*b*)

$$\int_0^{A/\lambda} x^{s_1}\mathcal{H}_{t_1}(x,\lambda x)\,\mathrm{d}x = o[\lambda^{-(r_1+1)}]. \tag{3.22}$$

*Proof.* As a result of assumption (3.6) and of the above definition of $S'''(x,u)$

$$\int_0^{A/\lambda} S'''(x,\lambda x)\,\mathrm{d}x = \int_0^{A/\lambda} f(x)|h(x)-c|^{t'_1} H_{t_1}[h(x)-c,\lambda x]\,\mathrm{d}x = o[\lambda^{-(r_1+1)}]. \tag{3.23}$$

Moreover, (3.20) leads to

$$\int_0^{A/\lambda} S''(x,\lambda x)\,\mathrm{d}x = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} I_{ij}(\lambda)$$

with complex

$$I_{ij}(\lambda) := \int_0^{A/\lambda} a^{ij}(\lambda x) f(x) x^s R^{ij}(x)\,\mathrm{d}x$$

and $s > r_1 - S_0(f)$. For $t := [s - r_1 + S_0(f)]/2 > 0$, $a^{ij}(\lambda x) f(x) x^s R^{ij}(x) = x^{r_1+t} a^{ij}(\lambda x) f(x) F_{ij}(x)$ where $F_{ij}(x) := x^{t-S_0(f)} f(x) R^{ij}(x)$ is bounded on the right of zero. The change of variable $u = \lambda x$ yields

$$\begin{aligned} I_{ij}(\lambda) &= \int_0^{A/\lambda} a^{ij}(\lambda x) f(x) x^s R^{ij}(x)\,\mathrm{d}x \\ &= \left[\int_0^{A} u^{r_1+t} a^{ij}(u) F_{ij}(u/\lambda)\,\mathrm{d}u\right]\lambda^{-(r_1+1+t)}. \end{aligned} \tag{3.24}$$

Because $S_0(a^{ij}) \geqslant -1 - r_1$ and $F_{ij}$ is bounded near zero the last integration exists and is bounded for large enough $\lambda$. Finally, $t > 0$ ensures that $I_{ij}(\lambda) = o[\lambda^{-(r_1+1)}]$ and thereby that

$$\int_0^{A/\lambda} S''(x, \lambda x)\,\mathrm{d}x = o[\lambda^{-(r_1+1)}].$$

If $m \in \mathbb{N}$ and $\beta \in C$ with $\mathrm{Re}(\beta) > r_1$, the choice of $R^{ij}(x) := 1$ or $R^{ij}(x) := F(x)$, $f(x) := \log^m(x)$ with $S_0(f) = 0$ shows that

$$J_{\beta,m}(\lambda) := \int_0^{A/\lambda} a^{ij}(\lambda x) x^{\beta} \log^m x\,\mathrm{d}x = o[\lambda^{-(r_1+1)}]$$

and

$$J'_{\beta,m}(\lambda) := \int_0^{A/\lambda} a^{ij}(\lambda x) F(x) x^{\beta} \log^m x\,\mathrm{d}x = o[\lambda^{-(r_1+1)}].$$

As a finite sum of such integrals $J_{\beta,m}(\lambda)$ or $J'_{\beta,m}(\lambda)$ the contributions

$$\int_0^{A/\lambda} S(x, \lambda x)\,\mathrm{d}x \quad \text{and} \quad \int_0^{A/\lambda} S'(x, \lambda x)\,\mathrm{d}x$$

are also equal to $o[\lambda^{-(r_1+1)}]$. Thus, relation (3.24) is proved.

(*c*) If $(a^{ij}) = (0)$ then the second sum on the right-hand side of (3.9) is zero. Else for this sum $S_0(f) + \gamma S \leqslant \mathrm{Re}(\tau^i_{p_i} + \delta_l) \leqslant r - 1 \leqslant r_1$ and the assumption $a^{ij} \in \mathcal{E}^{r_1+1}_{-1-\gamma S - S_0(f)}(]0, +\infty[, C)$ ensures that

$$\boldsymbol{fp} \int_0^{\infty} a^{ij}(u) u^{\tau^i_{p_i} + \delta_l} \log^{q_i+k-v}(u)\,\mathrm{d}u$$

is meaningful.

3. The complex function $W_{s_1,s_2}$ defined by

$$\mathcal{H}_{t_1}[x, u] = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} L_{nm}(x) u^{-\gamma_n} \log^m u + u^{-s_2} W_{s_1,s_2}(x, u)$$

is bounded for $(x, u) \in [0, \eta] \times [A, +\infty[$.

*Proof.* As a result of expansion (3.7) for each function $h^{ij}$ for $u \geqslant A$ and of the linearity with respect to $(F^{ij})$ of operators $T[x, (F^{ij})]$, $T'[x, (F^{ij})]$ and $T''[x, (F^{ij})]$, one obtains for $(x, u) \in [0, \eta] \times [A, +\infty[$

$$\begin{aligned} x^{s_1}\mathcal{H}_{t_1}[x, u] &= T[x, (a^{ij}(u))] + T'[x, (a^{ij}(u))] + T''[x, (a^{ij}(u))] \\ &\qquad + x^{\gamma t'_1} f(x) h_0(x)^{t'_1} H_{t_1}[h(x) - c, u] \\ &= \sum_{n=0}^{N} \sum_{m=0}^{M(n)} \{[T + T' + T''][x, (K^{ij}_{nm})]\} u^{-\gamma_n} \log^m u \\ &\qquad + u^{-s_2}\{[T + T' + T''][x, (O^{ij}(u))]\} + x^{\gamma t'_1} f(x) h_0(x)^{t'_1} \\ &\qquad \times \left\{ \sum_{n=0}^{N} \sum_{m=0}^{M(n)} V_{nm}[x^{\gamma} h_0(x)] u^{-\gamma_n} \log^m u + u^{-s_2} w_{t'_1,s_2}[h(x) - c, u] \right\}. \end{aligned}$$

Use of property 4 of definition 11 allows us to write for $(x,u)\in[0,\eta]\times[A,+\infty[$

$$\begin{aligned} x^{s_1}\mathcal{H}_{t_1}[h(x)-c,u] &= \sum_{n=0}^{N}\sum_{m=0}^{M(n)}\{[T+T'+T''][x,(K^{ij}_{nm})]+x^{\gamma t'_1}f(x)h_0(x)^{t'_1}V_{nm}[x^{\gamma}h_0(x)]\}u^{-\gamma_n}\log^m u \\ &\quad +u^{-s_2}\{[T+T'+T''][x,(O^{ij}(u))]+x^{\gamma t'_1}f(x)h_0(x)^{t'_1}w_{t'_1,s_2}[h(x)-c,u]\}. \end{aligned}$$

Each factor associated to $u^{-\gamma_n}\log^m u$ turns out to be $x^{s_1}L_{nm}(x)$ (see (3.13), (3.14), (3.15)). Inspection of the definitions of

$$T[x,(O_{ij}(u))],\qquad T'[x,(O_{ij}(u))],\qquad T''[x,(O_{ij}(u))]$$

with the assumption $O_{ij}$ bounded for $u\geqslant A$ ensures that

$$[T+T'+T''][x,(O_{ij}(u))]=x^{s_1}R(x,u)$$

with $R(x,u)$ bounded for $(x,u)\in[0,\eta]\times[A,+\infty[$. Moreover, one gets

$$x^{\gamma t'_1}h_0(x)^{t'_1}f(x)=x^{s_1}G(x)$$

with $G$ bounded for $x\in[0,\eta]$. Consequently, the complex function $W_{s_1,s_2}$ satisfies

$$W_{s_1,s_2}(x,u)=R(x,u)+G(x)w_{t'_1,s_2}[h(x)-c,u]$$

and is bounded for $(x,u)\in[0,\eta]\times[A,+\infty[$.

Gathering all the properties satisfied by pseudo-function $\overline{K}(x,u)$, it appears that $\overline{K}\in\mathcal{F}^{r_2}_{r_1}(]0,b[,C)$ where this set is defined in Sellier (1994). Application of theorem 1 of this latter paper allows us to expand

$$I(\lambda)=\boldsymbol{fp}\int_0^b\overline{K}(x,\lambda x)\,\mathrm{d}x$$

and thereafter ensures the stated result. ■

As outlined right after definition 5 the case of a function $h$ which is constant in a neighbourhood on the right of zero is not taken into account by definition 11 and previous theorem 12. Assume that $(f,h,K)$ obeys the next modified properties of definition 11:

1. Property 1 is unchanged.
2. Properties $2a$ and $2b$ are unchanged with this time $V_{r_2}(c,u)$ bounded near infinity, property $2c$ replaced by: $g_{nm}(x):=f(x)K_{nm}[h(x)]\in L^1_{\mathrm{loc}}(]0,b],C)$, $h(x)=c$ for $x\in[0,\eta]$ where $\eta$ remains the real number introduced by properties $2a$ and $2b$ and (3.5) is replaced by $K_{nm}[c]=K_{nm}[c]$, i.e. $i=0=J(0)$, $\alpha_0=0$, $K^{00}_{nm}=K_{nm}[c]$ and also $V_{nm}=0$. Note that these features agree with expansion (2.7) introduced in definition 5 if $p_0=Q_0(0)=0$, $\Delta^0_{00}=1$, $\Delta^0_r=0$.
3. Expansion (3.6) and assumption (3.7) hold with $a^0(u)=K[c,u]$ and $H_{t_1}=0$. Finally, one assumes that

$$\lim_{u\to 0}K[c,u]=\sum_{q,\mathrm{Re}(\beta_p)\leqslant R}A_{pq}u^{\beta_p}\log^q u$$

and (3.9) is deduced by applying (3.3) for $X=c$. Thus $O_0(u)=V_{r_2}(c,u)$, which is bounded in a neighbourhood of infinity.

4. Naturally property 4 is true with $w_{t'_1,s_2}=0$.

Under this set of assumptions, one gets for $r \leqslant \max[r_1+1, r_2]$ the following asymptotic expansion:

$$
\begin{aligned}
I(\lambda) &= \boldsymbol{fp}\int_0^b f(x)K[h(x),\lambda x]\,\mathrm{d}x \\
&= \sum_{m,\mathrm{Re}(\gamma_n)\leqslant r}\sum_{e=0}^{m} C_m^e\left[\boldsymbol{fp}\int_0^b f(x)K_{nm}[h(x)]x^{-\gamma_n}\log^{m-e}(x)\,\mathrm{d}x\right]\lambda^{-\gamma_n}\log^e\lambda \\
&\quad+\sum_{k,\mathrm{Re}(\delta_l)\leqslant r-1}\sum_{v=0}^{k} C_k^v(-1)^v f_{lk}^0\left\{\boldsymbol{fp}\int_0^\infty K[c,u]u^{\delta_l}\log^{k-v}(u)\,\mathrm{d}u\right. \\
&\quad-\sum_{\{p;\beta_p=-\delta_l-1\}}\sum_{q=0}^{Q(p)}\frac{A_{pq}}{1+k+q-v}\log^{1+k+q-v}\lambda \\
&\quad\left.+\sum_{\{n;\gamma_n=\delta_l+1\}}\sum_{m=0}^{M(n)}\frac{K_{nm}[c]}{1+k+m-v}\log^{1+k+m-v}\lambda\right\}\lambda^{-(\delta_l+1)}\log^v\lambda+o(\lambda^{-r}).
\end{aligned}
\tag{3.25}
$$

It is indeed possible to write $I(\lambda)=I_1(\lambda)+I_2(\lambda)$, where

$$
I_1(\lambda)=\boldsymbol{fp}\int_0^\eta f(x)K[c,\lambda x]\,\mathrm{d}x;\quad I_2(\lambda)=\int_\eta^b f(x)K[h(x),\lambda x]\,\mathrm{d}x. \tag{3.26}
$$

Note that, if $\eta > b$, then one sets $I_2(\lambda)=0$, else properties $2a$ and $2b$ ensure that

$$
I_2(\lambda)=\sum_{n=0}^{N}\sum_{m=0}^{M(n)}\int_\eta^b f(x)K_{nm}[h(x)](\lambda x)^{-\gamma_n}\log^m[\lambda x]\,\mathrm{d}x+O(\lambda^{-s_2}). \tag{3.27}
$$

To conclude, application of theorem 12 to integral $I_1(\lambda)$ with the above definitions of $V_{nm}$, $H_{t_1}$, $\alpha_i$, $\Delta_{p_iq_i}^{ij}$ leads to the asymptotic expansion of $I(\lambda)$.

## 4. Application to a large class of complex kernels

At this stage, the properties required by definition 11 may appear strong and thereby somewhat restrictive to the reader. Nevertheless, this section shows that previous results apply to a large class of complex pseudo-functions $K(x,u)$. Not only the case of '$\mu$ homogeneous' kernel is included but also the case of other related integrals. Finally, many examples are given in order to illustrate the expansions that have been derived above.

*Definition 13.* For a real value $c$ and $(N,I,M)\in\mathbb{N}^3$, $\mathcal{D}_c^{(N,I,M)}$ is the set of complex functions $L(x,y)$ such that if $\partial_1^k L(x,y) := \partial^k L/\partial x^k(x,y)$ and $\partial_2^k L(x,y) := \partial^k L/\partial y^k(x,y)$ for $k\in\mathbb{N}$ then

1. If $k_{max} := \max(N+1, I+1)$, then $\forall k\in\{0,\ldots,k_{max}\}$, $\partial_2^k L(1,y)$ exists and is bounded in a neighbourhood of $y=0$.
2. There exist $A>0$ and $U_c$ a neighbourhood of point $c$ such that $\partial_2^{I+1}L(x,y)$ exists and is bounded for $(x,y)\in[0,A]\times U_c$.
3. $\forall i\in\{0,\ldots,I\}$, the functions $\partial_1^m\partial_2^i L(u,c)$ for $0\leqslant m\leqslant M$ exist and are bounded in a neighbourhood of $u=0$.

If the kernel $L$ is singular at the point $(0,0)$ then the above property 3 induces that $L$ cannot belong to $\mathcal{D}_0^{(N,I,M)}$. However, for $c \neq 0$ it is often possible to find $(N,I,M)$ such that $L \in \mathcal{D}_c^{(N,I,M)}$.

**Theorem 14.** *Consider $0 < b < +\infty$, $f \in D_+(0,C) \cap L^1_{\mathrm{loc}}(]0,b],C)$ with $S_0(f) := S_0^+(f)$, $L(x,y)$ a complex and 'μ homogeneous' kernel (see (1.2)) and $h$ a real function bounded in $[0,b]$ with $h(0)=c$ and fulfilling property 1 of definition 5, i.e. such that there exist $\gamma \in \mathbb{R}_+^*$, $\eta > 0$, $h_0$ with $h(x) - c \overset{+}{=} x^\gamma h_0(x)$, $h_0(x) > 0$,*

$$h_0(x) = \sum_{m=0}^{M} a_m \log^m(x) + o(x) \quad in \quad ]0,\eta]$$

*with $a_M \neq 0$. For $r \geqslant 0$, if $r$, $f$, $h$ and $L$ satisfy each of the following properties:*

1. *for $\lambda$ large enough, $g_\lambda(x) := f(x)L[\lambda x, h(x)] \in \mathcal{P}(]0,b[,C)$;*
2. *$r \geqslant \max[-\mu, 1 + S_0(f)]$ and $\lim_{x\to 0^+} f(x) = \sum_{k,\mathrm{Re}(\delta_l)\leqslant r-1} x^{\delta_l}\log^k(x)$;*
3. *if $N := [\![r+\mu]\!]$, $I := [\![\gamma^{-1}(r-1-S_0(f))]\!]$ and $M := \max([\![-1-S_0(f)]\!]+1, 0)$ then $L \in \mathcal{D}_c^{(N,I,M)}$;*

*4 $\forall i \in \{0,\ldots,I\}$, $[h(x)-c]^i \in D_+^{r-1-S_0(f)}(0,C)$ with (see decomposition (2.7))*

$$[h(x)-c]^i = \sum_{p_i=0}^{P_i}\sum_{q_i=0}^{Q_i(p_i)} \Delta^i_{p_i q_i} x^{\tau^i_{p_i}} \log^{q_i}(x) + x^{s_i}\Delta^i(x) \tag{4.1}$$

*for $x \in ]0,\eta_i]$, $i\gamma \leqslant \mathrm{Re}(\tau^i_{p_i}) \leqslant r-1-S_0(f)$, $s_i > r-1-S_0(f)$ and $\Delta^i$ bounded; then*

$$\begin{aligned}
I(\lambda) &= \boldsymbol{fp}\int_0^b f(x)L[\lambda x, h(x)]\,\mathrm{d}x \\
&= \sum_{n=0}^{N} \frac{\partial_2^n L(1,0)}{n!}\left[\boldsymbol{fp}\int_0^b f(x)[h(x)]^n x^{\mu-n}\,\mathrm{d}x\right]\lambda^{-(n-\mu)} \\
&\quad + \sum_{\mathrm{Re}(\tau^i_{p_i}+\delta_l)\leqslant r-1}^{l,k,i,p_i,q_i} f^0_{lk}\Delta^i_{p_i q_i}\sum_{v=0}^{q_i+k} C^v_{q_i+k}(-1)^v \\
&\quad \times\left\{\boldsymbol{fp}\int_0^\infty \frac{\partial_2^i L(u,c)}{i!} u^{\tau^i_{p_i}+\delta_l}\log^{q_i+k-v}(u)\,\mathrm{d}u\right. \\
&\quad - \sum_{\{j;j=-\tau^i_{p_i}-\delta_l-1\}} \frac{\partial_1^j\partial_2^i L(0,c)}{i!j!}\frac{\log^{1+q_i+k-v}\lambda}{1+q_i+k-v} \\
&\quad \left. + \sum_{\{j;j=1-i+\mu+\tau^i_{p_i}+\delta_l\}} \frac{c^j\partial_2^{i+j}L(1,0)}{i!j!}\frac{\log^{1+q_i+k-v}\lambda}{1+q_i+k-v}\right\}\lambda^{-(\tau^i_{p_i}+\delta_l+1)}\log^v\lambda \\
&\quad + o(\lambda^{-r}).
\end{aligned} \tag{4.2}$$

*Proof.* The derivation consists in proving that under the proposed assumptions on $r, f, h$ and the kernel $L$ then $(f,h,K) \in \mathcal{L}^{r_2}_{r_1}(]0,b[,C)$ where $K(x,u) := L(u,x)$, $r_1 := r-1$ and $r_2 := r$. Property 1 of definition 11 naturally holds. Each remaining property of this latter definition is carefully checked below.

Relation $K(x,u) = u^\mu L(1, x/u)$ combined with the definition of integer $N$ and the assumption bearing on $\partial_2^k L(1,y)$ for $0 \leqslant k \leqslant N+1$ lead for $(X,u) \in h(]0,b[)\times[\eta,+\infty[$ to

$$K(X,u) = \sum_{n=0}^{N} \frac{\partial_2^n L(1,0)}{n!} X^n u^{-(n-\mu)} + u^{-(N+1-\mu)} V_{r_2}(X,u) \tag{4.3}$$

with

$$V_{r_2}(X,u) := X^{N+1}(N!)^{-1} \int_0^1 (1-t)^N \partial_2^{N+1} L(1, tX/u)\,\mathrm{d}t.$$

Thus decomposition (3.2) holds with $\gamma_n = n-\mu$, $M(n) = 0$, $s_2 = N+1-\mu > r$ because $N = [\![r+\mu]\!]$ and also $K_n(X) = \partial_2^n L(1,0)X^n/n!$.

Note that if $\eta > b$, the introduction of the function $\mathcal{F}(x) := H_e(b-x)f(x)$ allows us to write

$$I(\lambda) = \boldsymbol{fp}\int_0^b f(x)L[\lambda x, h(x)]\,\mathrm{d}x = \boldsymbol{fp}\int_0^\eta \mathcal{F}(x)L[\lambda x, h'(x)]\,\mathrm{d}x$$

if $h'(x) = h(x)$ for $0 < x < b$, else $h'(x) = cste > 0$. In such a case, $b = \eta$ and inequality (3.3) holds. If $\eta < b, h$ bounded in $[0,b]$ ensures that $th(x)/(\lambda x) \to 0$ as $\lambda \to +\infty$ for any $(x,t) \in [\eta,b]\times[0,1]$. Consequently,

$$H_\lambda(x) := (N!)^{-1}\int_0^1 (1-t)^N \partial_2^{N+1} L[1, th(x)/(\lambda x)]\,\mathrm{d}t$$

is bounded for $\lambda$ large enough and $x \in [\eta,b]$. Moreover, $x^{-(N+1-\mu)}[h(x)]^n$ turns out to be bounded too on $[\eta,b]$. Therafter and since $f \in L^1_{\mathrm{loc}}(]0,b],C)$ there exists a real $B$ such that $0 \leqslant B < +\infty$ and

$$\left|\int_\eta^b f(x)x^{-s_2}V_{r_2}[h(x),\lambda x]\,\mathrm{d}x\right| = \left|\int_\eta^b f(x)x^{-(N+1-\mu)}[h(x)]^n H_\lambda(x)\,\mathrm{d}x\right| \leqslant B < +\infty, \tag{4.4}$$

i.e. inequality (3.3) is satisfied.

Recall that $h$ is bounded in $[0,b]$ and $f \in L^1_{\mathrm{loc}}(]0,b],C)$. As a consequence, $\forall n \in \{0,\dots,N\}$, $g_n(x) = f(x)\partial_2^n L(1,0)[h(x)]^n/n! \in L^1_{\mathrm{loc}}(]0,b],C)$. Moreover,

$$t_n := \max[r_1 - S_0(f), \qquad \mathrm{Re}(\gamma_n) - 1 - S_0(f)] = r - 1 - S_0(f) \geqslant 0.$$

For $t_1 := \gamma^{-1}t_n \geqslant 0$ and $I := [\![t_1]\!] \geqslant 0$, then in a neighbourhood of $c$ expansion (3.4) holds and

$$K_n(X) = \sum_{i=0}^{I} K_n^i (X-c)^i + |X-c|^{t_1'} V_n(X-c), \tag{4.5}$$

with $t_1' = I+1 > t_1$, $\alpha_i = i$, $J(i) = 0$ and also the two different cases below:

(i) if $n \leqslant I$ then $V_n = 0$, $K_n^i = 0$ for $n < i \leqslant I$ and $K_n^i = \partial_2^n L(1,0)c^{n-i}/[i!(n-i)!]$ for $0 \leqslant i \leqslant n$.

(ii) if $n > I$ for $0 \leqslant i \leqslant I < n$ then $K_n^i = \partial_2^n L(1,0)c^{n-i}/[i!(n-i)!]$ and here

$$|X-c|^{t_1'}V_n(X-c) = \frac{\partial_2^n L(1,0)(X-c)^{t_1'}}{I!(n-I-1)!}\int_0^1 (1-t)^I[c+t(X-c)]^{n-I-1}\,\mathrm{d}t, \tag{4.6}$$

with $t_1' = I+1 > t_1$. Assumption 4 of theorem 14 guarantees that $(h, K_n) \in \mathit{\Pi}_+^{r-1-S_0(f),1}(0,C)$.

When $X \to c$, formula (2.3) yields in a neighbourhood of $c$ to an expansion (3.5)

$$K(X,u) = \sum_{i=0}^{I} \frac{\partial_2^i L(u,c)}{i!}(X-c)^i + |X-c|^{t_1'} H_{t_1}(X-c,u), \tag{4.7}$$

with $t_1' = I+1$, $a^i(u) = \partial_2^i L(u,c)/i!$ and also

$$|X-c|^{t_1'} H_{t_1}(X-c,u) = (I!)^{-1}(X-c)^{I+1} \int_0^1 (1-t)^I \partial_2^{I+1} L[u, c+t(X-c)]\,\mathrm{d}t.$$

For real $A$ introduced by property 2 of definition 13, we consider

$$\begin{aligned} E &= \int_0^{A/\lambda} f(x)|h(x)-c|^{t_1'} H_{t_1}[h(x)-c, \lambda x]\,\mathrm{d}x \\ &= \int_0^{A/\lambda} f(x) x^{\gamma(I+1)} h_0(x)^{I+1} S_\lambda(x)\,\mathrm{d}x \end{aligned} \tag{4.8}$$

if

$$S_\lambda(x) := (I!)^{-1} \int_0^1 (1-t)^I \partial_2^{I+1} L[\lambda x, c + t x^\gamma h_0(x)]\,\mathrm{d}t.$$

For $\lambda$ large enough and $(x,t) \in [0, A/\lambda] \times [0,1]$ then $t x^\gamma h_0(x) \to 0$ and since $\partial_2^{I+1} L(x,y)$ is bounded for $(x,y) \in [0,A] \times U_c$ then $S_\lambda$ is bounded too. Since $\gamma(I+1) > r - 1 - S_0(f)$ then $\eta' := [\gamma(I+1) + 1 + S_0(f) - r]/3 > 0$ and

$$D(x) := f(x) x^{\gamma(I+1)} h_0(x) S_\lambda(x) = x^{r-1+\eta'}[x^{\eta + S_0(f)} f(x)][x^\eta h_0(x)] S_\lambda(x).$$

As $\lim_{x\to 0^+} x^{\eta' + S_0(f)} f(x) = 0 = \lim_{x \to 0^+} x^{\eta'} h_0(x)$ and $S_\lambda$ is bounded then there exists $C > 0$ such that, for $\lambda$ large,

$$|E| = \left| \int_0^{A/\lambda} D(x)\,\mathrm{d}x \right| \leqslant C \left| \int_0^{A/\lambda} x^{r-1+\eta'}\,\mathrm{d}x \right| = o[\lambda^{-r}] = o[\lambda^{-(r_1+1)}], \tag{4.9}$$

i.e. property (3.6) is satisfied.

Here $S \geqslant 0$, $r_1 = r - 1 \geqslant S_0(f)$ and $r_2 = r \geqslant 0$. Thus, $R := \max[-1 - r_1, -1 - \gamma S - S_0(f)] \leqslant -1 - S_0(f)$ and $R' := \max(r_1 + 1, r_2) = r$. For a given $i \in \{0, \ldots, I\}$, $\partial_2^i L(u,c)$ is bounded for $u$ near zero so that $S_0(a^i) \geqslant 0 \geqslant -1 - r_1 = -r$. If $-1 - S_0(f) < 0$, then $a^i(u) = u^{-1-S_0(f)} F_i(u)$ with $F_i(u) = u^{1+S_0(f)} a^i(u)$ bounded near zero; else $P := [\![-1 - S_0(f)]\!] \geqslant 0$, $M := \max([\![-1-S_0(f)]\!] + 1, 0) = P + 1 > R$ and near zero

$$a^i(u) = \sum_{p=0}^{P} \frac{\partial_1^p \partial_2^i L(0,c)}{i!p!} u^p + \frac{u^{P+1}}{i!P!} \int_0^1 (1-t)^P \partial_1^{P+1} \partial_2^i L(tu, c)\,\mathrm{d}t. \tag{4.10}$$

The above expansion yields coefficients $A_p^i = \partial_1^p \partial_2^i L(0,c)/[i!p!]$. Since the new kernel $\partial_2^i L(x,y)$ satisfies a '$\mu - i$ homogeneous' property: $\partial_2^i L(tx, ty) = t^{\mu - i} \partial_2^i L(x,y)$ for $t \in \mathbb{R}_+^*$, $a^i(u)$ may be rewritten as $a^i(u) = u^{-(i-\mu)} \partial_2^i L(1, c/u)/i!$. This latter form is convenient to study the behaviour of $a^i$ near infinity. One obtains

$$a^i(u) = \sum_{n=0}^{N} D_n^i u^{-\gamma_n} + u^{-s_2} O_i(u), \tag{4.11}$$

with $O_i$ bounded near infinity, $\gamma_n = n - \mu$, $s_2 = N + 1 - \mu > r_2 = r$. More precisely, two cases occur:

(i) if $i \geqslant N+1, i-\mu \geqslant N+1-\mu = s_2$. Thus $D_n^i = 0$ and

$$O_i(u) = u^{-(i-\mu-s_2)}\partial_2^i L(1,c/u)/i!$$

is indeed bounded near infinity.

(ii) if $0 \leqslant i \leqslant N$, then $J_i := N-i \in \mathbb{N}$ and formula (2.3) leads to

$$a^i(u) = \sum_{j=0}^{J_i} \frac{\partial_2^{i+j}L(1,0)}{i!j!}c^j u^{-(i+j-\mu)} + u^{-s_2}O_i(u) \tag{4.12}$$

with

$$O_i(u) = (J_i!)^{-1}c^{J_i+1}\int_0^1 (1-t)^{J_i}\partial_2^{N+1}L(1,tc/u)\,dt$$

bounded near infinity. The change of index $n = i+j$ yields

$$a^i(u) = \sum_{n=0}^{N} D_n^i u^{-\gamma_n} + u^{-s_2}O_i(u)$$

with $D_n^i = 0$ if $0 \leqslant i \leqslant n$; else $D_n^i = \partial_2^n L(1,0)c^{n-i}/[i!(n-i)!]$ for $i \leqslant n$. These results show that $\forall n \in \{0,\dots,N\}, \forall i \in \{0,\dots,I\}$ then $D_n^i = K_n^i$.

(iii) Finally the last property 4 of definition 11 is considered. Here $t_1' = I+1$, $s_2 = N+1-\mu$ and also

$$|X-c|^{t_1'}H_{t_1}[X-c,u] = \frac{(X-c)^{t_1'}u^{-(I+1-\mu)}}{I!}\int_0^1 (1-t)^I\partial_2^{I+1}L\left[1,\frac{c+t(X-c)}{u}\right]dt. \tag{4.13}$$

Two cases are discussed.

If $I \geqslant N$, then $\forall n \in \{0,\dots,N\}$, $I \geqslant N \geqslant n$ and $V_n = 0$. Hence, the function $w_{t_1',s_2}$ obeys

$$w_{t_1',s_2}(X-c,u) =^{+}_{-} (I!)^{-1}u^{(N-I)}\int_0^1 (1-t)^I\partial_2^{I+1}L[1,(c+t(X-c))/u]\,dt$$

and is bounded for $(X,u) \in h([0,\eta]) \times [A,+\infty[$.

If $I < N$ for $J := N-I-1 \geqslant 0$, formula (2.3) yields

$$|X-c|^{t_1'}H_{t_1}[X-c,u] = \sum_{j=0}^{J} d_j(X-c)u^{-(I+1+j-\mu)} + u^{-s_2}|X-c|^{t_1'}w_{t_1',s_2}(X-c,u), \tag{4.14}$$

where the function $w_{t_1',s_2}$ defined as

$$w_{t_1',s_2}(X-c,u) = \pm\frac{1}{I!J!}\int_0^1 (1-t)^I[c+t(X-c)]^J \times\left[\int_0^1 (1-v)^J\partial_2^{N+1}L[1,v(\frac{c+t(X-c)}{u})]\,dv\right]dt$$

is bounded for $(X,u) \in h([0,\eta]) \times [A,+\infty[$ and

$$d_j(X-c) = (I!j!)^{-1}\partial_2^{I+1+j}L(1,0)(X-c)^{t_1'}\int_0^1 (1-t)^I[c+t(X-c)]^j\,dt.$$

With the new index $n = I + 1 + j$, (4.14) may be rewritten as

$$|X-c|^{t_1'} H_{t_1}[X-c,u] = \sum_{n=0}^{N} W_n(X-c)|X-c|^{t_1'} u^{-(n-\mu)} + u^{-s_2}|X-c|^{t_1'} w_{t_1',s_2}(X-c,u)$$

with $W_n(X-C) = 0 = V_n(X-C)$ for $0 \leqslant n \leqslant I$; else

$$|X-c|^{t_1'} w_{t_1',s_2}(X-c,u) = (X-c)^{t_1'} \partial_2^n L(1,0)[I!(n-I-1)!]^{-1} \times \int_0^1 (1-t)^I [c+t(X-c)]^{n-I-1}\, \mathrm{d}t = |X-c|^{t_1'} V_n(X-c).$$

Consequently, property 4 is satisfied.

To conclude, application of theorem 12 to

$$I(\lambda) = \boldsymbol{fp} \int_0^b f(x) L[\lambda x, h(x)]\, \mathrm{d}x$$

ensures the stated asymptotic expansion (4.2). ■

*Example 1.* For $q \in \mathbb{N}$, $\beta \in C$, $0 < b < +\infty$ and $r \geqslant \max[1, 1-\mathrm{Re}(\beta)]$ then

$$\begin{aligned}
\boldsymbol{fp} \int_0^1 \frac{\log^q(x)\, \mathrm{d}x}{x^\beta (1 + x\mathrm{e}^x + \lambda x)} &= \sum_{n=0}^{[\![r-1]\!]} \left[ \boldsymbol{fp} \int_0^1 [-1 - x\mathrm{e}^x]^n x^{-(\beta+n+1)} \log^q(x)\, \mathrm{d}x \right] \lambda^{-(n+1)} \\
&+ \sum_{p=0}^{I} \sum_{i=0}^{p} \sum_{v=0}^{q} \frac{i^{p-i} C_q^v (-1)^v}{(p-i)!} \left\{ \boldsymbol{fp} \int_0^\infty \frac{(-1)^i u^{p-\beta}}{(u+1)^{i+1}} \log^{q-v}(u)\, \mathrm{d}u \right. \\
&\quad - E(\beta) \sum_{\{j; j=-p+\beta-1\}} \frac{(-1)^{i+j}(i+j)!}{i!j!} \frac{\log^{1+q-v} \lambda}{1+q-v} \\
&\quad \left. + E(I-\beta-1) \sum_{\{j; j=p-i-\beta\}} \frac{(-1)^{i+j}(i+j)!}{i!j!} \frac{\log^{1+q-v} \lambda}{1+q-v} \right\} \lambda^{-(p-\beta+1)} \log^v \lambda \\
&+ o(\lambda^{-r}),
\end{aligned} \tag{4.15}$$

where $I := [\![r - 1 + \mathrm{Re}(\beta)]\!]$ and for complex $z$, $E(z) := 0$ except if $z \in \mathbb{N} \setminus \{0\}$, then $E(z) := 1$.

Observe that expansion (4.15) contains logarithmic terms $\lambda^\delta \log^k \lambda$ if and only if $\beta$ is a non-zero and positive integer.

For a given $\lambda$, the functions $H_\lambda(x) = 1 + x\mathrm{e}^x + \lambda x$ and $g(X) = X^{-1}$ are smooth respectively near zero and near $H_\lambda(0) = 1$. Thus, Faa de Bruno's formula (2.11) ensures that $g \circ H_\lambda$ admits derivatives at zero up to any order. Consequently, $g_\lambda(x) = \log^q(x)[g \circ H_\lambda](x) \in \mathcal{P}(]0, b[, C)$ and

$$I_1(\lambda) = \boldsymbol{fp} \int_0^b g_\lambda(x)\, \mathrm{d}x$$

exists. Moreover, theorem 14 applies to $I_1(\lambda)$ with $f(x) = x^{-\beta} \log^q(x)$, $S_0(f) = -\mathrm{Re}(\beta)$, $L(x,y) = (x+y)^{-1}$, $\mu = -1$, $h(x) = 1 + x\mathrm{e}^x$, $c = 1 \neq 0$, $\gamma = 1$, $N = [\![r-1]\!]$, $I = [\![r - 1 + \mathrm{Re}(\beta)]\!]$, $M = \max([\![-1 + \mathrm{Re}(\beta)]\!] + 1, 0)$, $L \in \mathcal{D}_1^{(N,I,M)}$, $q_i = 0 = Q_i(p_i)$, $\tau_{p_i}^i = p_i + i$ and $\Delta_{p_i 0}^i = i^{p_i}/p_i!$ with $P_i = [\![r - 1 + \mathrm{Re}(\beta) - i]\!]$.

*Example 2.* If $c \in \mathbb{R}^*$, $\beta \in C$ and $r \geqslant \max[1, 1-\mathrm{Re}(\beta)]$ then

$$\begin{aligned}
\boldsymbol{fp}\int_0^\infty \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x^\beta(c+\sqrt{x}+\lambda x)} &= \sum_{n=0}^{[\![r-1]\!]}\left[\boldsymbol{fp}\int_0^\infty (-1)^n[c+\sqrt{x}]^n x^{-(\beta+n+1)}\mathrm{e}^{-x}\,\mathrm{d}x\right]\lambda^{-(n+1)} \\
&+\sum_{i=0}^{I}\sum_{l=0}^{e(i)}\frac{(-1)^l}{(l)!}\left\{\boldsymbol{fp}\int_0^\infty \frac{(-1)^i u^{i/2+l-\beta}}{(u+c)^{i+1}}\,\mathrm{d}u\right. \\
&\quad -E(\beta-i/2-l)\sum_{\{j;j=-i/2+\beta-l-1\}}\frac{(-1)^{i+j}(i+j)!}{i!j!}c^{-(i+j+1)}\log\lambda \\
&\quad \left.+E(l+1-i/2-\beta)\sum_{\{j;j=-i/2-\beta+l\}}\frac{(-1)^{i+j}(i+j)!}{i!j!}c^j\log\lambda\right\}\lambda^{-(i/2+l-\beta+1)} \\
&+o(\lambda^{-r}), \qquad (4.16)
\end{aligned}$$

where $I := [\![2(r-1+\mathrm{Re}(\beta)]\!]$ and $e(i) := [\![r-1+\mathrm{Re}(\beta)-i/2]\!]$.

This integral is in fact split into two terms

$$I_2(\lambda) := \boldsymbol{fp}\int_0^1 \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x^\beta(c+\sqrt{x}+\lambda x)} \quad \text{and} \quad J(\lambda) := \int_1^\infty \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x^\beta(c+\sqrt{x}+\lambda x)}. \qquad (4.17)$$

For $N := [\![r-1]\!]$, expansion of $J(\lambda)$ with respect to $\lambda$ is obtained by expanding the function $[1+(c+\sqrt{x})/\lambda x]^{-1}$. One easily gets

$$J(\lambda) = \sum_{n=0}^{[\![r-1]\!]}\left[\int_1^\infty (-1)^n[c+\sqrt{x}]^n x^{-(\beta+n+1)}\mathrm{e}^{-x}\,\mathrm{d}x\right]\lambda^{-(n+1)} + R(\lambda), \qquad (4.18)$$

where

$$\begin{aligned}
R(\lambda) &= \lambda^{-(N+2)}(-1)^{N+1}\int_1^\infty \mathrm{e}^{-x}x^{-(\beta+N+1)}[c+\sqrt{x}]^{N+1}\left[1+\frac{c+\sqrt{x}}{\lambda x}\right]^{-1}\mathrm{d}x \\
&= o(\lambda^{-r}),
\end{aligned}$$

since $N+2 > r$. The remaining integral $I_2(\lambda)$ is treated by application of theorem 14 with $f(x) = x^{-\beta}\mathrm{e}^{-x}$, $S_0(f) = -\mathrm{Re}(\beta)$, $L(x,y) = (x+y)^{-1}$, $\mu = -1$,

$$\lim_{x\to 0^+} f(x) = \sum_{l=0}^{K}(-1)^l x^{l-\beta}/l!,$$

where $K := [\![r-1+\mathrm{Re}(\beta)]\!]$, $h(x) = c+\sqrt{x}$, $c \neq 0$, $\gamma = 1/2$, $N = [\![r-1]\!]$, $I = [\![2(r-1+\mathrm{Re}(\beta))]\!]$, $M = \max([\![-1+\mathrm{Re}(\beta)]\!]+1, 0)$, $L \in \mathcal{D}_c^{(N,I,M)}$, $q_i = 0 = Q_i(p_i)$, $[h(x)-c]^i = x^{i/2}$.

This result allows one to deal also with

$$I_3(\lambda) := \int_0^\infty \mathrm{e}^{-x}\log(c+\sqrt{x}+\lambda x)\,\mathrm{d}x$$

for $c > 0$. Use of an integration by parts indeed yields

$$I_3(\lambda) = \log(c) + \lambda\int_0^\infty \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{(c+\sqrt{x}+\lambda x)} + \frac{1}{2}\int_0^\infty \frac{x^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x}{(c+\sqrt{x}+\lambda x)}.$$

Use of theorem 14 allows us to answer the question (see the introduction) of finding the expansion of

$$\begin{aligned} J_h(\epsilon) &= \boldsymbol{fp}\int_0^b f(x)L[x,\epsilon h(x)]\,\mathrm{d}x \\ &= \epsilon^{\mu}\boldsymbol{fp}\int_0^b f(x)L[\epsilon^{-1}x,h(x)]\,\mathrm{d}x \\ &= \epsilon^{\mu} I(\epsilon^{-1}) \end{aligned}$$

when weak assumptions are assumed for pseudo-functions $f$, $h$ and $L$. In usual applications, the special case $f(x) = x^{-\beta}\log^q(x)g(x)$ for $q \in \mathbb{N}$, $\beta \in C$ and $g$ and $h$ admitting derivatives up to a certain order near zero is often encountered. The proposition below provides the expansion of $J_h(\epsilon)$ in such circumstances.

**Proposition 15.** *Consider $q \in \mathbb{N}$, $\beta \in C$, $0 < b < +\infty$, $L(x,y)$ a complex and '$\mu$ homogeneous' kernel, $g \in D_+(0,C) \cap L^1_{\mathrm{loc}}(]0,b],C)$ with $S_0(g) \in \mathbb{N}$ and $h$ a real and bounded function on $[0,b]$ with $h(0)=c$ and fulfilling property 1 of definition 5 which one introduces real $\gamma > 0$. For real value $R$ such that $R \geqslant \max(0, 1-\mathrm{Re}(\beta)+\mu+S_0(g))$ and*

*1. for $\lambda$ large enough, $g_\lambda(x) := x^{-\beta}\log^q(x)g(x)L[\lambda x, h(x)] \in \mathcal{P}(]0,b[,C)$;*

*2. if $N := [\![R]\!]$, $I := [\![\gamma^{-1}(R-\mu-1+\mathrm{Re}(\beta)-S_0(g))]\!]$ and $M := \max([\![-1+\mathrm{Re}(\beta)-S_0(g)]\!]+1, 0)$ then $L \in \mathcal{D}_c^{(N,I,M)}$;*

*3. if $K := [\![R-\mu-1+\mathrm{Re}(\beta)]\!]$ and $P := [\![R-\mu-1+\mathrm{Re}(\beta)-S_0(g)]\!]$ then $g \in \mathcal{C}_+^{K+1}(0,C)$ and $h(x)-h(0) = x\phi(x)$, with $\phi \in \mathcal{C}_+^{P+1}(0,\mathbb{R})$; then*

$$\begin{aligned} N_h(\epsilon) &= \boldsymbol{fp}\int_0^b x^{-\beta}\log^q(x)g(x)L[x,\epsilon h(x)]\,\mathrm{d}x \\ &= \sum_{n=0}^{[\![R]\!]} \frac{\partial_2^n L(1,0)}{n!}\left[\boldsymbol{fp}\int_0^b g(x)\log^q(x)[h(x)]^n x^{-(\beta+n-\mu)}\,\mathrm{d}x\right]\epsilon^n \\ &\quad + \sum_{m=S_0(g)}^{[\![R-\mu-1+\mathrm{Re}(\beta)]\!]} \sum_{l=S_0(g)}^{m}\sum_{i=0}^{m-l}\sum_{v=0}^{q} \frac{C_q^v g^{(l)}(0)}{(-1)^q l!} a^i_{m-l-i} \\ &\quad \times\left\{\boldsymbol{fp}\int_0^\infty \frac{\partial_2^i L(u,c)}{i!} u^{m-\beta}\log^{q-v}(u)\,\mathrm{d}u \right. \\ &\quad - \sum_{\{j;j=\beta-m-1\}} \frac{\partial_1^j\partial_2^i L(0,c)}{i!j!}\frac{\log^{1+q-v}(\epsilon^{-1})}{1+q-v} \\ &\quad \left. + \sum_{\{j;i+j=1+\mu-\beta+m\}} \frac{c^j\partial_2^{i+j}L(1,0)}{i!j!}\frac{\log^{1+q-v}(\epsilon^{-1})}{1+q-v}\right\}\epsilon^{m-\beta+\mu+1}\log^v\epsilon \\ &\quad + o(\epsilon^R), \end{aligned} \tag{4.19}$$

*where the coefficients $a_p^i$ are defined for $0 \leqslant i \leqslant I$ by*

$$\left[\frac{h(x)-h(0)}{x}\right]^i = [\phi(x)]^i = \sum_{p=0}^{P-i} a_p^i x^p + o(x^{P-i}), \tag{4.20}$$

*and $[\phi(x)]^0 = 1, a_p^0 = \delta_{p0}$.*

*Proof.* By setting $f(x) = x^{-\beta}\log^q(x)g(x)$, $S_0(f) = -\operatorname{Re}(\beta) + S_0(g)$, $r = R - \mu$, application of theorem 14 to

$$I(\epsilon) := \boldsymbol{fp}\int_0^b f(x)L[\epsilon^{-1}x, h(x)]\,\mathrm{d}x$$

leads to the expansion (4.19). The reader may check that the assumptions bearing on $f$, $h$ and $L$ guarantee the equalities (4.20) and all the properties required for this theorem. Observe that each expansion (4.20) may be obtained by using Faa de Bruno's formula for $\phi(x) = x^{-1}[h(x) - h(0)]$. ■

*Example 3.* Consider $0 < b < +\infty$, $g \in D_+(0, C) \cap L^1_{\mathrm{loc}}(]0, b], C)$ with $S_0(g) \in \mathbb{N}$ and $h$ a real positive and bounded function on $[0, b]$. For real $R \geqslant 1/2 + S_0(g)$, $K := [\![R-1/2]\!]$ and $P := [\![R-1/2-S_0(g)]\!]$ if $g \in \mathcal{C}_+^{K+1}(0, C)$ and $h(x)-h(0) = x\phi(x)$ with $\phi \in \mathcal{C}_+^{P+1}(0, \mathbb{R})$ then

$$\begin{aligned}\int_0^b \frac{g(x)\,\mathrm{d}x}{\sqrt{x+\epsilon h(x)}} &= \sum_{n=0}^{[\![R]\!]} \frac{(-1)^n b_n}{2^n n!}\left[\boldsymbol{fp}\int_0^b g(x)[h(x)]^n x^{-(n+1/2)}\,\mathrm{d}x\right]\epsilon^n \\ &+ \sum_{m=S_0(g)}^{[\![R-1/2]\!]}\sum_{l=S_0(g)}^{m}\sum_{i=0}^{m-l} a^i_{m-l-i}\frac{(-1)^i g^l(0) b_i}{2^i i! l!}\left[\boldsymbol{fp}\int_0^\infty \frac{u^m\,\mathrm{d}u}{[u+h(0)]^{i+1/2}}\right]\epsilon^{m+1/2} \\ &+ o(\epsilon^R), \end{aligned} \tag{4.21}$$

where the real $a^i_p$ obeys definitions (4.20) and $b_0 := 1$, $b_n := 1\times 3\times\ldots\times[1+2(n-1)]$ for $n \geqslant 1$.

If $h(0) \neq 0$, application of proposition 15 with $\beta = q = 0$ and

$$L(x, y) = \frac{1}{\sqrt{x+y}}, \qquad \mu = -1/2,$$

$$\partial_2^i L(u, c) = \frac{(-1)^i b_i}{2^i (u+c)^{i+1/2}}, \qquad \gamma = S_0(\phi), \qquad N = [\![R]\!]$$

,

$$I = [\![\gamma^{-1}(R - 1/2 - S_0(g))]\!], \qquad M = 0, \qquad L \in \mathcal{D}^{(N,I,M)}_{h(0)}$$

leads to the result. If $h(0) = 0$, observe that

$$\boldsymbol{fp}\int_0^\infty u^{m-i-1/2}\,\mathrm{d}u := \boldsymbol{fp}\left[\int_\epsilon^{\epsilon^{-1}} u^{m-i-1/2}\,\mathrm{d}u\right] = 0$$

and for $\epsilon\phi(x) \to 0$ use of the Taylor expansion (2.3)

$$\frac{1}{\sqrt{1+\epsilon\phi(x)}} = \sum_{n=0}^{N} \frac{(-1)^n b_n [\phi(x)]^n}{2^n n!}\epsilon^n + \frac{[\epsilon\phi(x)]^{N+1}}{N!}\int_0^1 \frac{(1-t)^N(-1)^{N+1}b_{N+1}\,\mathrm{d}t}{2^{N+1}[1+t\epsilon\phi(x)]^{N+3/2}} \tag{4.22}$$

yields the first sum on the right-hand side of (4.21).

In the previous integral, no logarithmic term $\lambda^\delta \log^k \lambda$ appears in the asymptotic expansion. The next example exhibits a case where this is not true any more.

*Example 4.* For $c > 0$, $0 < b < +\infty$, $k \in \mathbb{N} \setminus \{0\}$ and real $R \geqslant 0$, then

$$\begin{aligned}
&\boldsymbol{fp} \int_0^b \frac{\cos(x)\,\mathrm{d}x}{x^{3/2}\sqrt{x + \epsilon(c + x^k)}} \\
&= \sum_{n=0}^{[\![R]\!]} \frac{(-1)^n b_n}{2^n n!} \left[ \boldsymbol{fp} \int_0^b \cos(x)[c + x^k]^n x^{-(n+1/2)}\,\mathrm{d}x \right] \epsilon^n \\
&\quad + \sum_{m=0}^{[\![R+1]\!]} \sum_{l'=0}^{F(m)} \sum_{\{i; ik = m - 2l'\}} \frac{(-1)^{i+l'} b_i}{2^i i! (2l')!} \left\{ \boldsymbol{fp} \int_0^\infty \frac{u^{m-3/2}\,\mathrm{d}u}{[u + c]^{i+1/2}} \right. \\
&\quad - \left. \sum_{\{j; j = m-1-i\}} \frac{(-1)^j c^j}{2^j j!} (2i+1)(2i+3)\ldots(2i+1+2j-2) \log \epsilon \right\} \epsilon^{m-1} \\
&\quad + o(\epsilon^R),
\end{aligned} \tag{4.23}$$

where $b_n$ is defined in example 3 and for a positive integer $m$, $F(2m) = F(2m+1) := m$.

Here $g(x) = \cos(x)$, $S_0(g) = 0$, $g^{(2l')}(0) = (-1)^{l'}$, $g^{(2l'+1)}(0) = 0$, $\beta = 3/2$, $q = 0$, $\gamma = k$, $N = [\![R]\!]$, $I = [\![k^{-1}R]\!]$, $M = 0$, $K = [\![R+1]\!]$, $P = [\![R]\!]$,

$$L(x, y) = \frac{1}{\sqrt{x + y}} \in \mathcal{D}_c^{(N,I,M)}$$

and $a_p^i = 0$ except if $p = i(k-1)$ then $a_p^i = 1$. Note that if $h(x) = c + x^s$ with $s > 0$ and not a positive integer then $h(x) - c$ does not belong to

$$\mathcal{C}_+^{[\![R]\!]+1}(0, C)$$

for $[\![R]\!] > s - 1$. In such circumstances, the expansion of the integral up to order $o(\epsilon^R)$ is supplied by the general theorem 14. Observe that each integral of the type

$$\boldsymbol{fp} \int_0^\infty \frac{u^\alpha}{(u + c)^\beta}\,\mathrm{d}u,$$

arising on the right-hand side of (4.23), may be calculated by using Mellin transforms.

At this stage it is possible to handle the motivating case of integral $M_h(\epsilon)$ (see (1.2)). Since the change of scale $x' = -x$ induces no corrective term for an integration in the finite part sense of Hadamard (consult Sellier 1994, lemma 2) and under $Q$ pseudo-homogeneous property of kernel $K$ one actually gets for $-\infty < a < 0 < b < +\infty$

$$M_h(\epsilon) = \boldsymbol{fp} \int_0^b f(x) K[x, \epsilon h(x)]\,\mathrm{d}x + (-1)^Q S(-1) \boldsymbol{fp} \int_0^{-a} g(x') K[x', \epsilon H(x')]\,\mathrm{d}x' \tag{4.24}$$

with $g(x) = f(-x)$ and also $H(x) = -h(-x)$. Application of general theorem 14 leads to the asymptotic expansion of $M_h(\epsilon)$ which is seen to involve the behaviour of the pseudo-functions $f$ and real function $h$ at zero on the right and on the left. These behaviours may be different. For the sake of simplicity we restrict the study to the usual case of functions $f$ and $h$ which are smooth enough in a neighbourhood of zero and other behaviours are left to the reader to examine.

**Theorem 16.** *Consider* $-\infty < a < 0 < b < +\infty, Q$ *an integer,* $K$ *a 'Q pseudo-*

*homogeneous' kernel, $f \in D(0,C)$ with $S_0(f) \in \mathbb{N}$ and $h$ a real and bounded function on $[a,b]$ such that $S_0[h-h(0)]>0$. If $R \geqslant \max(0,1+Q+S_0(f))$ and*

*1. for $\lambda$ large enough then $f(x)K[\lambda x, h(x)] \in \mathcal{P}(]a,b[,C)$;*

*2. if $N = [\![R]\!]$, $I = [\![S_0^{-1}[h-h(0)](R-Q-1-S_0(f))]\!]$ then $K \in \mathcal{D}_{h(0)}^{(N,I,0)}$;*

*3. if $E = [\![R-Q-1]\!]$, $P = [\![R-Q-1-S_0(f)]\!]$ then $f \in \mathcal{C}^{E+1}(0,C)$ and $h(x) = h(0) + x\phi(x)$ with $\phi \in \mathcal{C}^{P+1}(0,\mathbb{R})$ then*

$$\begin{aligned}
M_h(\epsilon) &= \boldsymbol{fp}\int_a^b f(x)K[x,\epsilon h(x)]\,\mathrm{d}x \\
&= o(\epsilon^R) + \sum_{n=0}^{[\![R]\!]} \frac{\partial_2^n K(1,0)}{n!}\left[\boldsymbol{fp}\int_a^b S(x)f(x)[h(x)]^n x^{Q-n}\,\mathrm{d}x\right]\epsilon^n \\
&\quad + \sum_{m=S_0(f)}^{[\![R-Q-1]\!]} \sum_{l=S_0(f)}^{m} \frac{f^{(l)}(0)}{l!}\left[\sum_{i=0}^{m-l}\frac{a^i_{m-l-i}}{i!}\right]\left[\boldsymbol{fp}\int_{-\infty}^{\infty}\partial_2^i K(u,h(0))u^m\,\mathrm{d}u\right]\epsilon^{Q+m+1} \\
&\quad -[1-S(-1)]\left[\sum_{n=\max[0,S_0(g)+Q+1]}^{[\![R]\!]} \sum_{l=S_0(g)}^{n-Q-1}\sum_{i=0}^{n-Q-l-1}\frac{[h(0)]^{n-i}f^{(l)}(0)}{i!(n-i)!l!}a^i_{n-Q-l-i-1}\right] \\
&\quad \times \partial_2^n K(1,0)\epsilon^n \log\epsilon, 
\end{aligned} \tag{4.25}$$

*where the coefficients $a^i_p$ are defined by relations (4.20).*

*Proof.* First, application of proposition 15 with $\mu = Q$, $q = 0 = \beta$ and $L = K$ immediately yields the asymptotic behaviour of the first integral, $M_h^1(\epsilon)$, arising on the right-hand side of (4.24). More precisely, the reader may check that, under the specified assumptions

$$\begin{aligned}
M_h^1(\epsilon) &= \boldsymbol{fp}\int_0^b f(x)K[x,\epsilon h(x)]\,\mathrm{d}x \\
&= \sum_{n=0}^{[\![R]\!]} \frac{\partial_2^n L(1,0)}{n!}\left[\boldsymbol{fp}\int_a^b f(x)[h(x)]^n x^{Q-n}\,\mathrm{d}x\right]\epsilon^n \\
&\quad + \sum_{m=S_0(f)}^{[\![R-Q-1]\!]} \sum_{l=S_0(f)}^{m} \frac{f^{(l)}(0)}{l!}\left[\sum_{i=0}^{m-l}\frac{a^i_{m-l-i}}{i!}\right]\left[\boldsymbol{fp}\int_0^{\infty}\partial_2^i L(u,h(0))u^m\,\mathrm{d}u\right]\epsilon^{Q+m+1} \\
&\quad -\left[\sum_{n=\max[0,S_0(g)+Q+1]}^{[\![R]\!]} \sum_{l=S_0(g)}^{n-Q-1}\sum_{i=0}^{n-Q-l-1}\frac{[h(0)]^{n-i}f^{(l)}(0)}{i!(n-i)!l!}a^i_{n-Q-l-i-1}\right] \\
&\quad \times \partial_2^n L(1,0)\epsilon^n \log\epsilon + o(\epsilon^R).
\end{aligned} \tag{4.26}$$

The remaining integral $M_h^2(\epsilon) = M_h(\epsilon) - M_h^1(\epsilon)$ is treated by taking into account the above formula (4.26) and also definitions $g(x) = f(-x)$, $H(x) = -h(-x)$ which ensure $g^{(l)}(0) = (-1)^l f^{(l)}(0)$,

$$x^{-i}[H(x)-H(0)]^i = \sum_{p=0}^{P-i}(-1)^p a^i_p x^p + o(x^{P-i})$$

and also property $\partial_2^i L[u,-h(0)] = (-1)^{Q-i}S(-1)\partial_2^i L[-u,h(0)]$. ∎

To conclude, a theorem extending the results proposed by theorem 14 is given.

**Theorem 17.** *Consider* $0 < b < +\infty$, $f \in D_+(0, C) \cap L^1_{\text{loc}}(]0, b], C)$ *with* $S_0(f) := S_0^+(f)$, $g \in D_+(0, C) \cap L^1_{\text{loc}}(]0, +\infty[, C)$ *with* $S_0(g) := S_0^+(g) = 0$ *and* $S_\infty(g) := S_0^+(g_1)$ *if* $g_1(x) := g(x^{-1})$ *for* $x > 0$, $L(x, y)$ *a complex and 'μ homogeneous' kernel (see (1.2)). Moreover* $h$ *is a real function bounded in* $[0, b]$ *with* $h(0) = c$ *and fulfilling property 1 of definition 5, i.e. such that there exist* $\gamma \in \mathbb{R}_+^*$, $\eta > 0$, $h_0$ *with* $h(x) - c =^{+}_{-} x^\gamma h_0(x)$, $h_0(x) > 0$,

$$h_0(x) = \sum_{m=0}^{M} a_m \log^m(x) + o(x)$$

*in* $]0, \eta]$ *with* $a_M \neq 0$. *For* $r \geqslant 0$, *if* $r$, $f$, $g$, $h$ *and* $L$ *satisfy each of the next properties:*

*1. for* $\lambda$ *large enough,*

$$v_\lambda(x) := f(x) g(\lambda x) L[\lambda x, h(x)] \in \mathcal{P}(]0, b[, C);$$

*2.* $r \geqslant \max[-\mu + S_\infty(g), 1 + S_0(f)]$,

$$\lim_{x \to 0^+} f(x) = \sum_{k, \text{Re}(\delta_l) \leqslant r-1} x^{\delta_l} f^0_{lk} \log^k(x)$$

*and also*

$$\lim_{x \to 0^+} g(x) = \sum_{q, \text{Re}(\beta_p) \leqslant -1 - S_0(f)} g^0_{pq} x^{\beta_p} \log^q(x);$$

*3. if*

$$N := [\![ r + \mu - S_\infty(g) ]\!],$$
$$I := [\![ \gamma^{-1}(r - 1 - S_0(f)) ]\!]$$

*and*

$$M := \max([\![ -1 - S_0(f) ]\!] + 1, 0)$$

*then* $L \in \mathcal{D}_c^{(N,I,M)}$;

*4. if there exists* $n \in \{0, \ldots, N\}$ *with* $\partial_2^n L(1, 0) \neq 0$ *then* $n_1 := \min\{n \in \{0, \ldots, N\}, \partial_2^n L(1, 0) \neq 0\}$ *and* $g \in \mathcal{E}^{r+\mu-n_1}_{-1-S_0(f)}(]0, +\infty[, C)$ *with* $T > r + \mu - n_1$ *and* $G^\infty$ *bounded in a neighbourhood of infinity where*

$$g(u) = \sum_{e=0}^{E} \sum_{m=0}^{M(e)} g^\infty_{em} u^{-\Lambda_e} \log^m(e) + u^{-T} G^\infty(u), \tag{4.27}$$

*with* $S_\infty(g) = \text{Re}(\Lambda_0) < \cdots < \text{Re}(\Lambda_E) \leqslant T$, *else if* $\partial_2^n L(1, 0) = 0$, $\forall n \in \{0, \ldots, N\}$ *then* $n_1 := N + 1$ *and*

$$\sum_{n=n_1}^{N} F(n) := 0;$$

*5.* $\forall i \in \{0, \ldots, I\}$, $[h(x) - c]^i \in D_+^{r-1-S_0(f)}(0, C)$ *with (see (4.1))*

$$[h(x) - c]^i = \sum_{p_i=0}^{P_i} \sum_{q_i=0}^{Q_i(p_i)} \Delta^i_{p_i q_i} x^{\tau^i_{p_i}} \log^{q_i}(x) + x^{s_i} \Delta^i(x);$$

then

$$
\begin{aligned}
I(\lambda) &= \boldsymbol{fp}\int_0^b f(x)g(\lambda x)L[\lambda x, h(x)]\,\mathrm{d}x \\
&= o(\lambda^{-r}) + \sum_{m,\mathrm{Re}(\Lambda_e)+n-\mu\leqslant r}^{n,e,m}\sum_{v=0}^{m} C_m^v g_{em}^{\infty}\frac{\partial_2^n L(1,0)}{n!} \\
&\quad\times\left[\boldsymbol{fp}\int_0^b f(x)[h(x)]^n x^{\mu-\Lambda_e-n}\log^{m-v}(x)\,\mathrm{d}x\right]\lambda^{-(n+\Lambda_e-\mu)}\log^v\lambda \\
&\quad+\sum_{\mathrm{Re}(\tau_{p_i}^i+\delta_l)\leqslant r-1}^{l,k,i,p_i,q_i} f_{lk}^0 \Delta_{p_i q_i}^i \sum_{v=0}^{q_i+k} C_{q_i+k}^v(-1)^v \\
&\quad\times\Bigg\{\boldsymbol{fp}\int_0^\infty \frac{g(u)\partial_2^i L(u,c)}{i!}u^{\tau_{p_i}^i+\delta_l}\log^{q_i+k-v}(u)\,\mathrm{d}u \\
&\qquad-\sum_{\{j;j+\beta_p=-\tau_{p_i}^i-\delta_l-1\}}^{j,p}\sum_{q=0}^{Q(p)} g_{pq}^0\frac{\partial_1^j\partial_2^i L(0,c)}{i!j!}\frac{\log^{1+q_i+k+q-v}\lambda}{1+q_i+k+q-v} \\
&\qquad+\sum_{\{j;j+\Lambda_e=1-i+\mu+\tau_{p_i}^i+\delta_l\}}^{j,e}\sum_{m=0}^{M(e)}\frac{g_{em}^{\infty}c^j\partial_2^{i+j}L(1,0)\log^{1+q_i+k+m-v}\lambda}{i!j![1+q_i+k+m-v]}\Bigg\} \\
&\quad\times\lambda^{-(\tau_{p_i}^i+\delta_l+1)}\log^v\lambda.
\end{aligned}
\tag{4.28}
$$

Note that this theorem allows us to treat also the case of $S_0(g)\neq 0$. This is achieved by choosing $G(u)=g(u)u^{-S_0(g)}$ and $F(x):=x^{S_0(g)}f(x)$. Hence,

$$
\begin{aligned}
I(\lambda) &= \boldsymbol{fp}\int_0^b f(x)g(\lambda x)L[\lambda x, h(x)]\,\mathrm{d}x \\
&= \lambda^{S_0(g)}\boldsymbol{fp}\int_0^b F(x)G(\lambda x)L[\lambda x, h(x)]\,\mathrm{d}x
\end{aligned}
$$

with $S_0(G)=0$.

*Proof.* Naturally $I(\lambda)$ exists and it is now understood that each notation $r_1=r-1$, $r_2=r$, $K_n^i$, $V_n$, $t_1$, $t_1'$, $H_{t_1}$, $S_\lambda$, $\eta'$, $w_{t_1',s_2}$, $d_m(X-c)$ keeps its meaning as introduced for the derivation of theorem 14, unless it is clearly modified. For $K(x,u):=g(u)L(u,x)$ the proof is similar to the one employed for theorem 14 and it is therefore only briefly reported below. ■

If $N:=[\![r+\mu-S_\infty(g)]\!]$, when $u\to+\infty$, combination of expansion (4.27) for function $g$ and for '$\mu$ homogeneous' function $L(u,X)$ of the following decomposition

$$
\begin{aligned}
L(X,u) &= \sum_{n=0}^{N}\frac{\partial_2^n L(1,0)}{n!}X^n u^{-(n-\mu)} \\
&\quad+\frac{u^{-(N+1-\mu)}X^{N+1}}{N!}\int_0^1(1-t)^N\partial_2^{N+1}L(1,tX/u)\,\mathrm{d}t
\end{aligned}
\tag{4.29}
$$

yields the equality (3.2) in the form

$$K(X,u)=\sum_{\mathrm{Re}(\Lambda_e)+n-\mu\leqslant r}^{n,e}\ \sum_{m=0}^{M(e)}\frac{g_{em}^{\infty}\partial_2^n L(1,0)}{n!}X^n u^{-(n+\Lambda_e-\mu)}\log^m u+A(X,u),\quad (4.30)$$

where the complex function $A(X,u)$ obeys

$$\begin{aligned}A(X,u)&=g(u)u^{-(N+1-\mu)}\frac{X^{N+1}}{N!}\int_0^1(1-t)^N\partial_2^{N+1}L(1,tX/u)\,\mathrm{d}t\\&\quad+\sum_{\mathrm{Re}(\Lambda_e)+n-\mu>r}^{n,e,m}\frac{g_{em}^{\infty}\partial_2^n L(1,0)}{n!}X^n u^{-(n+\Lambda_e-\mu)}\log^m u\\&\quad+\sum_{n=n_1}^{N}\frac{\partial_2^n L(1,0)}{n!}X^n u^{-(n+T-\mu)}G^{\infty}(u).\end{aligned}\quad (4.31)$$

By introducing successively real values $s>r$ with $s:=\inf\{\mathrm{Re}(\Lambda_e)+n-\mu$ for $(n,e)\in\{n_1,N\}\times\{0,\dots,E\}$ such that $\mathrm{Re}(\Lambda_e)+n-\mu>r\}$, $t:=[N+1-\mu+S_\infty(g)-r]/2>0$, $s':=\min(s,T+n_1-\mu,r+t)>r$ and finally $s_2:=r+(s'-r)/2>r$ then $r+t>s_2$ and $A(X,u)$ may be rewritten $A(X,u)=u^{-s_2}V_{r_2}(X,u)$ with $V_{r_2}(X,u)=V_1(X,u)+V_2(X,u)+V_3(X,u)$ if $u^{-s_2}V_2(X,u)$ is the last sum on the right-hand side of (4.31) and

$$\begin{aligned}V_1(X,u)&=\sum_{\mathrm{Re}(\Lambda_e)+n-\mu>r}^{n,e,m}\frac{g_{em}^{\infty}\partial_2^n L(1,0)}{n!}X^n u^{-(n+\Lambda_e-\mu-s_2)}\log^m u,\\V_3(X,u)&=[g(u)u^{S_\infty(g)-t}]u^{-(r+t-s_2)}\frac{X^{N+1}}{N!}\int_0^1(1-t)^N\partial_2^{N+1}L(1,tX/u)\,\mathrm{d}t.\end{aligned}\quad (4.32)$$

For $\eta\geqslant b$, (3.3) holds (see theorem 14). If $\eta<b$, the above decomposition of $A(X,u)$ and the choice of $s_2$ prove that (3.3) is true.

As a result of (4.30), $\gamma_{ne}=n+\Lambda_e-\mu$, $K_{nem}(X)=g_{em}^{\infty}\partial_2^n L(1,0)X^n/n!$ and $g_{nem}(x):=f(x)K_{nem}[h(x)]\in L^1_{\mathrm{loc}}(]0,b[,C)$. For $t_n:=\max[r-1-S_0(f),\mathrm{Re}(\gamma_{ne})-1-S_0(f)]=r-1-S_0(f)\geqslant 0$ and $I:=[\![\gamma^{-1}t_n]\!]\geqslant 0$, (4.5) is replaced, in a neighbourhood of $c$, for $t_1'=I+1>t_1=\gamma^{-1}t_n$ by

$$K_{nem}(X)=\sum_{i=0}^{I}K_{nem}^i(X-c)^i+|X-c|^{t_1'}V_{nem}(X-c),\quad (4.33)$$

with $K_{nem}^i=g_{em}^{\infty}K_n^i$, $V_{nem}(X-c)=g_{em}^{\infty}V_n(X-c)$. Expansion (4.1) ensures that $(h,K_{nem})\in\Pi_+^{r-1-S_0(f),1}(0,C)$.

If $X\to c$, expansion (4.7) of $K(X,u)$ is replaced by

$$K(X,u)=g(u)L(u,X)=\sum_{i=0}^{I}a^i(u)(X-c)^i+g(u)|X-c|^{t_1'}H_{t_1}(X-c,u)$$

with $a^i(u):=g(u)\partial_2^i L(u,c)/i!$.

Note that $D_g(x):=g(\lambda x)f(x)|h(x)-c|^{t_1'}H_{t_1}[h(x)-c,\lambda x]=g(\lambda x)x^{r-1+\eta'}B(x)$ with the function $B$ bounded near zero and $\eta':=[\gamma(I+1)+1+S_0(f)-r]/3>0$. Moreover, $S_0(g)=0$ and $g\in L^1_{\mathrm{loc}}(]0,+\infty[,C)$ ensures that $g\in L^1_{\mathrm{loc}}([0,+\infty[,C)$.

Thereafter, relation (3.6) is fulfilled because

$$\left|\int_0^{A/\lambda} g(\lambda x)f(x)|h(x)-c|^{t_1'}H_{t_1}[h(x)-c,\lambda x]\,\mathrm{d}x\right| \leqslant \left[\int_0^A \frac{|g(u)B(u/\lambda)|\,\mathrm{d}u}{u^{1-r-\eta'}}\right]\lambda^{-(r+\eta')}.$$

$S_0(g)=0$ and the definition of $a^i$ show that $S_0(a^i)\geqslant 0\geqslant -r=-r_1-1$. Here $R'=r$, $R=-1-S_0(f)$. Under the proposed assumptions it is possible to write near zero

$$a^i(u)=\sum_{\mathrm{Re}(\beta_p)+j\leqslant -1-S_0(f)}^{p,q,j} g^0_{pq}\partial_1^j\partial_2^i L(0,c)u^{\beta_p+j}\log^q(x)/[i!j!]+o(u^R)$$

with $0\leqslant j\leqslant M:=\max([\![-1-S_0(f)]\!]+1,0)$. If $i\geqslant N+1$ then $a^i(u)=u^{-s_2}O_i(u)$ with

$$i!O_i(u)=[g(u)u^{S_\infty(g)-t}][u^{-(i-N-1)}\partial_2^i L(1,c/u)]u^{-(r+t-s_2)},$$

i.e. $O_i$ is bounded near infinity. For $0\leqslant i\leqslant N$, introduction of the positive integer $J_i=N-i$ leads to

$$\frac{a^i(u)}{g(u)}=\sum_{j=0}^{J_i}\frac{\partial_2^{i+j}L(1,0)}{i!j!}c^j u^{-(i+j-\mu)}+\frac{u^{-(N+1-\mu)}}{J_i!}\int_0^1(1-t)^{J_i}\partial_2^{N+1}L(1,tc/u)\,\mathrm{d}t. \tag{4.34}$$

The above approach for the expansion of $K(X,u)$ when $u\to+\infty$ (see (4.29), (4.30) and (4.31)) and a change of index $n=i+j$ show that

$$a^i(u)=\sum_{\mathrm{Re}(\Lambda_e)+n-\mu\leqslant r}^{n,e,m} E_{(n,i)}\frac{g^\infty_{em}\partial_2^n L(1,0)}{i!(n-i)!}c^{n-i}u^{-(n+\Lambda_e-\mu)}\log^m(u)+u^{-s_2}O_i(u), \tag{4.35}$$

with $O_i(u)$ bounded near infinity and $E_{(n,i)}:=1$ if $n\geqslant i$, else $E_{(n,i)}:=0$. Thus, (3.7) is true.

Consider the new function

$$F(X,u)=g(u)u^{-(I+1-\mu)}\frac{(X-c)^{t_1'}}{I!}\int_0^1(1-t)^I\partial_2^{I+1}L[1,c+t(X-c)/u]\,\mathrm{d}t. \tag{4.36}$$

If $I\geqslant N$, relation

$$g(u)u^{-(I+1-\mu)}=g(u)u^{-(N+1-\mu)}u^{-(I-N)}=u^{-s_2}[g(u)u^{S_\infty(g)-t}]u^{-(r+t-s_2)}u^{-(I-N)}$$

and $V_{nem}=0$ show that $W_{t_1',s_2}(X-c,u)=g(u)w_{t_1',s_2}(X-c,u)$ is bounded for $(X,u)\in h([0,\eta])\times[A,+\infty[$.

If $I\leqslant N-1$, use of decomposition (4.14) ensures that this time with $J:=N-I-1\geqslant 0$ and $n=I+1+m$

$$\frac{F(X,u)}{g(u)}=\sum_{m=0}^{J}d_m(X-c)u^{-(I+1+m-\mu)}+u^{-(N+1-\mu)}\frac{(X-c)^{t_1'}}{I!J!}R(X,u)$$

$$=\sum_{n=I+1}^{N}d_{n-I-1}(X-c)u^{-(n-\mu)}+u^{-(N+1-\mu)}\frac{(X-c)^{t_1'}}{I!J!}R(X,u),$$

$$R(X,u)=\int_0^1(1-t)^I[c+t(X-c)]^J\left[\int_0^1(1-v)^J\partial_2^{N+1}L[1,v(c+\frac{t(X-c)}{u})]\,\mathrm{d}v\right]\mathrm{d}t.$$

Hence, it is possible to choose $W_{t'_1,s_2}$ bounded for $(X,u) \in h([0,\eta]) \times [A,+\infty[$ and such that

$$|X-c|^{t'_1} W_{t'_1,s_2}(X-c,u) = \sum_{\operatorname{Re}(\Lambda_e)+n-\mu>r}^{n,e,m} g^{\infty}_{em} d_{n-I-1}(X-c) u^{-(n+\Lambda_e-\mu-s_2)} \log^m u$$
$$+ \sum_{n=I+1}^{N} d_{n-I-1}(X-c) u^{-(T+n-\mu-s_2)} G^{\infty}(u)$$
$$+ g(u) u^{-(N+1-\mu-s_2)} \frac{(X-c)^{t'_1}}{I!J!} R(X-c,u).$$

*Example 5.* For $q \in \mathbb{N}$, $\beta \in C$, $c \in \mathbb{R}^*$, $0 < b < +\infty$ and $r \geqslant \max[2, 1-\operatorname{Re}(\beta)]$ then

$$\boldsymbol{fp} \int_0^1 \frac{\log^q(x)\,\mathrm{d}x}{x^{\beta}(1+\lambda x)(c+x\log x+\lambda x)}$$
$$= \sum_{j=0}^{[\![r-2]\!]} (-1)^j \sum_{n=0}^{j} \left[ \boldsymbol{fp} \int_0^1 [c+x\log x]^n x^{-(\beta+j+2)} \log^q(x)\,\mathrm{d}x \right] \lambda^{-(j+2)}$$
$$+ \sum_{i=0}^{I} \sum_{v=0}^{q+i} C^v_{q+i} (-1)^v \left\{ \boldsymbol{fp} \int_0^{\infty} \frac{(-1)^i u^{i-\beta} \log^{q+i-v}(u)\,\mathrm{d}u}{(1+u)(u+c)^{i+1}} \right.$$
$$+ \frac{\log^{1+q+i-v}\lambda}{1+q+i-v} \left[ E(-\beta)(-1)^{i-\beta-1} \sum_{e=0}^{-\beta-1} \frac{(i-\beta-1-e)!\,c^{-\beta-1-e}}{i!(-\beta-1-e)!} \right.$$
$$\left.\left. -E(\beta-i)(-1)^{\beta-1} \sum_{p=0}^{\beta-1-i} \frac{(\beta-1-p)!\,c^{p-\beta}}{i!(\beta-1-i-p)!} \right] \right\}$$
$$\times \lambda^{-(i-\beta+1)} \log^v \lambda + o(\lambda^{-r}), \tag{4.37}$$

where $I = [\![r-1+\operatorname{Re}(\beta)]\!]$ and for $z \in C, E(z) := 1$ if $z \in \mathbb{N} \setminus \{0\}$, else $E(z) := 0$. Here $f(x) = x^{-\beta}\log^q(x)$, $S_0(f) = -\operatorname{Re}(\beta)$, $h(x) = c + x\log x$, $\gamma = 1$, $h(0) = c \neq 0$, $[h(x)-h(0)]^i = x^i \log^i x$, $P_i = i$, $Q_i(p_i) = i$, $\Delta^i_{p_i q_i} = \delta_{(p_i,i)}$, $\tau^i_{p_i} = i\delta_{(p_i,i)}$, $g(u) = 1/[1+u]$, $S_0(g) = 0$, $\beta_p = p$, $Q(p) = 0$, $g^0_p = (-1)^p$, $\Lambda_e = e+1$, $M(e) = 0$, $g^{\infty}_e = (-1)^e$, $L(x,y) = 1/[x+y]$, $\mu = -1$, $N = [\![r-2]\!]$, $I = [\![r-1+\operatorname{Re}(\beta)]\!]$, $M = \max([\![-1+\operatorname{Re}(\beta)]\!]+1, 0)$ and $L \in \mathcal{D}_c^{(N,I,M)}$.

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