Application of the Hadamard's finite part concept to the asymptotic expansion of a class of multidimensional integrals

By A. Sellier

LADHYX, Ecole Polytechnique, 91128 Palaiseau Cedex, France

Contents

The aim of this paper is to study the expansion with respect to the large and real parameter λ of the integral

$$I_n^f(\lambda) := \int_0^1 \cdots \int_0^1 x_1^{\alpha_1} \dots x_n^{\alpha_n} \log^{l_1}(x_1) \dots \log^{l_n}(x_n) \times f(x_1, \dots, x_n) g(\lambda x_1^{a_1} \dots x_n^{a_n}) \, \mathrm{d}x_1 \dots \mathrm{d}x_n,$$

where for $i \in \{1, \dots, n\}$: $a_i > 0$, $l_i \in \mathbb{N}$ and α_i is complex with $\mathrm{Re}(\alpha_i) > -1$. Moreover, f is a smooth enough function and g belongs to $\mathcal{B}_r(]0, +\infty[)$, a space defined below. The derivation of such an asymptotic expansion is established by induction on the integer n and makes use of a basic concept: the integration in the finite part sense of Hadamard.

1. Introduction

In many problems, it is worth finding the asymptotic expansion with respect to the large and real parameter λ of the integral

$$J_1^h(\lambda) := \int_0^1 h(x_1) g(\lambda x_1^{a_1}) \, \mathrm{d}x_1, \tag{1.1}$$

where $a_1 > 0$ and the pseudofunctions h and g present more or less restrictive behaviour, respectively, near the critical point zero on the right and at infinity. Several methods are available to deal with $J_1^h(\lambda)$. The reader is for instance referred to Bleistein & Handelsman (1975) or Wong (1989) for an approach based on the Mellin transform and also to Estrada & Kanwal (1990) for a distributional point

Phil. Trans. R. Soc. Lond. A (1996) **354**, 1195–1246

Printed in Great Britain

TEX Paper

of view (here obtained by writing $J_1^h(\lambda) = \langle G(\lambda' x_1), h(x_1)\rangle$ if $\lambda' := \lambda^{1/a_1}$, G is the generalized function such that $G(X) := g(X^{a_1})$ and $\langle , \rangle$ designates a duality bracket). Moreover (see Sellier 1994, theorem 3), the concept of integration in the finite part sense of Hadamard (1932) turns out to be an alternative and powerful tool to treat the integral $J_1^h(\lambda)$ when pseudo–functions h and g offer, respectively, near zero and at infinity an expansion involving complex powers of x_1 and integer powers of $\log(x_1)$. In such a case and for given λ, the function $w(x_1) := h(x_1)g(\lambda x_1^{a_1})$ may be not integrable on a neighbourhood of zero and $J_1^h(\lambda) = \boldsymbol{fp}\int_0^1 h(x_1)g(\lambda x_1^{a_1})\,\mathrm{d}x_1$, where $\boldsymbol{fp}$ means an integration in the finite part sense of Hadamard (see § 2).

A special case of (1.1) is provided by

$$I_1^h(\lambda) := \int_0^1 x_1^{\alpha_1}\log^{l_1}(x_1) f(x_1) g(\lambda x_1^{a_1})\,\mathrm{d}x_1, \tag{1.2}$$

with $\alpha_1 \in C$ if C designates the set of complex numbers, $\mathrm{Re}(\alpha_1) > -1, l_1 \in \mathbb{N}$, $f \in \mathcal{C}^\infty(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, the Schwartz space. Under these strong assumptions the asymptotic expansion of $I_1^h(\lambda)$ is easy to obtain (the reader may consult Bruning (1984), theorem 1, case $n = 1$). Nevertheless, it is possible to weaken the assumptions both bearing on functions f and g by using Hadamard's concept. This is achieved by introducing for $r \geqslant 0$ a set containing $\mathcal{S}(\mathbb{R})$ which is the space of complex pseudo-functions $\mathcal{B}_r(]0,+\infty[) :=$ {complex function g bounded on $\mathbb{R}_+$ and there exists a neighbourhood of infinity in which the function g obeys $g(x) = \sum_{e=0}^{E}\sum_{m=0}^{M(e)} g_{em}x^{-\Lambda_e}\log^m(x) + x^{-s}G_r^\infty(x)$ with function G_r^∞ bounded and $\mathrm{Re}(\Lambda_0) < \cdots < \mathrm{Re}(\Lambda_E) \leqslant r < s$}. Then (see § 3, theorem 11), one obtains up to $o(\lambda^{-r})$ the asymptotic expansion of $I_1^h(\lambda)$ when f is bounded on $[0,1]$, smooth enough on a neighbourhood on the right of zero and $g \in \mathcal{B}_r(]0,+\infty[)$.

The question of finding the asymptotic expansion of multidimensional integrals is more difficult and not so much investigated. When $g(t) := \mathrm{e}^{\mathrm{i}\beta t}$, the method of steepest descents is used (see Bleistein & Handelsman 1975; Wong 1989). Observe that Estrada & Kanwal (1992, 1994) dealt with the asymptotic expansion of certain multidimensional generalized functions. In this paper we are interested in expanding for $n \geqslant 1$, $(a_1,\ldots,a_n) \in \mathbb{R}_+^{*n}$, $(l_1,\ldots,l_n) \in \mathbb{N}^n$ and $(\alpha_1,\ldots,\alpha_n) \in C^n$ with $\mathrm{Re}(\alpha_i) > -1$ the integral

$$I_n^f(\lambda) := \int_0^1 \cdots \int_0^1 x_1^{\alpha_1}\ldots x_n^{\alpha_n}\log^{l_1}(x_1)\ldots\log^{l_n}(x_n) \times f(x_1,\ldots,x_n) g(\lambda x_1^{a_1}\ldots x_n^{a_n})\,\mathrm{d}x_1\ldots\mathrm{d}x_n, \tag{1.3}$$

which is an extension of $I_1^f(\lambda)$ to $n \geqslant 2$. According to Bruning & Heintze (1984) and also Bruning (1984), such a question is crucial when deriving an asymptotic expansion of the trace of the equivariant heat kernel. Barlet (1982), and also Bruning (1984), indeed showed that for $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R})$ and $(\alpha_1,\ldots,\alpha_n) \in \mathbb{R}_+^{*n}$,

$$I_n^f(\lambda) = \sum_{i=1}^{n}\sum_{p\in\mathbb{N}}\sum_{j=0}^{J} < T_n^{i,p,j}, f > \log^j(\lambda)\lambda^{-(\alpha_i+p+1)/a_i}, \tag{1.4}$$

with $J := l_1+\cdots+l_n+n-1$ and $T_n^{i,p,j}$ designates a generalized function with support in the set $\{x = (x_1,\ldots,x_n) \in [0,1]^n; \prod_{j=1}^n x_j = 0\}$. Unfortunately, these works do not provide the distributions $T_n^{i,p,j}$. This was done by using Mellin transform for $n = 2$, $(l_1,l_2) = (0,0)$ and $(\alpha_1,\alpha_2) \in \mathbb{R}_+^{*2}$ by McClure & Wong (1987). Accordingly, this study was mainly motivated by the paper by Bruning (1984), i.e. the very first aim was to detail the above-mentioned generalized functions $T_n^{i,p,j}$.

Since the proposed approach also permits us to deal with a function g belonging to a space larger than $\mathcal{S}(\mathbb{R})$, this paper presents the derivation of the asymptotic expansion of $I_n^f(\lambda)$ up to order $o(\lambda^{-r})$ when $g \in \mathcal{B}_r(]0,+\infty[)$ and f is a smooth enough function. Not only each generalized function $T_n^{i,p,j}$ but also new coefficients $\langle G_n^{e,j}, f\rangle$ (related to the behaviour of function g near infinity) will be given. This expansion is proposed for $(l_1,\ldots,l_n) \in \mathbb{N}^n$, $(\alpha_1,\ldots,\alpha_n) \in C^n$ with $\mathrm{Re}(\alpha_i) > -1$ and is based on the use of the integration in the finite part sense of Hadamard.

This work is organized as follows. In § 2, multidimensional integration in the finite part sense of Hadamard and a useful lemma are introduced. Two basic theorems are exhibited in § 3. As a consequence, the case $n = 1$ (see theorem 11) and an important lemma (see lemma 12) are proved. The asymptotic expansion of $I_n^f(\lambda)$ and a key proposition are presented in § 4. Finally, the derivation of the stated theorem is achieved in several steps and by induction in §§ 5 and 6.

2. The concept of integration in the finite part sense of Hadamard

As outlined in the above introduction, the important tool for this paper is the concept of multidimensional integration in the finite part sense of Hadamard. Thereby this section not only introduces both this notion and appropriate functional spaces but also useful results. For detailed explanations regarding the one-dimensional case the reader may successively consult Hadamard (1932), Schwartz (1966) and Sellier (1994). By now, C denotes the set of complex numbers.

Definition 1. For $r \succ 0$, the complex function h is of the second kind on the set $]0,r[$ if, and only if, there exist a complex function H, a family of positive integers $(M(n))$, and two complex families (β_n) and (g_{nm}) such that

$$\forall \epsilon \in]0,r[, \quad h(\epsilon) = \sum_{n=0}^{N} \sum_{m=K(n)}^{M(n)} h_{nm}\epsilon^{\beta_n} \log^m(\epsilon) + H(\epsilon), \tag{2.1}$$

$$\begin{gathered}\mathrm{Re}(\beta_N) < \mathrm{Re}(\beta_{N-1}) < \ldots < \mathrm{Re}(\beta_1) < \mathrm{Re}(\beta_0) := 0, \\ \lim_{\epsilon\to 0} H(\epsilon) \in C \text{ and } h_{00} := 0 \text{ for } \beta_0 = 0.\end{gathered} \tag{2.2}$$

According to Hadamard's concept (see Hadamard 1932; Schwartz 1966) the finite part in the Hadamard sense of the quantity $h(\epsilon)$, noted $\boldsymbol{fp}[h(\epsilon)]$, is the complex $\lim_{\epsilon\to 0} H(\epsilon)$.

Definition 2. A complex function $f \in L^1_{\mathrm{loc}}(]0,+\infty[,C)$ belongs to $\mathcal{E}(]0,+\infty[,C)$ if, and only if, the function h_f defined on $]0,r]$ by $h_f(\epsilon) := \int_\epsilon^{1/\epsilon} f(x)\,\mathrm{d}x$ is of the second kind. Moreover, the linear transformation $\boldsymbol{fp}$ acting on $\mathcal{E}(]0,+\infty[,C)$ is such that for any $f \in \mathcal{E}(]0,+\infty[,C)$: $\boldsymbol{fp}\int_0^\infty f(x)\,\mathrm{d}x := \boldsymbol{fp}[h_f(\epsilon)]$.

Definition 3. A complex function f belongs to the set $\mathcal{P}(]0,+\infty[,C)$ if, and only if, $f \in L^1_{\mathrm{loc}}(]0,+\infty[,C)$ and there exist positive reals η_f and A_f, two functions $F^0 \in L^1([0,\eta_f],C)$ and $F^\infty \in L^1([A_f,+\infty[,C)$, two families of positive integers

$(J(i))$, $(K(l))$ and complex families (α_i), (f^0_{ij}), (γ_l) and (f^∞_{lk}) such that

$$f(x)=\sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^0_{ij}x^{\alpha_i}\log^j x+F^0(x),\quad \text{a.e. in }]0,\eta_f],$$
$$\mathrm{Re}(\alpha_I)<\mathrm{Re}(\alpha_{I-1})<\ldots<\mathrm{Re}(\alpha_1)<\mathrm{Re}(\alpha_0):=-1; \tag{2.3}$$

$$f(x)=\sum_{l=0}^{L}\sum_{k=0}^{K(l)} f^\infty_{lk}x^{-\gamma_l}\log^k x+F^\infty(x),\quad \text{a.e. in } [A_f,+\infty[,$$
$$\mathrm{Re}(\gamma_L)<\mathrm{Re}(\gamma_{L-1})<\ldots<\mathrm{Re}(\gamma_1)<\mathrm{Re}(\gamma_0):=1. \tag{2.4}$$

Observe that definition 3 implies a specific behaviour of the function f near zero and near infinity. For instance, if there exists (i,j) with $f^0_{ij}\neq 0$ then f is singular at zero (i.e. admits no limit as x tends to zero) and is not integrable on the set $]0,\eta_f]$.

Proposition 4. *For $j\in\mathbb{N}$, $\alpha\in C$ and two real values a and b with $0\prec a\prec b$, then*

$$\int_a^b x^\alpha\log^j x\,\mathrm{d}x=P^j_\alpha(b)-P^j_\alpha(a), \tag{2.5}$$

$$\textit{with } P^j_{-1}(t):=\frac{\log^{j+1}(t)}{j+1},\quad \textit{else } P^j_\alpha(t):=t^{\alpha+1}\sum_{k=0}^{j}\frac{(-1)^{j-k}j!}{k!(\alpha+1)^{1+j-k}}\log^k(t). \tag{2.6}$$

This result leads to the next proposition.

Proposition 5. *$\mathcal{P}(]0,+\infty[,C)\subset\mathcal{E}(]0,+\infty[,C)$ and for any $f\in\mathcal{P}(]0,+\infty[,C)$,*

$$\boldsymbol{fp}\int_0^\infty f(x)\,\mathrm{d}x:=\boldsymbol{fp}\left[\int_\epsilon^{1/\epsilon}f(x)\,\mathrm{d}x\right]=\int_\delta^B f(x)\,\mathrm{d}x$$
$$+\int_0^\delta F^0(x)dx+\int_B^\infty F^\infty(x)\,\mathrm{d}x+\sum_{i=0}^{I}\sum_{j=0}^{J(i)}f^0_{ij}P^j_{\alpha_i}(\delta)-\sum_{l=0}^{L}\sum_{k=0}^{K(l)}f^\infty_{lk}P^k_{-\gamma_l}(B), \tag{2.7}$$

for any pair (δ,B) such that $0\prec\delta\leqslant\eta_f$ and $A_f\leqslant B\prec+\infty$.

For $f\in\mathcal{P}(]0,+\infty[,C)$ and according to definition 3 and proposition 4 the function $h(\epsilon):=\int_\epsilon^{1/\epsilon}f(x)\,\mathrm{d}x$ is indeed of the second kind. Application of definition 1 and some algebra ensure the equality (2.7).

Change of variable when dealing with this specific integration may generate corrective terms. The following lemma gives those additional terms for two useful changes of variable when the function belongs to $\mathcal{P}(]0,+\infty[,C)$.

Lemma 6. *Consider $f\in\mathcal{P}(]0,+\infty[,C)$ and two real values λ and a with $\lambda\neq 0$ and $a>0$. If we set $\mathrm{sgn}(\lambda):=\lambda/|\lambda|$,*

$$\boldsymbol{fp}\int_0^\infty f(x)\,\mathrm{d}x=\boldsymbol{fp}\int_0^{sgn(\lambda)\infty}f(\lambda t)d(\lambda t)$$
$$+\sum_{j=0}^{J(0)}\delta_{-1,\alpha_0}f^0_{0j}\frac{\log^{j+1}|\lambda|}{j+1}-\sum_{k=0}^{K(0)}\delta_{1,\gamma_0}f^\infty_{0k}\frac{\log^{k+1}|\lambda|}{k+1}, \tag{2.8}$$

where for complex values z_1 *and* z_2; $\delta_{z_1,z_2} = 0$ *except if* $z_1 = z_2 : \delta_{z_1,z_1} = 1$. *Moreover, for the change of variable* $x := t^a$

$$\boldsymbol{fp}\int_0^\infty f(x)\,\mathrm{d}x = \boldsymbol{fp}\int_0^\infty f(t^a)\,\mathrm{d}(t^a) = \boldsymbol{fp}\int_0^\infty a t^{a-1} f(t^a)\,\mathrm{d}t. \tag{2.9}$$

Proof. These results are deduced by applying definition 3. Equality (2.8) is proved in Sellier (1994). Consider now the relation (2.9). For $f \in \mathcal{P}(]0,+\infty[,C)$ and $\epsilon \succ 0$, we introduce the complex functions

$$h_1(\epsilon) := \int_\epsilon^{1/\epsilon} f(x)\,\mathrm{d}x, \quad h_2(\epsilon) := \int_\epsilon^{1/\epsilon} f(t^a) d(t^a), \quad \text{and} \quad \Delta(\epsilon) := h_2(\epsilon) - h_1(\epsilon). \tag{2.10}$$

Since $f \in L^1_{\mathrm{loc}}(]0,+\infty[,C)$, change of variable $x := t^a$ is legitimate for $\Delta(\epsilon)$ and leads to

$$\Delta(\epsilon) = \int_{\epsilon^a}^{1/\epsilon^a} f(x)\,\mathrm{d}x - \int_\epsilon^{1/\epsilon} f(x)\,\mathrm{d}x = \int_{\epsilon^a}^{\epsilon} f(x)\,\mathrm{d}x + \int_{1/\epsilon}^{1/\epsilon^a} f(x)\,\mathrm{d}x. \tag{2.11}$$

Use of expansions (2.3) and (2.4) for ϵ small enough makes it possible to write

$$\Delta(\epsilon) = \int_{\epsilon^a}^{\epsilon} F^0(x)\,\mathrm{d}x + \int_{1/\epsilon}^{1/\epsilon^a} F^\infty(x)\,\mathrm{d}x + \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f_{ij}^0 \int_{\epsilon^a}^{\epsilon} x^{\alpha_i}\log^j x\,\mathrm{d}x + \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^\infty \int_{1/\epsilon}^{1/\epsilon^a} x^{-\gamma_l}\log^k x\,\mathrm{d}x. \tag{2.12}$$

Because $a > 0$, $F^0 \in L^1([0,\eta_f],C)$ and $F^\infty \in L^1([A_f,+\infty[,C)$ the function

$$D(\epsilon) := \int_{\epsilon^a}^{\epsilon} F^0(x)\,\mathrm{d}x + \int_{1/\epsilon}^{1/\epsilon^a} F^\infty(x)\,\mathrm{d}x$$

tends to zero with ϵ. Application of proposition 4 allows us to cast $\Delta(\epsilon)$ in the next form

$$\Delta(\epsilon) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f_{ij}^0[P_{\alpha_i}^j(\epsilon) - P_{\alpha_i}^j(\epsilon^a)] + \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f_{lk}^\infty[P_{-\gamma_l}^k(\epsilon^{-a}) - P_{-\gamma_l}^k(\epsilon^{-1})] + D(\epsilon). \tag{2.13}$$

Since $f \in \mathcal{P}(]0,+\infty[,C)$, h_1 is of the second kind. Moreover, combination of equalities (2.6) and (2.13) shows that Δ is of the second kind. Consequently, $h_2 = h_1 + \Delta$ is also of the second kind and the new function g defined by $g(t) := a t^{a-1} f(t^a)$ belongs to $\mathcal{E}(]0,+\infty[,C)$. Finally, the assumptions $\mathrm{Re}(\alpha_i) \leqslant -1$, $\mathrm{Re}(\gamma_l) \leqslant 1$ and the definitions (2.6) associated with $\lim_{\epsilon\to 0} D(\epsilon) = 0$ and assumption $a > 0$ ensure that $\boldsymbol{fp}[\Delta(\epsilon)] = 0$, i.e. provide the result (2.9).

For $0 < b < +\infty$, it is also easy to define for an appropriate function f the quantities $\boldsymbol{fp}\int_0^b f(x)\,\mathrm{d}x$ or $\boldsymbol{fp}\int_b^\infty f(x)\,\mathrm{d}x$. Application of definition 2 leads to the introduction of the sets $\mathcal{P}(]0,b[,C) := \{f \in L^1_{\mathrm{loc}}(]0,b],C)$ and if f_- obeys $f_-(x) := f(x)$ a.e. in $]0,b]$ and $f_-(x) := 0$ a.e. in $]b,+\infty[$, then $f_- \in \mathcal{P}(]0,+\infty[,C)\}$ and $\mathcal{P}(]b,+\infty[,C) := \{f \in L^1_{\mathrm{loc}}([b,+\infty[,C)$ and if f_+ obeys $f_+(x) := f(x)$ a.e. in $[b,+\infty[$ and $f_+(x) := 0$ a.e. in $]0,b[$, then $f_+ \in \mathcal{P}(]0,+\infty[,C)\}$. For $f \in \mathcal{P}(]0,b[,C)$,

$$\boldsymbol{fp}\int_0^b f(x)\,\mathrm{d}x := \boldsymbol{fp}\int_0^\infty f_-(x)\,\mathrm{d}x$$

and for $f \in \mathcal{P}(]b, +\infty[, C)$,

$$\boldsymbol{fp}\int_b^\infty f(x)\,\mathrm{d}x := \boldsymbol{fp}\int_0^\infty f_+(x)\,\mathrm{d}x.$$

If $0 < c \leqslant +\infty$ and $f \in \mathcal{P}(]0, c[, C)$ then for any real b such that $0 < b < c$:

$$\boldsymbol{fp}\int_0^c f(x)\,\mathrm{d}x = \boldsymbol{fp}\int_0^b f(x)\,\mathrm{d}x + \boldsymbol{fp}\int_b^c f(x)\,\mathrm{d}x$$

(this latter integral reducing to a usual integration if $c < +\infty$). For instance, if $f \in \mathcal{P}(]0, b[, C)$ and λ is real and non-zero, lemma 6 yields

$$\boldsymbol{fp}\int_0^b f(x)\,\mathrm{d}x = \boldsymbol{fp}\int_0^{b/\lambda} f(\lambda t)\,\mathrm{d}(\lambda t) + \sum_{j=0}^{J(0)} \delta_{-1,\alpha_0} f_{0j}^0 \frac{\log^{j+1}|\lambda|}{j+1}. \tag{2.14}$$

This study also requires the extension of this concept of integration in the finite part sense of Hadamard to the case of multidimensional integration. More precisely, we are interested in defining for $n \geqslant 2$ and f a smooth enough function, the quantities L_n^f such that

$$L_2^f = \boldsymbol{fp}\int_0^{b_1} g_1(x_1)\left[\boldsymbol{fp}\int_0^{b_2} g_2(x_2) f(x_1, x_2)\,\mathrm{d}x_2\right]\mathrm{d}x_1, \tag{2.15}$$

$$L_{n\geqslant 3}^f = \boldsymbol{fp}\int_0^{b_1} g_1(x_1)$$
$$\times\left[\boldsymbol{fp}\int_0^{b_2} g_2(x_2)\ldots\left[\boldsymbol{fp}\int_0^{b_n} g_n(x_n) f(x_1, \ldots, x_n)\,\mathrm{d}x_n\right]\ldots\mathrm{d}x_2\right]\mathrm{d}x_1,$$

where for all $i \in \{1, \ldots, n\} : 0 < b_i < +\infty$ and $g_i \in \mathcal{P}(]0, b_i[, C)$. Observe that, for any function g_i, it is possible to find the positive integer $q_i := \inf\{m \in \mathbb{N}, x^m g_i(x) \in L^1([0, b_i], C)\}$. By now, it is assumed that f is continuous on an open set $\mathcal{U}$ containing the compact $[0, b_1] \times \ldots \times [0, b_n]$. If $q := \max(q_1, \ldots, q_n) = 0$ each function g_i belongs to $L^1([0, b_i], C)$ and L_n^f reduces to a usual integration.

For $n \geqslant 1$, some convenient notations are introduced. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$, it is understood that $|p| := p_1 + \ldots + p_n$, $x^p := x_1^{p_1} \ldots x_n^{p_n}$ and that the differential operator D^p, of order $|p|$, satisfies $D^p := \partial^{p_1}/\partial x_1^{p_1} \ldots \partial^{p_n}/\partial x_n^{p_n} = D^{|p|}/\partial x_1^{p_1} \ldots \partial x_n^{p_n}$. Moreover, for $q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ if Ω designates a bounded and open subset of $\mathbb{R}^n$ and $\partial\Omega$ its boundary, $\mathcal{D}^{(q_1,\ldots,q_n)}(\Omega)$ is the set of complex functions f such that for any $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ with $0 \leqslant p_i \leqslant q_i$ ($\forall i \in \{1, \ldots, n\}$) $D^p(f)$ exists and is continuous on an open set $\mathcal{U}$ countaining the compact $\Omega \cup \partial\Omega$. When $n = 1$ and $q \in \mathbb{N}^*$, one may remember that for $f \in \mathcal{D}^q(\Omega)$ and $(x, y) \in \Omega \times \Omega$ the Taylor polynomial expansion of this function f of order $q - 1$ at point y, noted $T_{f,y}^q$, and the associated remainder $R_{f,y}^q$ obey the well-known relations

$$T_{f,y}^q(x) := \sum_{i=0}^{q-1} \frac{f^{(i)}(y)}{i!}(x - y)^i, \tag{2.16}$$

$$R_{f,y}^q(x) := f(x) - T_{f,y}^q(x) = \frac{(x-y)^q}{(q-1)!}\int_0^1 (1-t)^{q-1} f^{(q)}[y + t(x - y)]\,\mathrm{d}t. \tag{2.17}$$

Application of the previous notations allows one to define the integral L_2^f for f

belonging to $\mathcal{D}^{(q_1,q_2)}(]0,b_1[\times]0,b_2[)$. As already outlined, L_2^f turns out to be a usual integration if $(q_1,q_2)=(0,0)$. Assume that $q_2 \geqslant 1$ (the case $q_2=0$ will be later discussed), then the function $B_2[f](x_1) := \boldsymbol{fp}\int_0^{b_2} g_2(x_2)f(x_1,x_2)\,\mathrm{d}x_2$ takes the form

$$B_2[f](x_1)=\int_0^{b_2} g_2(x_2)R_2^{q_2}[f](x_1,x_2)\,\mathrm{d}x_2+\sum_{i_2=0}^{q_2-1}\frac{\partial^{i_2}f}{\partial x_2^{i_2}}(x_1,0)\boldsymbol{fp}\int_0^{b_2} g_2(x_2)\frac{x_2^{i_2}}{i_2!}\,\mathrm{d}x_2 \tag{2.18}$$

for any $x_1\in[0,b_1]$, and with equality (2.17)

$$R_2^{q_2}[f](x_1,x_2)=\frac{x_2^{q_2}}{(q_2-1)!}\int_0^1 (1-t)^{q_2-1}\frac{\partial^{q_2}f}{\partial x_2^{q_2}}(x_1,tx_2)\,\mathrm{d}t. \tag{2.19}$$

Since $f\in\mathcal{D}^{(q_1,q_2)}(]0,b_1[\times]0,b_2[)$, the derivatives $\partial^{i_1}R_2^{q_2}[f](x_1,x_2)/\partial x_1^{i_1}$ exist on $[0,b_1]\times[0,b_2]$ for $0\leqslant i_1\leqslant q_1$ (they are obtained by derivating equality (2.19) because as continuous on the compact $[0,b_1]\times[0,b_2]$ the function $\partial^{i_1}\partial^{i_2}f/\partial x_1^{i_1}\partial x_2^{i_2}(x_1,x_2)$ is bounded on it) and $B_2[f]\in\mathcal{D}^{q_1}(]0,b_1[)$. Thus, $L_2^f=\boldsymbol{fp}\int_0^{b_1} g_1(x_1)B_2[f](x_1)\,\mathrm{d}x_1$ admits a sense. If $q_1\geqslant 1$, some algebra yields

$$\begin{aligned}
L_2^f=&\int_0^{b_1} g_1(x_1)\Bigg[\int_0^{b_2}\Bigg\{f(x_1,x_2)-\sum_{i_1=0}^{q_1-1}\frac{\partial^{i_1}f}{\partial x_1^{i_1}}(0,x_2)\frac{x_1^{i_1}}{i_1!}\\
&-\sum_{i_2=0}^{q_2-1}\frac{\partial^{i_2}f}{\partial x_2^{i_2}}(x_1,0)\frac{x_2^{i_2}}{i_2!}+\sum_{i_1=0}^{q_1-1}\sum_{i_2=0}^{q_2-1}\frac{\partial^{i_1+i_2}f}{\partial x_1^{i_1}\partial x_2^{i_2}}(0,0)\frac{x_1^{i_1}x_2^{i_2}}{i_1!i_2!}\Bigg\}g_2(x_2)\,\mathrm{d}x_2\Bigg]\mathrm{d}x_1\\
&+\sum_{i_1=0}^{q_1-1}\frac{1}{i_1!}\left[\boldsymbol{fp}\int_0^{b_1} x_1^{i_1}g_1(x_1)\,\mathrm{d}x_1\right]\left[\boldsymbol{fp}\int_0^{b_2}\frac{\partial^{i_1}f}{\partial x_1^{i_1}}(0,x_2)g_2(x_2)\,\mathrm{d}x_2\right]\\
&+\sum_{i_2=0}^{q_2-1}\frac{1}{i_2!}\left[\boldsymbol{fp}\int_0^{b_2} x_2^{i_2}g_2(x_2)\,\mathrm{d}x_2\right]\left[\boldsymbol{fp}\int_0^{b_1}\frac{\partial^{i_2}f}{\partial x_2^{i_2}}(x_1,0)g_1(x_1)\,\mathrm{d}x_1\right]\\
&-\sum_{i_1=0}^{q_1-1}\sum_{i_2=0}^{q_2-1}\frac{1}{i_1!i_2!}\frac{\partial^{i_1+i_2}f}{\partial x_1^{i_1}\partial x_2^{i_2}}(0,0)\left[\boldsymbol{fp}\int_0^{b_1} x_1^{i_1}g_1(x_1)\,\mathrm{d}x_1\right]\left[\boldsymbol{fp}\int_0^{b_2} x_2^{i_2}g_2(x_2)\,\mathrm{d}x_2\right].
\end{aligned} \tag{2.20}$$

It is clear that the first integral on the right-hand side of (2.20) is a usual one. Consequently, Fubini's theorem applies to this first term. Each of the remaining contributions to L_2^f involve one-dimensional integrations in the finite part sense of Hadamard which are all legitimate (for instance, the function $\partial^{i_1}f/\partial x_1^{i_1}(0,x_2)$ belongs to $\mathcal{D}^{q_2}(]0,b_2[)$). If $q_2\geqslant 1$ and $q_1=0$, one obtains

$$\begin{aligned}
L_2^f=&\int_0^{b_1} g_1(x_1)\Bigg[\int_0^{b_2}\Bigg\{f(x_1,x_2)-\sum_{i_2=0}^{q_2-1}\frac{\partial^{i_2}f}{\partial x_2^{i_2}}(x_1,0)\frac{x_2^{i_2}}{i_2!}\Bigg\}g_2(x_2)\,\mathrm{d}x_2\Bigg]\mathrm{d}x_1\\
&+\sum_{i_2=0}^{q_2-1}\frac{1}{i_2!}\left[\boldsymbol{fp}\int_0^{b_2} x_2^{i_2}g_2(x_2)\,\mathrm{d}x_2\right]\left[\int_0^{b_1}\frac{\partial^{i_2}f}{\partial x_2^{i_2}}(x_1,0)g_1(x_1)\,\mathrm{d}x_1\right],
\end{aligned} \tag{2.21}$$

and for $q_1\geqslant 1$, $q_2=0$ the function $B_2[f]$ rewrites $B_2[f](x_1)=\int_0^{b_2} g_2(x_2)f(x_1,x_2)\,\mathrm{d}x_2$

and

$$L_2^f = \int_0^{b_1} g_1(x_1)\left[\int_0^{b_2}\left\{f(x_1,x_2) - \sum_{i_1=0}^{q_1-1}\frac{\partial^{i_1} f}{\partial x_1^{i_1}}(0,x_2)\frac{x_1^{i_2}}{i_1!}\right\}g_2(x_2)\,\mathrm{d}x_2\right]\mathrm{d}x_1$$
$$+\sum_{i_1=0}^{q_1-1}\frac{1}{i_1!}\left[\boldsymbol{fp}\int_0^{b_1} x_1^{i_1} g_1(x_1)\,\mathrm{d}x_1\right]\left[\int_0^{b_1}\frac{\partial^{i_1} f}{\partial x_1^{i_1}}(0,x_2)g_2(x_2)\,\mathrm{d}x_2\right]. \quad (2.22)$$

These expansions (2.20)–(2.22) of the integral L_2^f also show that for $x = (x_1, x_2)$

$$\boldsymbol{fp}\int_0^{b_1} g_1(x_1)\left[\boldsymbol{fp}\int_0^{b_2} g_2(x_2) f(x)\,\mathrm{d}x_2\right]\mathrm{d}x_1$$
$$= \boldsymbol{fp}\int_0^{b_2} g_2(x_2)\left[\boldsymbol{fp}\int_0^{b_1} g_1(x_1) f(x)\,\mathrm{d}x_1\right]\mathrm{d}x_2.$$

■

For $n \geqslant 3$, the following proposition holds.

Proposition 7. *Consider $n \geqslant 3$ and families $(b_i), (g_i), (q_i)$ with $\forall i \in \{1,\dots,n\}$: $0 < b_i < +\infty$, $g_i \in \mathcal{P}(]0,b_i[,C), q_i := \inf\{m \in \mathbb{N}, x^m g_i(x) \in L^1([0,b_i],C)\}$. It is possible to define by induction for $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,b_1[\times\dots\times]0,b_n[)$ the complex*

$$L_n^f = \boldsymbol{fp}\int_0^{b_1} g_1(x_1)\left[\boldsymbol{fp}\int_0^{b_2} g_2(x_2)\dots\right.$$
$$\left.\times\left[\boldsymbol{fp}\int_0^{b_n} g_n(x_n) f(x_1,\dots,x_n)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 \quad (2.23)$$

and if σ denotes any permutation of the set $\{1,\dots,n\}$

$$L_n^f = \boldsymbol{fp}\int_0^{b_{\sigma(1)}} g_{\sigma(1)}(x_{\sigma(1)})\left[\boldsymbol{fp}\int_0^{b_{\sigma(2)}} g_{\sigma(2)}(x_{\sigma(2)})\dots\right.$$
$$\left.\times\left[\boldsymbol{fp}\int_0^{b_{\sigma(n)}} g_{\sigma(n)}(x_{\sigma(n)}) f(x_1,\dots,x_n)\,\mathrm{d}x_{\sigma}(n)\right]\dots\mathrm{d}x_{\sigma(2)}\right]\mathrm{d}x_{\sigma(1)}. \quad (2.24)$$

Proof. For $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,b_1[\times\dots\times]0,b_n[)$ and $(x_1,\dots,x_n) = (y,x_n)$, the function $B_n[f](y) := \boldsymbol{fp}\int_0^{b_n} g_n(x_n) f(y,x_n)\,\mathrm{d}x_n$ rewrites

$$B_n[f](y) = \int_0^{b_n} g_n(x_n) f(y,x_n)\,\mathrm{d}x_n, \quad \text{if } q_n = 0, \quad (2.25)$$

$$B_n[f](y) = \sum_{i_n=0}^{q_n-1}\frac{\partial^{i_n} f}{\partial x_n^{i_n}}(y,0)\left[\boldsymbol{fp}\int_0^{b_n} g_n(x_n)\frac{x_n^{i_n}}{i_n!}\,\mathrm{d}x_n\right]$$
$$+\int_0^{b_n}\frac{g_n(x_n)x_n^{q_n}}{(q_n-1)!}\left[\int_0^1 (1-t)^{q_n-1}\frac{\partial^{q_n} f}{\partial x_n^{q_n}}(y,tx_n)\,\mathrm{d}t\right]\mathrm{d}x_n, \quad \text{if } q_n \geqslant 1. \quad (2.26)$$

Consequently, the function $B_n[f]$ belongs to $\mathcal{D}^{(q_1,\dots,q_{n-1})}(]0,b_1[\times\dots\times]0,b_{n-1}[)$ and $L_n^f = L_{n-1}^{B_n[f]}$. This latter relation provides L_n^f by induction and also leads to $L_n^f =$

$L_{n-p}^{B_n\dots B_{n-p+1}[f]}$, for $2 \leqslant p \leqslant n-1$ and $B_jB_i[f] := B_j[B_i[f]]$. The reader may easily prove (2.24) by induction for $n \geqslant 3$, since it is true for $n=2$ and $L_n^f = L_{n-1}^{B_n[f]}$. ∎

Proposition 7 introduces L_n^f but it is worth obtaining a formula to calculate such a quantity. For $k \in \{1,\dots,n\}$, $\boldsymbol{\delta}_k$ and $\boldsymbol{t}_k$ are two operators acting on $\mathbb{R}^n$ and such that if $(y_1,\dots,y_n) = \boldsymbol{\delta}_k(x_1,\dots,x_n)$ and $(z_1,\dots,z_n) = \boldsymbol{t}_k(x_1,\dots,x_n)$ then $y_k = 0$, $z_k = t_kx_k$ and for $i \neq k$, $y_i = z_i = x_i$. The next proposition gives the required expansion.

Proposition 8. *Following proposition 7, consider for $n \geqslant 3$ families (b_i), (g_i), (q_i), $E := \{i \in \{1,\dots,n\}; q_i \geqslant 1\} = (j(l))_{l\in\{1,\dots,m\}}$ if $m = \mathrm{card}(E) \geqslant 1$, and also $\mathcal{S}_m = \{\sigma, \sigma$ is a permutation of $\{1,\dots,m\}\}$. Then, for*

$$f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,b_1[\times\dots\times]0,b_n[)$$

$$\begin{aligned}
L_n^f &= \boldsymbol{fp}\int_0^{b_1} g_1(x_1)\Bigg[\boldsymbol{fp}\int_0^{b_2} g_2(x_2)\dots \\
&\quad\times\left[\boldsymbol{fp}\int_0^{b_n} g_n(x_n) f(x_1,\dots,x_n)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\Bigg]\mathrm{d}x_1 \\
&= \int_0^{b_1} g_1(x_1)\left[\int_0^{b_2} g_2(x_2)\dots\left[\int_0^{b_n} g_n(x_n)\prod_{j\in E} R^{q_j}[f](x_1,\dots,x_n)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 \\
&\quad+\sum_{i_{j(1)}=0}^{q_{j(1)}-1}\cdots\sum_{i_{j(m)}=0}^{q_{j(m)}-1}\prod_{l=1}^{m}\left[\int_0^{b_{j(l)}} g_{j(l)}(u)\frac{u^{i_{j(l)}}}{i_{j(l)}!}\,\mathrm{d}u\right] \\
&\quad\times J_{i\in\{1,\dots,n\}\setminus E}^{(b_i,g_i)}\left[\frac{\partial^{i_{j(1)}}\dots\partial^{i_{j(m)}} f}{\partial x_{j(1)}^{i_{j(1)}}\dots\partial x_{j(m)}^{i_{j(m)}}}(\boldsymbol{\delta}_{j(1)}\dots\boldsymbol{\delta}_{j(m)}x)\right] \\
&\quad+\frac{1-\delta_{m,1}}{m!}\sum_{\sigma\in\mathcal{S}_m}\sum_{p=1}^{m-1} C_m^p\sum_{i_{j(\sigma(1))}=0}^{q_{j(\sigma(1))}-1}\cdots\sum_{i_{j(\sigma(p))}=0}^{q_{j(\sigma(p))}-1} \\
&\quad\times\prod_{k=1}^{p}\left[\int_0^{b_{j(\sigma(k))}} g_{j(\sigma(k))}(u)\frac{u^{i_{j(\sigma(k))}}}{i_{j(\sigma(k))}!}\,\mathrm{d}u\right] J_{i\in(j(\sigma(l)))_{p+1\leqslant l\leqslant m}}^{(b_i,g_i)} \\
&\quad\times\left[\prod_{k=p+1}^{m} R^{q_{j(\sigma(k))}}\left[\frac{\partial^{i_{j(\sigma(1))}+\dots+i_{j(\sigma(p))}} f}{\partial x_{j(\sigma(1))}^{i_{j(\sigma(1))}}\dots\partial x_{j(\sigma(p))}^{i_{j(\sigma(p))}}}(\boldsymbol{\delta}_{j(\sigma(1))}\dots\boldsymbol{\delta}_{j(\sigma(p))}x)\right]\right]
\end{aligned}$$

where $C_m^p := m!/[p!(m-p)!]$ for positive integers m, p such that $0 \leqslant p \leqslant m$; the operators T^{q_k}, R^{q_k} obey for $q_k \geqslant 1$ and $h \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,b_1[\times\dots\times]0,b_n[)$ the definition

$$T^{q_k}[h](x) = \sum_{i_k=0}^{q_k-1}\frac{x_k^{i_k}}{i_k!}\frac{\partial^{i_k}h}{\partial x_k^{i_k}}(\boldsymbol{\delta}_k x), \quad R^{q_k}[h](x) = h(x) - T^{q_k}[h](x), \tag{2.27}$$

and the integral $J_{i\in\mathcal{A}\subset\{1,\dots,n\}}^{(b_i,g_i)}[h]$ is defined later (see (2.31)).

Proof. First, it is indeed assumed that $m = \mathrm{card}(E) \geqslant 1$ so that L_n^f is not a usual integration. Consider $h \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,b_1[\times\dots\times]0,b_n[)$ and $(l,p) \in \{1,\dots,n\}^2$ with $l \neq p$. According to (2.27), $T^{q_l}T^{q_p}[h] = T^{q_l}[T^{q_p}[h]] = T^{q_p}T^{q_l}[h]$. Moreover, $R^{q_l}T^{q_p}[h] = T^{q_p}[h] - T^{q_l}T^{q_p}[h] = T^{q_p}[h - T^{q_l}[h]] = T^{q_p}R^{q_l}[h]; T^{q_l}R^{q_p}[h] = T^{q_l}[h - T^{q_p}[h]] = R^{q_p}T^{q_l}[h]$ and $R^{q_l}R^{q_p}[h] = h - T^{q_l}[h] - T^{q_p}[h] + T^{q_l}T^{q_p}[h] = R^{q_p}R^{q_l}[h]$. Thus, the identity $f = [R^{q_{j(1)}} + T^{q_{j(1)}}]\dots[R^{q_{j(m)}} + T^{q_{j(m)}}][f]$ may be cast into the useful and well-known form

$$f = \prod_{j\in E} R^{q_j}[f] + \prod_{j\in E} T^{q_j}[f] + \frac{1-\delta_{m,1}}{m!} \times \sum_{\sigma\in\mathcal{S}_m}\sum_{p=1}^{m-1} C_m^p T^{q_{j(\sigma(1))}}\dots T^{q_{j(\sigma(p))}} R^{q_{j(\sigma(p+1))}}\dots R^{q_{j(\sigma(m))}}[f], \tag{2.28}$$

where $\delta_{m,1} = 0$ except if $m = 1$: $\delta_{1,1} = 1$. Relation (2.17) combined with the definition of operator $\boldsymbol{t}_k$ ensures

$$R^{q_{j(\sigma(m))}}[f](x) = \frac{x_{j(\sigma(m))}^{q_{j(\sigma(m))}}}{[q_{j(\sigma(m))}-1]!}\int_0^1 [1-t_{j(\sigma(m))}]^{q_{j(\sigma(m))}-1}\frac{\partial^{q_{j(\sigma(m))}} f}{\partial_{x_{j(\sigma(m))}}^{q_{j(\sigma(m))}}}[\boldsymbol{t}_{j(\sigma(m))}x]\,\mathrm{d}t_{j(\sigma(m))}, \tag{2.29}$$

and thereafter $R^{q_{j(\sigma(m))}}[f] \in \mathcal{D}^{\boldsymbol{r}^n_{j(\sigma(m))}(q_1,\dots,q_n)}(\prod_{i\neq j(\sigma(m))}]0,b_i[)$, where for $n \geqslant 2$ and $1 \leqslant k \leqslant n$ the operator r_k^n acting on $\mathbb{R}^n$ is defined by $r_k^n(x) = r_k^n(x_1,\dots,x_n) = v = (v_1,\dots,v_{n-1})$ with:

(i) if $k = 1$ then $v_i = x_{i+1}$, $1 \leqslant i \leqslant n-1$;
(ii) if $k = n$ then $v_i = x_i$, $1 \leqslant i \leqslant n-1$;
(iii) if $2 \leqslant k \leqslant n-1$ then $v_i = x_i$ for $1 \leqslant i \leqslant k-1$ and $v_i = x_{i+1}$ for $k \leqslant i \leqslant n-1$.

It is also clear that the function

$$g_{j(\sigma(p+1))}\cdots g_{j(\sigma(m))}R^{q_{j(\sigma(p+1))}}\dots R^{q_{j(\sigma(m))}}[f]$$

belongs to the set

$$\mathcal{D}^{\boldsymbol{r}^{n-(m-p-1)}_{j(\sigma(p+1))}\cdots \boldsymbol{r}^n_{j(\sigma(m))}(q_1,\dots,q_n)}(\mathcal{B})$$

if $\mathcal{B}$ is defined by

$$\mathcal{B} = \prod_{i\ni\{j(\sigma(p+1)),\dots,j(\sigma(m))\}}]0,b_i[$$

thanks to the relation

$$R^{q_{j(\sigma(p+1))}}\dots R^{q_{j(\sigma(m))}}[f](x) = \prod_{k=p+1}^{m}\frac{x_{j(\sigma(k))}^{q_{j(\sigma(k))}}}{[q_{j(\sigma(k))}-1]!}\int_0^1[1-t_{j(\sigma(p+1))}]^{q_{j(\sigma(p+1))}-1}\dots \times\Bigg[\int_0^1[1-t_{j(\sigma(m))}]^{q_{j(\sigma(m))}-1}\frac{\partial^{q_{j(\sigma(p+1))}}\dots\partial^{q_{j(\sigma(m))}} f}{\partial_{x_{j(\sigma(p+1))}}^{q_{j(\sigma(p+1))}}\dots\partial_{x_{j(\sigma(m))}}^{q_{j(\sigma(m))}}} \times[\boldsymbol{t}_{j(\sigma(p+1))}\cdots\boldsymbol{t}_{j(\sigma(m))}x]\,\mathrm{d}t_{j(\sigma(m))}\Bigg]\cdots\mathrm{d}t_{j(\sigma(p+1))} \tag{2.30}$$

and definition $q_i := \inf\{m \in \mathbb{N}, x^m g_i(x) \in L^1([0,b_i],C)\}$. For an integer $1 \leqslant d \leqslant n$, $\mathcal{A} = (p_i)_{i\in\{1,\dots,d\}} \subset \{1,\dots,n\}$ and families $(c_i)_{i\in\{1,\dots,n\}}, (h_i)_{i\in\{1,\dots,n\}}$ such that $\forall l \in \{1,\dots,n\} : 0 < c_l < \infty$, h_l is a complex function; the complex $J^{(c_i,h_i)}_{i\in\mathcal{A}}[F]$ is defined for any function F such that $[\prod_{j\in\{1,\dots,d\}} h_{p_j}(x_{p_j})]F(x) \in L^1(\prod_{j\in\{1,\dots,d\}}[0,c_{p_j}],C)$ by

$$J_{i\in\mathcal{A}}^{(c_i,h_i)}[F] = \int_0^{c_{p_1}} h_{p_1}(u)F(u)\,\mathrm{d}u, \quad \text{if } d=1, \tag{2.31}$$

$$J_{i\in\mathcal{A}}^{(c_i,h_i)}[F] = \int_0^{c_{p_1}} h_{p_1}(x_{p_1})\cdots \times\left[\int_0^{c_{p_d}} h_{p_d}(x_{p_d})F(x_{p_1},\dots,x_{p_d})dx_{p_d}\right]\cdots \mathrm{d}x_{p_1}, \quad \text{for } d\geqslant 2.$$

Observe that Fubini's theorem applies to the above integral. Equality (2.30) proves that each derivative $\partial^{i_{j(\sigma(1))}+\cdots+i_{j(\sigma(p))}}[R^{q_{j(\sigma(p+1))}}\cdots R^{q_{j(\sigma(m))}}[f]]/\partial_{x_{j(\sigma(1))}}^{i_{j(\sigma(1))}}\cdots\partial_{x_{j(\sigma(p))}}^{i_{j(\sigma(p))}}$ exists for $0\leqslant i_{j(\sigma(l))}\leqslant q_{j(\sigma(l))}$, $\forall l\in\{1,\dots,p\}$ and that each integral L_n^w admit a sense for $w=T^{q_{j(\sigma(1))}}\cdots T^{q_{j(\sigma(p))}}R^{q_{j(\sigma(p+1))}}\cdots R^{q_{j(\sigma(m))}}[f]$, and according to definitions (2.27) and (2.31) takes the form

$$\begin{aligned}
&\boldsymbol{fp}\int_0^{b_1} g_1(x_1)\cdots\left[\boldsymbol{fp}\int_0^{b_n} g_n(x_n)T^{q_{j(\sigma(1))}}\cdots T^{q_{j(\sigma(p))}}R^{q_{j(\sigma(p+1))}}\cdots\right.\\
&\left.\times R^{q_{j(\sigma(m))}}[f](x)\,\mathrm{d}x_n\right]\cdots\mathrm{d}x_1\\
&=\sum_{i_{j(\sigma(1))}=0}^{q_{j(\sigma(1))}-1}\cdots\sum_{i_{j(\sigma(p))}=0}^{q_{j(\sigma(p))}-1}\prod_{k=1}^{p}\left[\boldsymbol{fp}\int_0^{b_{j(\sigma(k))}} g_{j(\sigma(k))}(u)\frac{u^{j(\sigma(k))}}{i_{j(\sigma(k))}!}\,\mathrm{d}u\right]\\
&\times J_{i\in\{j(\sigma(p+1)),\dots,j(\sigma(m))\}}^{(b_i,g_i)}\left[R^{q_{j(\sigma(p+1))}}\cdots\right.\\
&\left.\times R^{q_{j(\sigma(m))}}\left[\frac{\partial^{i_{j(\sigma(1))}+\cdots+i_{j(\sigma(p))}}[f]}{\partial x_{j(\sigma(1))}^{i_{j(\sigma(1))}}\cdots\partial x_{j(\sigma(p))}^{i_{j(\sigma(p))}}}(\delta_{j(\sigma(1))}\cdots\delta_{j(\sigma(p))}x)\right]\right].
\end{aligned}\tag{2.32}$$

To conclude, use of equality (2.28) leads to the result. ■

3. The asymptotic expansion for the one-dimensional case

This section presents two important theorems which give the expansion, with respect to a large and real parameter λ, of a class of integrals. The one-dimensional case reduces to an application of those theorems. A basic lemma will be also derived.

Definition 4. For a complex function h, two complex families $(\alpha_n),(a_{nm})$ such that the sequence $(\mathrm{Re}(\alpha_n))$ is strictly increasing $(\mathrm{Re}(\alpha_0)\prec\mathrm{Re}(\alpha_1)\prec\dots\prec\mathrm{Re}(\alpha_N))$ and a family of positive integers $(M(n))$, the following and abridged notations are used:

(i) for two reals r and $\epsilon\succ 0$,

$$\sum_{m,\mathrm{Re}(\alpha)\leqslant r} a_{nm}\epsilon^{\alpha_n}\log^m\epsilon := \sum_{n=0}^{N}\sum_{m=0}^{M(n)} a_{nm}\epsilon^{\alpha_n}\log^m\epsilon,$$

where $N:=\sup\{n,\mathrm{Re}(\alpha_n)\leqslant r\}$;

(ii)

$$\lim_{u\to 0} h(u) = \sum_{n,\mathrm{Re}(\alpha_n)\leqslant r}^{m,M(n)} a_{nm}u^{\alpha_n}\log^m u$$

means that there exist a real $s \succ r$ and a complex function H_r^0 bounded in a neighbourhood on the right of zero in which

$$h(u) = \sum_{m,\mathrm{Re}(\alpha)\leqslant r} a_{nm}u^{\alpha_n}\log^m u + u^s H_r^0(u)$$

and if there exists (n,m) with $a_{nm} \neq 0$ then $S_0(h) := \inf\{\mathrm{Re}(\alpha_n)$, where n is such that there exists $m \in \{0,\dots,M(n)\}$ with $a_{nm} \neq 0\}$;

(iii)

$$\lim_{u\to +\infty} h(u) = \sum_{n,\mathrm{Re}(\alpha_n)\leqslant r}^{m,M(n)} a_{nm}u^{-\alpha_n}\log^m u$$

means that there exist a real $s \succ r$ and a complex function H_r^∞ bounded in a neighbourhood of infinity in which

$$h(u) = \sum_{m,\mathrm{Re}(\alpha)\leqslant r} a_{nm}[u^{-1}]^{\alpha_n}\log^m u + u^{-s} H_r^\infty(u)$$

and if there exists (n,m) with $a_{nm} \neq 0$, then $S_\infty(h) := \inf\{\mathrm{Re}(\alpha_n)$, where n is such that there exists $m \in \{0,\dots,M(n)\}$ with $a_{nm} \neq 0\}$.

This definition allows us to introduce two useful functional spaces.

Definition 5. For two real values r_1, r_2, and $0 < b \leqslant +\infty$, the sets $\mathcal{E}_{r_1}^{r_2}(]0,b[,C)$ and $\mathcal{F}_{r_1}^{r_2}(]0,b[,C)$, respectively, satisfy

(A) $\mathcal{E}_{r_1}^{r_2}(]0,b[,C) := \{f$, f is a complex pseudo-function and there exist complex families $(\alpha_i), (A_{ij})$ with $(\mathrm{Re}(\alpha_i))$ strictly increasing, a family of positive integers $J(i)$ with

$$\lim_{x\to 0} f(x) = \sum_{i,\mathrm{Re}(\alpha_i)\leqslant r_1}^{j,J(i)} A_{ij}x^{\alpha_i}\log^j x;$$

if $b \prec +\infty$ then $f \in L^1_{\mathrm{loc}}(]0,b],C)$ else $f \in L^1_{\mathrm{loc}}(]0,+\infty[,C)$ and there exist complex families $(\gamma_n), (B_{nm})$ with $(\mathrm{Re}(\gamma_n))$ strictly increasing, a family $(M(n))$ with

$$\lim_{x\to +\infty} f(x) = \sum_{n,\mathrm{Re}(\gamma_n)\leqslant r_2}^{m,M(n)} B_{nm}x^{-\gamma_n}\log^m x\};$$

(B) $\mathcal{F}_{r_1}^{r_2}(]0,b[,C)$ is the set of pseudo-functions $K(x,u)$ such that, for λ large enough, $K(x,\lambda x) \in \mathcal{P}(]0,b[,C)$ and $K(x,u)$ satisfies the following properties:

(1) There exist a positive integer N, a complex family (γ_n) with $\mathrm{Re}(\gamma_0) \prec \dots \prec \mathrm{Re}(\gamma_N) := r_2$, families of positive integers $(M(n))$ and of complex pseudo-functions $(K_{nm}(x))$, a real $s_2 \succ r_2$, a complex function $G_{r_2}(x,u)$, a real $B \geqslant 0$ and a real $\eta \succ 0$ such that for any $(x,u) \in]0,b[\times[\eta,+\infty[$:

$$K(x,u) = \sum_{n=0}^{N}\sum_{m=0}^{M(n)} K_{nm}(x)u^{-\gamma_n}\log^m u + u^{-s_2}G_{r_2}(x,u); \tag{3.1}$$

$$\left| \int_{\eta}^{b} x^{-s_2} G_{r_2}(x, \lambda x)\, \mathrm{d}x \right| \leqslant B \prec +\infty; \tag{3.2}$$

there exist a positive integer I, a complex family (α_i) with $\mathrm{Re}(\alpha_0) \prec \ldots \prec \mathrm{Re}(\alpha_I) := r_1$, a family of positive integers $(J(i))$, with for $n \in \{0, \ldots, N\}$, $m \in \{0, \ldots, M(n)\}$, $K_{nm} \in \mathcal{E}^{1-\mathrm{Re}(\gamma_n)}_{\mathrm{Re}(\gamma_n)-1}(]0, b[, C)$ and there exist a complex family (K^{ij}_{nm}), a real $s_1 \succ r_1$, a complex function L_{nm} bounded in a neighbourhood of zero in which

$$K_{nm}(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K^{ij}_{nm} x^{\alpha_i} \log^j x + x^{s_1} L_{nm}(x). \tag{3.3}$$

(2) For the same η, I, s_1 families (α_i) and $(J(i))$, there exist a real $A \geqslant \eta \succ 0$, a family of complex pseudo-functions (h^{ij}), a complex function $H_{r_1}(x, u)$, a real $B' \geqslant 0$ and a real $W \succ 0$ such that for $0 \prec x \leqslant W$ and $u \succ 0$:

$$K(x, u) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} h^{ij}(u) x^{\alpha_i} \log^j x + x^{s_1} H_{r_1}(x, u); \tag{3.4}$$

$$\left| \int_{0}^{A} u^{s_1} H_{r_1}(u/\lambda, u)\, \mathrm{d}u \right| \leqslant B' \prec +\infty; \tag{3.5}$$

for $i \in \{0, \ldots, I\}$ and $j \in \{0, \ldots, J(i)\}$,

$$h^{ij} \in \mathcal{E}^{1+\mathrm{Re}(\alpha_i)}_{-1-\mathrm{Re}(\alpha_i)}(]0, +\infty[, C).$$

More precisely,

$$\lim_{u \to 0} h^{ij}(u) = \sum_{p, \mathrm{Re}(\beta_p) \leqslant -1-\mathrm{Re}(\alpha_i)}^{q, Q(p)} H^{ij}_{pq} u^{\beta_p} \log^q u$$

and also

$$\lim_{u \to +\infty} h^{ij}(u) = \sum_{n, \mathrm{Re}(\gamma_n) \leqslant 1+\mathrm{Re}(\alpha_i)}^{m, M(n)} K^{ij}_{nm} u^{-\gamma_n} \log^m u.$$

Moreover, there exists a complex function O_{ij} bounded in a neighbourhood of infinity in which

$$h^{ij}(u) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K^{ij}_{nm} u^{-\gamma_n} \log^m u + u^{-s_2} O_{ij}(u). \tag{3.6}$$

(3) Finally there exists a complex function $W_{r_1,r_2}(x, u)$, bounded in $]0, \eta] \times [A, +\infty[$, defined as

$$x^{s_1} u^{-s_2} W_{r_1,r_2}(x, u) := K(x, u) - \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K_{nm}(x) u^{-\gamma_n} \log^m u$$
$$- \sum_{i=0}^{I} \sum_{j=0}^{J(i)} \left[h^{ij}(u) - \sum_{n=0}^{N} \sum_{m=0}^{M(n)} K^{ij}_{nm} u^{-\gamma_n} \log^m u \right] x^{\alpha_i} \log^j x. \tag{3.7}$$

Taking into account these definitions and notation $C^p_n := n!/[p!(n-p)!]$ (if n and p are positive integers with $p \leqslant n$), the two following theorems hold (for a derivation the reader is referred to Sellier (1994)).

Theorem 9. *For a real r, if there exist $r_1 \geqslant r-1$ and $r_2 \geqslant r$ such that $K(x,u) \in \mathcal{F}_{r_1}^{r_2}(]0,b[,C)$, then when $\lambda \to +\infty$*

$$\begin{aligned}
&\boldsymbol{fp}\int_0^b K(x,\lambda x)\,\mathrm{d}x \\
&= \sum_{m,\mathrm{Re}(\gamma)\leqslant r}\sum_{l=0}^{m} C_m^l\left[\boldsymbol{fp}\int_0^b K_{nm}(x)x^{-\gamma_n}\log^{m-l}(x)\,\mathrm{d}x\right]\lambda^{-\gamma_n}\log^l\lambda \\
&\quad + \sum_{j,\mathrm{Re}(\alpha)\leqslant r-1}\sum_{l=0}^{j} C_j^l(-1)^l\left[\boldsymbol{fp}\int_0^\infty h^{ij}(v)v^{\alpha_i}\log^{j-l}(v)\,\mathrm{d}v\right. \\
&\quad - \sum_{\{p;\beta_p=-\alpha_i-1\}}\sum_{q=0}^{Q(p)}\frac{H_{pq}^{ij}}{1+j+q-l}\log^{1+j+q-l}\lambda \\
&\quad \left. + \sum_{\{n;\gamma_n=1+\alpha_i\}}\sum_{m=0}^{M(n)}\frac{K_{nm}^{ij}}{1+j+m-l}\log^{1+j+m-l}\lambda\right]\lambda^{-(\alpha_i+1)}\log^l\lambda + o(\lambda^{-r}), \qquad (3.8)
\end{aligned}$$

where each sum $\sum_{m,\mathrm{Re}(\gamma)\leqslant v}$ is defined by definition 4 and $\sum_{\{p;\beta_p=z\}}$ means a contribution corresponding to the value of positive integer p which satisfies $\beta_p = z$ and reduces to zero if such a value of p does not exist.

Observe that theorem 9 provides the asymptotic expansion of a singular integral (in the finite part sense of Hadamard). For this work, this theorem will be only applied to regular integrals. It is also possible to use the results derived by Bruning & Seeley (1985). These latter results have been recently extended by Lesch (1993) to functions with finite expansions at zero and infinity.

Theorem 10. *Consider two complex pseudo-functions $h \in \mathcal{P}(]0,b[,C)$ and also $H \in \mathcal{P}(]0,+\infty[,C)$. Assume that there exist three reals t, v, w with $t \geqslant -S_0(H)$, $w \geqslant \max(-t,-1-S_0(h))$ and also if $b=+\infty$, $v \geqslant \max(1-S_\infty(H), 1-t)$ and $t \geqslant 1-S_\infty(h)$. Moreover, assume that also $h \in \mathcal{E}_{t-1}^v(]0,b[,C)$, $H \in \mathcal{E}_w^t(]0,+\infty[,C)$ and $h(x)H(\lambda x) \in \mathcal{P}(]0,b[,C)$. Then, for any real $r \leqslant t$, if $\lambda \to +\infty$*

$$\begin{aligned}
&\boldsymbol{fp}\int_0^b h(x)H(\lambda x)\,\mathrm{d}x \\
&= \sum_{m,\mathrm{Re}(\gamma)\leqslant r}\sum_{l=0}^{m} C_m^l H_{nm}^\infty\left[\boldsymbol{fp}\int_0^b h(x)x^{-\gamma_n}\log^{m-l}(x)\,\mathrm{d}x\right]\lambda^{-\gamma_n}\log^l\lambda \\
&\quad + \sum_{j,\mathrm{Re}(\alpha)\leqslant r-1}\sum_{l=0}^{j} C_j^l(-1)^l h_{ij}^0\left[\boldsymbol{fp}\int_0^\infty H(v)v^{\alpha_i}\log^{j-l}(v)\,\mathrm{d}v\right. \\
&\quad - \sum_{\{p;\beta_p=-\alpha_i-1\}}\sum_{q=0}^{Q(p)}\frac{H_{pq}^0}{1+j+q-l}\log^{1+j+q-l}\lambda \\
&\quad \left. + \sum_{\{n;\gamma_n=1+\alpha_i\}}\sum_{m=0}^{M(n)}\frac{H_{nm}^\infty}{1+j+m-l}\log^{1+j+m-l}\lambda\right]\lambda^{-(\alpha_i+1)}\log^l\lambda + o(\lambda^{-r}), \qquad (3.9)
\end{aligned}$$

with

$$\lim_{x\to 0} h(x) = \sum_{i,\mathrm{Re}(\alpha)\leqslant t-1}^{j,J(i)} h^0_{ij} x^{\alpha_i} \log^j x$$

and for function H,

$$\lim_{u\to\infty} H(u) = \sum_{n,\mathrm{Re}(\gamma)\leqslant t}^{m,M(n)} H^{\infty}_{nm} u^{-\gamma_n} \log^m u,$$

$$\lim_{u\to 0} H(u) = \sum_{p,\mathrm{Re}(\beta)\leqslant w}^{q,Q(p)} H^0_{pq} u^{\beta_p} \log^q u.$$

Note that theorem 10 consists of an application of theorem 9 for $K(x,u) = h(x)H(u)$ and that assumption $H \in \mathcal{E}^t_w(]0,b[,C)$ is sufficient but not necessary (see derivation in Sellier (1994)). Moreover, the coefficients occurring in the expansions (3.8) or (3.9) are integrals in the finite part sense of Hadamard even if the initial quantity $\boldsymbol{fp}\int_0^b K(x,\lambda x)\,\mathrm{d}x$ reduces to a usual integration (in Lebesgue's sense).

For $r \geqslant 0$ and $q \in \mathbb{N}$ we introduce the sets $\mathcal{B}_r(]0,+\infty[) := \{g;\ g$ is bounded on $\mathbb{R}_+$ and $\lim_{u\to+\infty} g(u) = \sum_{n,\mathrm{Re}(\Lambda_n)\leqslant r}^{m,M(n)} g_{nm} u^{-\Lambda_n} \log^m u\}$, $\mathcal{C}_r(]0,+\infty[) := \{g;\ \forall\gamma$ with $\mathrm{Re}(\gamma) \succ -1, \forall q \in \mathbb{N}, g(x)x^{\gamma}\log^q x \in L^1_{\mathrm{loc}}([0,+\infty[,C)$ and $\lim_{u\to+\infty} g(u) = \sum_{n,\mathrm{Re}(\Lambda_n)\leqslant r}^{m,M(n)} g_{nm} u^{-\Lambda_n}\log^m u\}$ and also the set $\mathcal{H}^q(]0,1[) := \{$complex functions f bounded on $[0,1]$ and such that there exists $0 < \eta \leqslant 1$ with $f \in \mathcal{D}^q(]0,\eta[)\}$. Application of theorem 10 leads to the expansion of $I^f_1(\lambda)$ for $g \in \mathcal{C}_r(]0,+\infty[)$.

Theorem 11. *Consider $a \succ 0, l \in \mathbb{N}$ and $\alpha \in C$ with $\mathrm{Re}(\alpha) > -1$. Suppose that $r > 0$ and set $q := \max([\![ar - \mathrm{Re}(\alpha)]\!], 0)$ where $[\![b]\!]$ denotes the integer part of the real b. If $f \in \mathcal{H}^q(]0,1[)$ and $g \in \mathcal{C}_r(]0,+\infty[)$, then the integral $I^f_1(\lambda)$ admits, as λ tends to infinity, the following expansion:*

$$\begin{aligned}
I^f_1(\lambda) &= \int_0^1 x^{\alpha}\log^l(x) f(x) g(\lambda x^a)\,\mathrm{d}x \\
&= \sum_{i=0}^{[\![ar-\mathrm{Re}(\alpha)]\!]-1} \sum_{k=0}^{l} C^k_l (-1)^k \left(\frac{1}{a}\right)^{l+1} \frac{f^i(0)}{i!} \\
&\quad\times \left[\boldsymbol{fp}\int_0^{\infty} u^{\frac{1}{a}(\alpha+i+1)-1} \log^{l-k}(u) g(u)\,\mathrm{d}u\right] \lambda^{-(1/a)(\alpha+i+1)} \log^k\lambda \\
&\quad+ \sum_{m,\mathrm{Re}(\Lambda)\leqslant r} g_{nm} \sum_{k=0}^{m} C^k_m a^{m-k} \left[\boldsymbol{fp}\int_0^1 f(x) x^{\alpha - a\Lambda_n} \log^{l+m-k}(x)\,\mathrm{d}x\right] \lambda^{-\Lambda_n}\log^k\lambda \\
&\quad+ \sum_{m,\mathrm{Re}(\Lambda)\leqslant r} \sum_{\{i;a\Lambda_n = 1+\alpha+i\}} \left(\frac{1}{a}\right)^{l+1} \frac{f^i(0)}{i!} \frac{(-1)^l l!m! g_{nm}}{(m+l+1)!} \lambda^{-\Lambda_n} \log^{m+l+1}\lambda \\
&\quad+ o(\lambda^{-r}),
\end{aligned} \tag{3.10}$$

where it is understood that $\sum_{i=0}^{p} := 0$ if $p < 0$.

Proof. If $A := \sup_{x\in[0,1]} |f(x)|$ then

$$|I_1^f(\lambda)| \leqslant -A \int_0^1 x^\alpha \log^l(x)|g(\lambda x^a)|\,\mathrm{d}x := -A \times D.$$

Moreover, after some algebra and use of the new variable $t := \lambda x^a$, one gets

$$D = \left(\frac{1}{a}\right)^{l+1} \lambda^{-(\alpha+1)/a} \sum_{e=0}^{l} C_l^e(-1)^e \ln^e \lambda \left[\int_0^\lambda t^{((\alpha+1)/a)-1} \log^{l-e}(t)|g(t)|\,\mathrm{d}t\right], \quad (3.11)$$

with $\mathrm{Re}[((\alpha+1)/a)-1] > -1$ because $\mathrm{Re}(\alpha) > -1$. Thus, the definition of $\mathcal{C}_r(]0,+\infty[)$ shows that $I_1^f(\lambda)$ exists. Under the proposed assumptions, there exist a neighbourhood of zero (on the right), a function F_{ar-1}^0 bounded in this neighbourhood and a real $T > ar-1$ such that

$$F(x) := x^\alpha \log^l(x) f(x) = \sum_{i=0}^{[\![ar-\mathrm{Re}(\alpha)]\!]-1} \frac{f^i(0)}{i!} x^{\alpha+i} \log^l(x) + x^T F_{ar-1}^0(x), \quad (3.12)$$

with

(i) if $ar-1 \geqslant \mathrm{Re}(\alpha)$ then $[\![ar-\mathrm{Re}(\alpha)]\!] \geqslant 1$ and thanks to decomposition (2.17),

$$x^T F_{ar-1}^0(x) = [x^{\alpha+[\![ar-\mathrm{Re}(\alpha)]\!]} \log^l(x) \int_0^1 (1-t)^{q-1} f^q(tx)\,\mathrm{d}t]/(q-1)!,$$

where $q := [\![ar-\mathrm{Re}(\alpha)]\!]$;

(ii) if $ar-1 < \mathrm{Re}(\alpha)$ then $\mathrm{Re}(\alpha)+1-ar = \eta > 0$ and choice of $T := ar-1+\frac{1}{2}\eta$ and of $F_{ar-1}^0(x) := x^{\alpha+1-ar-\eta/2} \log^l(x) f(x)$ is possible with $F(x) = x^T F_{ar-1}^0(x)$. Consequently, for any real v, $F \in \mathcal{E}_{ar-1}^v(]0,1[,C)$. Moreover, if G is defined by $G(u) := g(u^a)$ then $G \in \mathcal{C}_r(]0,+\infty[)$ with, in a neighbourhood of infinity,

$$G(u) = g(u^a) = \sum_{n=0}^{N} \sum_{m=0}^{M(n)} a^m g_{nm} u^{-a\Lambda_n} \log^m u + u^{-as} G_r^\infty(u^a), \quad s > r. \quad (3.13)$$

Observe that for $\lambda' := \lambda^{1/a}$,

$$I_1^f(\lambda) = \int_0^1 x^\alpha \log^l(x) f(x) G(\lambda' x)\,\mathrm{d}x.$$

Thus, theorem 10 applies and by keeping the definition $\sum_{i=0}^p := 0$ if $p < 0$, one obtains

$$\begin{aligned} I_1^f(\lambda) = &\sum_{m,\mathrm{Re}(\Lambda)\leqslant r} a^m g_{nm} \left[\boldsymbol{fp} \int_0^1 x^{\alpha-a\Lambda_n} \log^l(x) f(x) \log^m(\lambda' x)\,\mathrm{d}x\right] \lambda'^{-a\Lambda_n} \\ &+ \sum_{i=0}^{[\![ar-\mathrm{Re}(\alpha)]\!]-1} \sum_{k=0}^{l} C_l^k (-1)^k \frac{f^i(0)}{i!} \left\{\boldsymbol{fp} \int_0^\infty v^{\alpha+i} \log^{l-k}(v) G(v)\,\mathrm{d}v \right. \\ &\left. + \sum_{\{n;a\Lambda_n=1+\alpha+i\}} \sum_{m=0}^{M(n)} \frac{a^m g_{nm}}{1+l+m-k} \log^{1+m+l-k} \lambda' \right\} \lambda'^{-(\alpha+i+1)} \log^k \lambda' + o(\lambda'^{-ar}). \end{aligned} \quad (3.14)$$

Assumptions $q = \max([\![ar - \mathrm{Re}(\alpha)]\!], 0)$ and $f \in \mathcal{H}^q(]0,1[)$ combined with expansion (3.12) show that for $Re(\Lambda_n) \leqslant r$, each integral

$$\boldsymbol{fp}\int_0^1 x^{\alpha - a\Lambda_n}\log^l(x) f(x)\log^m(\lambda' x)\,\mathrm{d}x$$

admits a sense. Moreover, for any real $0 \prec R \prec \infty$,

$$\int_0^R v^{\alpha+i}\log^{l-k}(v)G(v)\,\mathrm{d}v = a^{k-l-1}\int_0^{R^a} u^{((\alpha+i+1)/a)-1}\log^{l-k}(u)g(u)\,\mathrm{d}u$$

exists (because $g \in \mathcal{C}_r(]0,+\infty[)$) and for $\mathrm{Re}(\alpha) + i \leqslant ar - 1$, the function $v^{\alpha+i-as}\log^{l-k}(v)G_r^{\infty}(v^a)$ is measurable on a neighbourhood of infinity (thanks to $s \succ r$ and $G_r^{\infty}(v^a)$ bounded for v large enough). This justifies the existence of

$$\boldsymbol{fp}\int_0^\infty v^{\alpha+i}\log^{l-k}(v)G(v)\,\mathrm{d}v = a^{k-l-1}\boldsymbol{fp}\int_0^\infty u^{((\alpha+i+1)/a)-1}\log^{l-k}(u)g(u)\,\mathrm{d}u$$

(see lemma 6) and also explains that in applying theorem 10 the third term on the right-hand side of (3.9) reduces to zero. To conclude, the following equality is used

$$S_{ml} = \sum_{k=0}^{l}\frac{C_l^k(-1)^k}{1+l+m-k} = \frac{(-1)^l l!m!}{(m+l+1)!}, \quad \text{for } (m,l) \in \mathbb{N}^2. \tag{3.15}$$

Introduction of function

$$F_{ml}(x) = \sum_{k=0}^{l} C_l^k(-1)^k x^{1+l+m-k}(1+l+m-k)^{-1},$$

such that $F_{ml}(0) = 0, S_{ml} = F_{ml}(1)$ and $F'_{ml}(x) = x^m(x-1)^l$, indeed leads to $S_{ml} = \int_0^1 x^m(x-1)^l\,\mathrm{d}x$. ■

When dealing with multidimensional integrals in the finite part sense of Hadamard it is not always legitimate to apply without correction the usual Fubini's theorem. Proposition 7 exhibits a case where Fubini's theorem applies. The subsequent very important lemma treats the case of a pecular two-dimensional integral and provides the extra term. For $r > 0$, and in order to deal with this lemma, the space $\mathcal{L}_r(]0,+\infty[)$ is defined by

$$\mathcal{L}_r(]0,+\infty[) := \Big\{ g; \forall\gamma \text{ with } \mathrm{Re}(\gamma) \succ -1, \forall q \in \mathbb{N}, \forall\eta > 0 \text{ then } g(x)x^\gamma\log^q x \in L^1_{\mathrm{loc}}([0,+\infty[, C), X^\eta\int_0^1 y^\gamma\log^q(y)g(Xy)\,\mathrm{d}y \text{ is bounded as } X \to 0^+ \text{ and } \lim_{u\to+\infty} g(u) = \sum_{n,\mathrm{Re}(\Lambda_n)\leqslant r}^{m,M(n)} g_{nm}u^{-\Lambda_n}\log^m u \Big\}.$$

Observe that $\mathcal{B}_r(]0,+\infty[) \subset \mathcal{L}_r(]0,+\infty[) \subset \mathcal{C}_r(]0,+\infty[)$.

Lemma 12. *Consider reals $a \succ 0$, $b \succ 0$, complex numbers α and β with* $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, *positive integers j and q and a function $g \in \mathcal{L}_r(]0,+\infty[)$, where $r \geqslant \max([\mathrm{Re}(\alpha)+1]/b, [\mathrm{Re}(\beta)+1]/a)$. If we set*

$$\mathcal{D}_j^q[g] := \boldsymbol{fp}\int_0^1 x^\alpha\log^j(x)\left[\boldsymbol{fp}\int_0^\infty u^\beta\log^q(u)g(u^a x^b)\,\mathrm{d}u\right]\mathrm{d}x, \tag{3.16}$$

$$\mathcal{D}'^q_j[g] := \boldsymbol{fp}\int_0^\infty u^\beta \log^q u\left[\int_0^1 x^\alpha \log^j(x) g(u^a x^b)\,\mathrm{d}x\right]\mathrm{d}u, \tag{3.17}$$

then $\Delta^q_j[g] := \mathcal{D}^q_j[g] - \mathcal{D}'^q_j[g] = 0$; *except if* $a(\alpha+1) = b(\beta+1)$. *In this latter case*

$$\Delta^q_j[g] = \frac{q!j!}{(q+j+1)!}\left(\frac{a}{b}\right)^{j+1}\left[\boldsymbol{fp}\int_0^\infty u^\beta \log^{q+j+1}(u) g(u^a)\,\mathrm{d}u\right]. \tag{3.18}$$

Proof. Consider $x \in]0,1[$ and $\lambda = x^{b/a} \succ 0$. Some algebra for the new function $G(X) := g(X^a)$ and the complex

$$e := \boldsymbol{fp}\int_0^\infty u^\beta \log^q(u) g(u^a x^b)\,\mathrm{d}u$$

yields

$$e = x^{-(b/a)(\beta+1)}\sum_{p=0}^{q} C^p_q \log^p(x^{-b/a})\left[\boldsymbol{fp}\int_0^\infty (\lambda u)^\beta \log^{q-p}(\lambda u) G(\lambda u)\,\mathrm{d}(\lambda u)\right]. \tag{3.19}$$

The assumptions $\mathrm{Re}(\beta) > -1$, $g \in \mathcal{L}_r(]0,+\infty[)$ with $r \geqslant [\mathrm{Re}(\beta)+1]/a$ ensure that for any real $0 < R < +\infty$,

$$\int_0^R t^\beta \log^{q-p}(t) G(t)\,\mathrm{d}t = a^{p-q-1}\int_0^{R^a} u^{((\beta+1)/a)-1}\log^{q-p}(u) g(u)\,\mathrm{d}u$$

exists and that there exists a real $s \succ r$ (i.e. such that $\mathrm{Re}(\beta) - as < -1$) with

$$t^\beta G(t) = \sum_{m,\mathrm{Re}(\Lambda)\leqslant r} g_{nm} a^m t^{\beta - a\Lambda_n}\log^m(t) + t^{\beta - as} G^\infty_r(t^a) \tag{3.20}$$

in a neighbourhood of infinity in which one G^∞_r is bounded. Consequently, the complex

$$\boldsymbol{fp}\int_0^\infty t^\beta \log^{q-p}(t) G(t)\,\mathrm{d}t$$

admits a sense and application of the change of variable $t = \lambda u$ to equality (3.19) (see lemma 6 for the corrective terms) leads to

$$\mathcal{D}^q_j[g] = \sum_{p=0}^{q} C^p_q\left(-\frac{b}{a}\right)^p \boldsymbol{fp}\int_0^1 x^{\alpha-(b/a)(\beta+1)}\log^{j+p}(x)$$
$$\times\left[\boldsymbol{fp}\int_0^\infty t^\beta \log^{q-p}(t) g(t^a)\,\mathrm{d}t + \sum_{\{n; a\Lambda_n = \beta+1\}}\sum_{m=0}^{M(n)} \frac{a^m g_{nm}\log^{q-p+m+1}(x^{b/a})}{q-p+m+1}\right]\mathrm{d}x. \tag{3.21}$$

Use of the change of variable $z := u^a x^b$ for the integral $d := \int_0^1 x^\alpha \log^j(x) g(u^a x^b)\,\mathrm{d}x$ yields

$$\mathcal{D}'^q_j[g] = \frac{1}{b}\sum_{l=0}^{j} C^l_j\left(-\frac{a}{b}\right)^l \boldsymbol{fp}\int_0^\infty u^{\beta-(a/b)(\alpha+1)}\log^{q+l}(u)$$
$$\times\left[\int_0^{u^a} z^{((\alpha+1)/b)-1}\log^{j-l}(z^{1/b}) g(z)\,\mathrm{d}z\right]\mathrm{d}u. \tag{3.22}$$

Step 1. Suppose that $a(\alpha+1) = b(\beta+1)$. Because $\boldsymbol{fp}\int_0^1 x^{-1}\log^i(x)\,\mathrm{d}x = 0$ for $i \in \mathbb{N}$, (3.21) gives $\mathcal{D}_j^q[g] = 0$. Taking into account this assumption $a(\alpha+1) = b(\beta+1)$ and proposition 4, an integration by parts of equality (3.26) (always valid when dealing with integration in the finite part sense of Hadamard) leads to

$$\mathcal{D}'^q_j(g) = \boldsymbol{fp}[\mathcal{F}(u)]_0^\infty - \left(\frac{a}{b}\right)^{j+1} \left[\sum_{l=0}^{j} \frac{C_j^l(-1)^l}{q+l+1}\right] \left[\boldsymbol{fp} \int_0^\infty u^\beta \log^{q+j+1}(u) g(u^a)\,\mathrm{d}u\right], \tag{3.23}$$

where

$$\begin{aligned}\mathcal{F}(u) &= \frac{1}{b} \sum_{l=0}^{j} C_j^l \left(-\frac{a}{b}\right)^l \frac{\log^{q+l+1}(u)}{q+l+1} \int_0^{u^a} z^{((\alpha+1)/b)-1} \log^{j-l}(z^{1/b}) g(z)\,\mathrm{d}z \\ &= \sum_{l=0}^{j} \sum_{n=0}^{j-l} \frac{(-1)^l C_j^l C_{j-l}^n}{q+l+1} \left(\frac{a}{b}\right)^{j-n} u^{\beta+1} \log^{q+j+1-n}(u) \left[\int_0^1 x^\alpha \log^n(x) g(u^a x^b)\,\mathrm{d}x\right].\end{aligned} \tag{3.24}$$

Note that

$$\int_0^1 x^\alpha \log^n(x) g(u^a x^b)\,\mathrm{d}x = b^{-(n+1)} \int_0^1 t^{((\alpha+1/b)-1} \log^n(t) g(u^a t)\,\mathrm{d}t$$

with $\mathrm{Re}[(\alpha+1)/a - 1] > -1$. Moreover, $\mathrm{Re}(\beta) + 1 > 0$ and $g \in \mathcal{L}_r(]0,+\infty[)$ ensure that $\mathcal{F}(0) = 0$. Because $a(\alpha+1) = b(\beta+1)$, $g \in \mathcal{L}_r(]0,+\infty[)$ with $r \geqslant [\mathrm{Re}(\beta)+1]/a = [\mathrm{Re}(\alpha)+1]/b$. Thus, application of theorem 11 with $f := 1$ ensures, as $u \to +\infty$, an asymptotic expansion of each integral $\int_0^1 x^\alpha \log^n(x) g(u^a x^b)\,\mathrm{d}x$ with a remainder $o(u^{-s})$, where $s = ar \geqslant \mathrm{Re}(\beta) + 1 > 0$. Consequently, as $u \to +\infty$, $\mathcal{F}(u)$ rewrites

$$\mathcal{F}(u) = \sum_{l=0}^{j} \sum_{n=0}^{j-l} \sum_{p} \sum_{l'} F_{ln}^{pl'} u^{-\gamma_p} \log^{q+l'+1+j-n}(u) + o(1), \tag{3.25}$$

with $\mathrm{Re}(\gamma_p) \leqslant 0$ and $q + l' + 1 + j - n \geqslant 1$. Due to these remarks, $\boldsymbol{fp}[\mathcal{F}(\epsilon^{-1})] = 0$. Finally,

$$\begin{aligned}\sum_{l=0}^{j} C_j^l(-1)^l/[q+l+1] &= (-1)^j \sum_{k=0}^{j} C_j^k(-1)^k/[1+q+j-k] \\ &= (-1)^j S_{qj} = j!q!/(q+j+1)!\end{aligned}$$

(see (3.15)) and equality (3.23) provides the announced result.

Step 2. Suppose now that $a(\alpha+1) \neq b(\beta+1)$. First we set $j = 0$. Equality (3.21) and proposition 4 yield

$$\begin{aligned}\mathcal{D}_0^q[g] = &\sum_{p=0}^{q} \left(\frac{b}{a}\right)^p \frac{C_q^p p!}{[\alpha+1-(b/a)(\beta+1)]^{p+1}} \left[\boldsymbol{fp} \int_0^\infty t^\beta \log^{q-p}(t) g(t^a)\,\mathrm{d}t\right] \\ &+ \sum_{p=0}^{q} C_q^p(-1)^p \sum_{\{n;\, a\Lambda_n = \beta+1\}} \sum_{m=0}^{M(n)} \left(\frac{b}{a}\right)^{q+m+1} \\ &\times \frac{a^m g_{nm}}{(q-p+m+1)} \frac{(-1)^{q+m+1}(q+m+1)!}{[\alpha+1-(b/a)(\beta+1)]^{q+m+2}}.\end{aligned} \tag{3.26}$$

Use of the relation

$$\sum_{p=0}^{q} \frac{C_q^p(-1)^p}{[1+q+m-p]} = S_{mq} = \frac{(-1)^q q!m!}{(q+m+1)!}$$

allows us to cast the second contribution of $\mathcal{D}_0^q[g]$ in the form

$$C_2 = \sum_{\{n;\, a\Lambda_n=\beta+1\}} \sum_{m=0}^{M(n)} \left(\frac{b}{a}\right)^{q+m+1} \frac{a^m g_{nm}(-1)^{m+1}q!m!}{[\alpha+1-(b/a)(\beta+1)]^{q+m+2}}. \tag{3.27}$$

An integration by parts of equality (3.26) with $j=0$ gives

$$\mathcal{D}'^{q}_{0}[g] = -\frac{a}{b}\boldsymbol{fp}\int_0^{\infty}\left[\sum_{m'=0}^{q}\frac{q!(-1)^{q-m'}u^{\beta}\log^{m'}(u)}{m'![1+\beta-(a/b)(\alpha+1)]^{1+q-m'}}\right]g(u^a)\,\mathrm{d}u + \boldsymbol{fp}[\Psi(u)]_0^{\infty}, \tag{3.28}$$

with

$$\Psi(u) := \sum_{m'=0}^{q}\frac{q!(-1)^{q-m'}u^{\beta+1}\log^{m'}(u)}{m'![\beta+1-(a/b)(\alpha+1)]^{1+q-m'}}\left[\int_0^1 x^{\alpha}g(u^a x^b)\,\mathrm{d}x\right]. \tag{3.29}$$

Here, $\Psi(0)=0$ (see treatment of $\mathcal{F}(0)$). If $u^a \to +\infty$, theorem 11 associated with the assumption $g \in \mathcal{L}_r(]0,+\infty[)$ with $r \geqslant \max([\mathrm{Re}(\alpha)+1]/b, [\mathrm{Re}(\beta)+1]/a)$ provides the asymptotic expansion of

$$\int_0^1 x^{\alpha}g(u^a x^b)\,\mathrm{d}x.$$

Such an expansion involves the terms $[u^a]^{-(1/b)(\alpha+1)} = u^{-(a/b)(\alpha+1)}$, associated to the first and third sums of the expansion (3.10), but also (see the second contribution on the right-hand side of (3.10)) the terms

$$\sum_{m,\mathrm{Re}(\Lambda)\leqslant r} g_{nm}\sum_{k=0}^{m} C_m^k b^{m-k}a^k[\boldsymbol{fp}\int_0^1 x^{\alpha-b\Lambda_n}\log^{m-k}(x)\,\mathrm{d}x]u^{-a\Lambda_n}\log^k u$$

and the remainder is $o(u^{-s})$ with $s \geqslant \mathrm{Re}(\beta)+1$. Thus, $\boldsymbol{fp}[\Psi(\epsilon^{-1})]$ is zero except if there exists n such that $-a\Lambda_n+\beta+1=0$ and in such a case the contribution is obtained by setting $k=0$ (and $m'=0$ in relation (3.29)). Hence,

$$\boldsymbol{fp}[\Psi(u)]_0^{\infty} = \sum_{\{n;a\Lambda_n=\beta+1\}}\sum_{m=0}^{M(n)} g_{nm}b^m[\boldsymbol{fp}\int_0^1 x^{\alpha-b(\beta+1)/a}\log^m(x)\,\mathrm{d}x].$$

Taking into account that $\alpha - b(\beta+1)/a \neq -1$,

$$\boldsymbol{fp}\int_0^1 x^{\alpha-b(\beta+1)/a}\log^m(x)\,\mathrm{d}x = (-1)^m m!/[\alpha+1-b(\beta+1)/a]^{m+1}$$

and

$$\boldsymbol{fp}[\Psi(u)]_0^{\infty} = \sum_{\{n;a\Lambda_n=\beta+1\}}\sum_{m=0}^{M(n)}\left(\frac{b}{a}\right)^{q+1}\frac{b^m g_{nm}(-1)^{m+1}q!m!}{[\alpha+1-b(\beta+1)/a]^{q+m+2}} = C_2. \tag{3.30}$$

To conclude, if we set $p=q-m'$, the first term on the right-hand side of (3.28) equals the first term on the right-hand side of (3.26). The case $j \neq 0$ is obtained by induction and by using the following property: if $g \in \mathcal{L}_r(]0,+\infty[)$ then the function h_g defined by $h_g(x) := g(x)\log(x)$ belongs to $\mathcal{L}_r(]0,+\infty[)$. Suppose that $a(\alpha+1) \neq b(\beta+1)$ and that for any positive integer q, and any function $g \in \mathcal{L}_r(]0,+\infty[) : \mathcal{D}_j^q[g] = \mathcal{D}'^q_j[g]$. This is true for $j=0$. Relation

$$\log^{j+1}(x) = \log^j(x)[\log(u^a x^b) - a\log(u)]/b$$

gives

$$\mathcal{D}^q_{j+1}[g] = -\frac{a}{b}\mathcal{D}^{q+1}_j[g] + \frac{1}{b}\boldsymbol{fp}\int_0^1 x^\alpha \log^j(x)\left[\boldsymbol{fp}\int_0^\infty u^\beta \log^q(u) h_g(u^a x^b)\,\mathrm{d}u\right]\mathrm{d}x,$$

$$\mathcal{D}'^q_{j+1}[g] = -\frac{a}{b}\mathcal{D}'^{q+1}_j[g] + \frac{1}{b}\boldsymbol{fp}\int_0^\infty u^\beta \log^q(u)\left[\int_0^1 x^\alpha \log^j(x) h_g(u^a y^b)\,\mathrm{d}x\right]\mathrm{d}u.$$

The assumption $\mathcal{D}^q_j[g] = \mathcal{D}'^q_j[g]$, $\forall q \in \mathbb{N}$ and the property $h_g \in \mathcal{L}_r(]0,+\infty[)$ show that $\mathcal{D}^q_{j+1}[g] = \mathcal{D}'^q_{j+1}[g]$. ■

4. The multidimensional case

By now if $n \geqslant 1$ and $1 \leqslant k \leqslant n$, we set $\delta^n_k = \delta_k$, where δ_k is the operator introduced in the previous section,

$$\prod_j^n A_j := \prod_{j=1}^n A_j$$

and also

$$\prod_{j;i_1,\dots,i_k}^n A_j := \prod_{j\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}} A_j,$$

where each i_l belongs to $\{1,\dots,n\}$. This product is set equal to one if $k=n$ and $\{i_1,\dots,i_k\} = \{1,\dots,n\}$. Moreover, the notation $\sum^n_{i_1,\dots,i_k} F(i_1,\dots,i_k)$ means a sum over all the subsets $\{i_1,\dots,i_k\}$ of $\{1,\dots,n\}$ such that the integers i_l are all different. If $k=1$,

$$\sum_{i_1}^n F(i_1) \text{ reduces to } \sum_{i=1}^n F(i).$$

Given a complex family $(\alpha_i)_{i\in\{1,\dots,n\}}$, a real family $(a_i)_{i\in\{1,\dots,n\}}$ and $(n_1,\dots,n_k) \in \mathbb{N}^k$ with $1 \leqslant k \leqslant n$, the real $\Delta(n_1,\dots,n_k)$ equals one if $k=1$ or if $a_i(\alpha_j+n_j+1) = a_j(\alpha_i+n_i+1)$, $\forall(i,j) \in \{1,\dots,k\}^2$; else $\Delta(n_1,\dots,n_k) := 0$. For a complex γ, the real $\Delta^\gamma(n_1,\dots,n_k)$ is zero except if $a_i\gamma = \alpha_i+n_i+1$, $\forall i \in \{1,\dots,k\}$ and in such circumstances $\Delta^\gamma(n_1,\dots,n_k) := 1$. Finally, if p and q are integers with $p \geqslant 0$ then

$$\sum_{i=p}^q F_i := 0$$

if $q \prec p$ and for any integers $(p_i)_{i\in\{1,\dots,k\}}$,

$$\sum_{i_1,\dots,i_k}^{p_{i_1},\dots,p_{i_k}} F(i_1,\dots,i_k) := \sum_{i_1=0}^{p_{i_1}} \cdots \sum_{i_k=0}^{p_{i_k}} F(i_1,\dots,i_k).$$

Theorem 13. *Consider $n \geqslant 1$, a real $r > 0$ and families (α_i), (a_i), (l_i), (q_i), (p_i) such that for any $i \in \{1, \dots, n\}$: $\alpha_i \in C$ with $\mathrm{Re}(\alpha_i) > -1$, $a_i \in \mathbb{R}_+^*$, $l_i \in \mathbb{N}$, $q_i = \max([\![a_i r - \mathrm{Re}(\alpha_i)]\!], 0)$ and $p_i = [\![a_i r - \mathrm{Re}(\alpha_i)]\!] - 1$. If $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$ and $g \in \mathcal{B}_r(]0,+\infty[)$, the integral $I_n^f(\lambda)$ admits, as λ tends to infinity, the following expansion:*

$$
\begin{aligned}
I_n^f(\lambda) &= \int_0^1 \cdots \int_0^1 x_1^{\alpha_1} \cdots x_n^{\alpha_n} \log^{l_1}(x_1) \cdots \log^{l_n}(x_n) \\
&\quad \times f(x_1, \dots, x_n) g(\lambda x_1^{a_1} \cdots x_n^{a_n}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \sum_{i_1,\dots,i_k}^{n} \sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}} \frac{\Delta(n_{i_1}, \dots, n_{i_k})}{n_{i_1}! \dots n_{i_k}!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \\
&\quad \times \frac{l_{i_1}! \cdots l_{i_k}!}{(\sum_{j=1}^{k} l_{i_j} + k - 1)!} \Bigg\{ \prod_{j; i_1,\dots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - (a_j/a_{i_1})(\alpha_{i_1} + n_{i_1} + 1)} \\
&\quad \times \log^{l_j}(x_j) f_{x_{i_1} \cdots x_{i_k}}^{n_{i_1} \cdots n_{i_k}} (\delta_{i_1}^n \cdots \delta_{i_k}^n x) \bigg[\boldsymbol{fp} \int_0^\infty u^{((\alpha_{i_1} + n_{i_1} + 1)/(a_{i_1})) - 1} \\
&\quad \times \left[\log\left(\lambda^{-1} u \prod_{j; i_1,\dots,i_k}^{n} x_j^{-a_j} \right) \right]^{(\sum_{j=1}^{k} l_{i_j} + k - 1)} g(u) \, \mathrm{d}u \bigg] \mathrm{d}x_j \Bigg\} \lambda^{-(\alpha_{i_1} + n_{i_1} + 1)/a_{i_1}} \\
&\quad + \sum_{\mathrm{Re}(\Lambda_e) \leqslant r} \sum_{m=0}^{M(e)} g_{em} (-1)^m \Bigg\{ \boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - a_1 \Lambda_e} \log^{l_1}(x_1) \bigg[\boldsymbol{fp} \int_0^1 x_2^{\alpha_2 - a_2 \Lambda_e} \log^{l_2}(x_2) \cdots \\
&\quad \times \bigg[\boldsymbol{fp} \int_0^1 x_n^{\alpha_n - a_n \Lambda_e} \log^{l_n}(x_n) f(x_1, \dots, x_n) \\
&\quad \times \log^m \left(\lambda^{-1} \prod_{j=1}^{n} x_j^{-a_j} \right) \mathrm{d}x_n \bigg] \cdots \mathrm{d}x_2 \bigg] \mathrm{d}x_1 \Bigg\} \lambda^{-\Lambda_e} \\
&\quad + \sum_{\mathrm{Re}(\Lambda_e) \leqslant r} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{i_1,\dots,i_k}^{n} \sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}} \frac{\Delta^{\Lambda_e}(n_{i_1}, \dots, n_{i_k})}{n_{i_1}! \dots n_{i_k}!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \\
&\quad \times \sum_{m=0}^{M(e)} \frac{(-1)^m g_{em} m! l_{i_1}! \cdots l_{i_k}!}{(\sum_{j=1}^{k} l_{i_j} + m + k)!} \Bigg\{ \prod_{j; i_1,\dots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - a_j \Lambda_e} \log^{l_j}(x_j) \\
&\quad \times f_{x_{i_1} \cdots x_{i_k}}^{n_{i_1} \cdots n_{i_k}} (\delta_{i_1}^n \cdots \delta_{i_k}^n x) \left[\log\left(\lambda^{-1} \prod_{j; i_1,\dots,i_k}^{n} x_j^{-a_j} \right) \right]^{(\sum_{j=1}^{k} l_{i_j} + m + k)} \mathrm{d}x_j \Bigg\} \lambda^{-\Lambda_e} \\
&\quad + o(\lambda^{-r}),
\end{aligned} \tag{4.1}
$$

where it is understood that $f_{x_{i_1} \cdots x_{i_k}}^{n_{i_1} \cdots n_{i_k}}(x) := \partial^{n_{i_1} + \cdots + n_{i_k}} f / \partial x_{i_1}^{n_{i_1}} \cdots \partial x_{i_k}^{n_{i_k}}(x)$ and that there exist a real $s \succ r$ and a function G_r^∞ bounded in a neighbourhood of infinity in which

$$
g(u) = \sum_{e=0}^{E} \sum_{m=0}^{M(e)} g_{em} u^{-\Lambda_e} \log^m u + u^{-s} G_r^\infty(u), \tag{4.2}
$$

with, since g is bounded on $\mathbb{R}_+$, $0 \prec \mathrm{Re}(\Lambda_e) \leqslant r \prec s$ ($\mathrm{Re}(\Lambda_0) \geqslant 0$ *if* $M(0) = 0$).

This theorem requires some remarks.

(i) The assumptions $\mathrm{Re}(\alpha_i) > -1$, f and g bounded respectively on $[0,1]^n$ and on $\mathbb{R}_+$ ensure the existence of $I_n^f(\lambda)$ for $\lambda \geqslant 0$.

(ii) Because $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$, $g \in \mathcal{B}_r(]0,+\infty[)$ and

$$\left[\log\left(\lambda^{-1}u^v \prod_{j;i_1,\dots,i_k}^{n} x_j^{-a_j}\right)\right]^m = \sum_{l=0}^{m}\sum_{q=0}^{l} C_m^l C_l^q \log^{m-l}(\lambda^{-1}) \log^{l-q}(u^v) \log^q \left[\prod_{j;i_1,\dots,i_k}^{n} x_j^{-a_j}\right],$$

each integral occurring on the right-hand side of (4.1) not only admits a sense but also (see proposition 7) satisfies Fubini's theorem and it is also understood that

$$\prod_{j;1,\dots,n}^{n} \boldsymbol{fp}\int_0^1 g(x_j) H[\delta_1^n \cdots \delta_n^n x] F\left(\prod_{j;1,\dots,n}^{n} x_j^{-a_j}\right) \mathrm{d}x_j := F(1) H[(0,\dots,0)]. \quad (4.3)$$

(iii) The proposed result is expressed with specific notations such as the sums $\sum_{i_1,\dots i_k}^{n}$,

$$\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}, \qquad \sum_{\mathrm{Re}(\Lambda_e)\leqslant r}$$

the product $\prod_{i_1,\dots,i_k}^{n}$ and the reals $\Delta(n_{i_1},\dots,n_{i_k})$ or $\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})$ which have been clearly defined. In applying this theorem, each of these notations must be carefully respected. For instance, formula (4.1) for $n=1$ gives theorem 11 for $g \in \mathcal{B}_r(]0,+\infty[) \subset \mathcal{C}_r(]0,+\infty[)$.

(iv) If the space $\mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$ is endowed with the classical norm $\|\cdot\|_n$ such that for $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$: $\|f\|_n := \sup\{|\partial^{i_1+\dots+i_n} f/\partial x_1^{i_1}\cdots\partial x_n^{i_n}(x)|, x \in [0,1]^n, \forall j \in \{1,\dots,n\} : 0 \leqslant i_j \leqslant q_j\}$, the dual space of generalized functions $\mathcal{D}'^{(q_1,\dots,q_n)}(]0,1[^n)$ is the set of linear and continuous (in the sense of the usual topology induced by this norm $\|\cdot\|_n$) functionals T on $\mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$ and the duality bracket $\langle T, f\rangle$ indicates the image of function f by distribution T. According to these definitions, $I_n^f(\lambda)$ rewrites $I_n^f(\lambda) = \langle T_n^g(\lambda), f\rangle$ with $T_n^g(\lambda) \in \mathcal{D}'^{(q_1,\dots,q_n)}(]0,1[^n)$. This asymptotic expansion of $\langle T_n^g(\lambda), f\rangle$ may also be shared into two kind of contributions: the terms related to the asymptotic behaviour of function g near infinity, i.e. the terms involving the coefficients g_{em} (see (4.2)), and the others. More precisely, inspection of equality (4.1) shows that $I_n^f(\lambda)$ rewrites

$$I_n^f(\lambda) = \langle T_n^g(\lambda), f\rangle = \sum_{i=1}^{n}\sum_{n_i=0}^{p_i}\sum_{j=0}^{J} \langle T_n^{i,n_i,j}, f\rangle \log^j(\lambda) \lambda^{-(\alpha_i+n_i+1)/a_i} + \sum_{\mathrm{Re}(\Lambda_e)\leqslant r}\sum_{j=0}^{J(e)} \langle G_n^{ej}, f\rangle \log^j(\lambda)\lambda^{-\Lambda_e} + o(\lambda^{-r}) \quad (4.4)$$

with $J := \sum_{i=1}^{n} l_i + n - 1$, $J(e) := \sum_{i=1}^{n} l_i + n + M(e)$ and $T_n^{i,n_i,j}$, G_n^{ej} belong to $\mathcal{D}'^{(q_1,\dots,q_n)}(]0,1[^n)$ with support in $\{x = (x_1,\dots,x_n) \in [0,1]^n; \prod_{i=1}^{n} x_i = 0\}$ or in $[0,1]^n$ (G_n^{ej} is concerned with the behaviour of g). This form (4.4) agrees with the results of Bruning (1984, theorem 1); here obtained by choosing $g \in \mathcal{S}(\mathbb{R})$, i.e. by setting $g_{em} = 0$ in formula (4.1).

For three real families $(c_i)_{i\in\{1,\dots,n\}}, (b_i)_{i\in\{1,\dots,n\}}$ and $(a_i)_{i\in\{1,\dots,n\}}$ such that $-\infty < c_i < 0 < b_i < +\infty$ and $a_i > 0$, one may be eager to find the asymptotic expansion of the integral

$$J_n^f(\lambda) = \int_{c_1}^{b_1} \cdots \int_{c_n}^{b_n} |y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n} \log^{l_1}(|y_1|) \cdots \log^{l_n}(|y_n|) \times f(y_1, \dots, y_n) g(\lambda |y_1|^{a_1} \cdots |y_n|^{a_n})\, \mathrm{d}y_1 \cdots \mathrm{d}y_n, \tag{4.5}$$

if $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]c_1, b_1[\times \cdots \times]c_n, b_n[)$ and $g \in \mathcal{B}_r(]0, +\infty[)$. Such a question reduces to an application of theorem 13. For $1 \leqslant i,\ j \leqslant n$ and $(s,t) \in \{-,+\}^2$, observe that $A_i^s A_j^t = A_j^t A_i^s$ if the operators A_i^+ and A_i^- are defined by

$$A_i^+[f](y_1,\dots,y_n) := f(y_1,\dots,y_n), \quad A_i^-[f](y_1,\dots,y_n) := 0, \quad \text{if } y_i \geqslant 0, \tag{4.6}$$

$$A_i^-[f](y_1,\dots,y_n) := f(y_1,\dots,y_n), \quad A_i^+[f](y_1,\dots,y_n) := 0, \quad \text{if } y_i < 0. \tag{4.7}$$

This remark indeed allows us not only to write (see formula (2.28)) for $n \geqslant 2$

$$f = \left[\prod_{i=1}^n A_i^+\right][f] + \left[\prod_{i=1}^n A_i^-\right][f] + \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_{\{1,\dots,n\}}} \sum_{p=1}^{n-1} C_n^p A_{\sigma(1)}^+ \cdots A_{\sigma(p)}^+ A_{\sigma(p+1)}^- \cdots A_{\sigma(n)}^-[f], \tag{4.8}$$

but also to cast, after adequate and legitimate changes of scale, the integral $J_n^f(\lambda)$ into the form

$$J_n^f(\lambda) = I_n^{f_0}(\lambda_0) + I_n^{f_n}(\lambda_n) + (n!)^{-1} \sum_{\sigma \in \mathcal{S}_{\{1,\dots,n\}}} \sum_{p=1}^{n-1} C_n^p I_n^{f_{\sigma,p}}(\lambda_{\sigma,p})$$

with the following relations:

$$\lambda_0 = \left[\prod_{i=1}^n b_i^{a_i}\right]\lambda, \quad \lambda_n = \left[\prod_{i=1}^n |c_i|^{a_i}\right]\lambda, \quad \lambda_{\sigma,p} = \left[\prod_{i=1}^p b_{\sigma(i)}^{a_{\sigma(i)}}\right]\left[\prod_{i=p+1}^n |c_{\sigma(i)}|^{a_{\sigma(i)}}\right]\lambda,$$

$$f_0(x_1,\dots,x_n) = \left[\prod_{i=1}^n b_i^{\alpha_i+1}\right]\left[\prod_{i=1}^n (\log(x_i) + \log b_i)^{l_i}\right] f(b_1x_1,\dots,b_nx_n), \tag{4.9}$$

$$f_n(x_1,\dots,x_n) = \left[\prod_{i=1}^n |c_i|^{\alpha_i+1}\right]\left[\prod_{i=1}^n (\log(x_i) + \log|c_i|)^{l_i}\right] f(c_1x_1,\dots,c_nx_n), \tag{4.10}$$

$$f_{\sigma,p}(x_1,\dots,x_n) = \left[\prod_{i=1}^p b_{\sigma(i)}^{\alpha_{\sigma(i)}+1} (\log(x_{\sigma(i)}) + \log b_{\sigma(i)})^{l_{\sigma(i)}}\right] \times \left[\prod_{i=p+1}^n |c_{\sigma(i)}|^{\alpha_{\sigma(i)}+1} (\log(x_{\sigma(i)}) + \log|c_{\sigma(i)}|)^{l_{\sigma(i)}}\right] \times f(b_{\sigma(1)}x_{\sigma(1)},\dots,b_{\sigma(p)}x_{\sigma(p)}, c_{\sigma(p+1)}x_{\sigma(p+1)},\dots,c_{\sigma(n)}x_{\sigma(n)}). \tag{4.11}$$

Equalities (4.9)–(4.11) show that, after an expansion of the logarithmic contributions, each function f_0, f_n or $f_{\sigma,p}$ is in fact a sum of terms $h(x_1,\dots,x_n) = [\prod_{i=1}^{n}\log^{e_i}(x_i)]w(x_1,\dots,x_n)$ with $e_i \in \mathbb{N}$ and $w \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$.

Before detailing the derivation of theorem 13 it is perhaps worth applying it to a special case. Since McClure & Wong (1987) dealt with the circumstances $n=2$, $(l_1,l_2)=(0,0)$, $(\alpha_1,\alpha_2)\in\mathbb{R}_+^{*2}$ and $g\in\mathcal{S}(\mathbb{R})$, the case of $n=3$ is proposed. Under assumptions and definitions introduced by theorem 13, one obtains

$$I_3^f(\lambda) = G(\lambda) + E(\lambda) + \sum_{\mathrm{Re}(\Lambda_e)\leqslant r}\sum_{m=0}^{M(e)} A_{em}[\log(\lambda)]\lambda^{-\Lambda_e} + o(\lambda^{-r}),$$

where asymptotic sequences $G(\lambda)$, $E(\lambda)$ and also $A_{em}[\log(\lambda)]$ are defined later. As in the general case of $I_n^l(\lambda)$, the asymptotic expansion of $I_3^f(\lambda)$ indeed contains two kind of terms: $G(\lambda)$ which is a sum of general terms in the sense it always arises and other extra terms $I_3^f(\lambda)-G(\lambda)$ which exist only when families $(a_i)_{i\in\{1,2,3\}}$, $(\alpha_i)_{i\in\{1,2,3\}}$ and $(\Lambda_e)_{\mathrm{Re}(\Lambda_e)\leqslant r}$ satisfy specific conditions. More precisely, one gets

$$\begin{aligned}
G(\lambda) = {} & \sum_{n_1=0}^{p_1}\Bigg\{\boldsymbol{fp}\int_0^1 x_2^{\alpha_2-(a_2/a_1)(\alpha_1+n_1+1)}\log^{l_2}(x_2) \\
& \times\Bigg[\boldsymbol{fp}\int_0^1 x_3^{\alpha_3-(a_3/a_1)(\alpha_1+n_1+1)}\log^{l_3}(x_3)\frac{f_{x_1}^{n_1}(0,x_2,x_3)}{n_1!a_1^{l_1+1}}\boldsymbol{fp}\int_0^\infty u^{((\alpha_1+n_1+1)/a_1)-1}g(u) \\
& \times\log^{l_1}(\lambda^{-1}ux_2^{-a_2}x_3^{-a_3})\,\mathrm{d}u\,\mathrm{d}x_3\Bigg]\,\mathrm{d}x_2\Bigg\}\lambda^{-(\alpha_1+n_1+1)/a_1} \\
& +\sum_{n_2=0}^{p_2}\Bigg\{\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-(a_1/a_2)(\alpha_2+n_2+1)}\log^{l_1}(x_1)\Bigg[\boldsymbol{fp}\int_0^1 x_3^{\alpha_3-(a_3/a_2)(\alpha_2+n_2+1)}\log^{l_3}(x_3) \\
& \times\frac{f_{x_2}^{n_2}(x_1,0,x_3)}{n_2!a_2^{l_2+1}}\boldsymbol{fp}\int_0^\infty u^{((\alpha_2+n_2+1)/a_2)-1}g(u) \\
& \times\log^{l_2}(\lambda^{-1}ux_1^{-a_1}x_3^{-a_3})\,\mathrm{d}u\,\mathrm{d}x_3\Bigg]\,\mathrm{d}x_1\Bigg\}\lambda^{-(\alpha_2+n_2+1)/a_2} \\
& +\sum_{n_3=0}^{p_3}\Bigg\{\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-(a_1/a_3)(\alpha_3+n_3+1)}\log^{l_1}(x_1)\Bigg[\boldsymbol{fp}\int_0^1 x_2^{\alpha_2-(a_2/a_3)(\alpha_3+n_3+1)}\log^{l_2}(x_2) \\
& \times\frac{f_{x_3}^{n_3}(x_1,x_2,0)}{n_3!a_3^{l_3+1}}\boldsymbol{fp}\int_0^\infty u^{((\alpha_3+n_3+1)/a_3)-1}g(u) \\
& \times\log^{l_3}(\lambda^{-1}ux_1^{-a_1}x_2^{-a_2})\,\mathrm{d}u\,\mathrm{d}x_2\Bigg]\,\mathrm{d}x_1\Bigg\}\lambda^{-(\alpha_3+n_3+1)/a_3} \\
& +\sum_{\mathrm{Re}(\Lambda_e)\leqslant r}\sum_{m=0}^{M(e)} g_{em}(-1)^m\Bigg\{\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\Lambda_e}\log^{l_1}(x_1)\Bigg[\boldsymbol{fp}\int_0^1 x_2^{\alpha_2-a_2\Lambda_e}\log^{l_2}(x_2) \\
& \times\Bigg[\boldsymbol{fp}\int_0^1 x_3^{\alpha_3-a_3\Lambda_e}\log^{l_3}(x_3)f(x_1,x_2,x_3) \\
& \times\log^m(\lambda^{-1}x_1^{-a_1}x_2^{-a_2}x_3^{-a_3})\,\mathrm{d}x_3\Bigg]\,\mathrm{d}x_2\Bigg]\,\mathrm{d}x_1\Bigg\}\lambda^{-\Lambda_e},
\end{aligned}$$

whereas sum $E(\lambda)$ is non-zero if there exists (n_1, n_2, n_3) with $0 \leqslant n_i \leqslant p_i$, $i \in \{1, 2, 3\}$ and

$$\Delta(n_1, n_2) + \Delta(n_1, n_3) + \Delta(n_2, n_3) \geqslant 1$$

and obeys

$$\begin{aligned}
E(\lambda) = &- \sum_{n_1=0}^{p_1} \sum_{n_2=0}^{p_2} \left\{ \frac{\Delta(n_1, n_2) l_1! l_2!}{(l_1 + l_2 + 1)! a_1^{l_1+1} a_2^{l_2+1}} \right. \\
&\times \boldsymbol{fp} \int_0^1 x_3^{\alpha_3 - (a_3/a_1)(\alpha_1 + n_1 + 1)} \log^{l_3}(x_3) \frac{f_{x_1 x_2}^{n_1 n_2}(0, 0, x_3)}{n_1! n_2!} \\
&\times \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_1 + n_1 + 1)/a_1) - 1} g(u) \right. \\
&\left. \left. \times [\log(\lambda^{-1} u x_3^{-a_3})]^{(l_1 + l_2 + 1)} \, \mathrm{d}u \right] \mathrm{d}x_3 \right\} \lambda^{-(\alpha_1 + n_1 + 1)/a_1} \\
&- \sum_{n_1=0}^{p_1} \sum_{n_3=0}^{p_3} \left\{ \frac{\Delta(n_1, n_3) l_1! l_3!}{(l_1 + l_3 + 1)! a_1^{l_1+1} a_3^{l_3+1}} \right. \\
&\times \boldsymbol{fp} \int_0^1 x_2^{\alpha_2 - (a_2/a_1)(\alpha_1 + n_1 + 1)} \log^{l_2}(x_2) \frac{f_{x_1 x_3}^{n_1 n_3}(0, x_2, 0)}{n_1! n_3!} \\
&\times \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_1 + n_1 + 1)/a_1) - 1} g(u) \right. \\
&\left. \left. \times [\log(\lambda^{-1} u x_2^{-a_2})]^{(l_1 + l_3 + 1)} \, \mathrm{d}u \right] \mathrm{d}x_2 \right\} \lambda^{-(\alpha_1 + n_1 + 1)/a_1} \\
&- \sum_{n_2=0}^{p_2} \sum_{n_3=0}^{p_3} \left\{ \frac{\Delta(n_2, n_3) l_2! l_3!}{(l_2 + l_3 + 1)! a_2^{l_2+1} a_3^{l_3+1}} \right. \\
&\times \boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - (a_1/a_2)(\alpha_2 + n_2 + 1)} \log^{l_1}(x_1) \frac{f_{x_2 x_3}^{n_2 n_3}(x_1, 0, 0)}{n_2! n_3!} \\
&\times \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_2 + n_2 + 1)/a_2) - 1} g(u) \right. \\
&\left. \left. \times [\log(\lambda^{-1} u x_1^{-a_1})]^{(l_2 + l_3 + 1)} \, \mathrm{d}u \right] \mathrm{d}x_1 \right\} \lambda^{-(\alpha_3 + n_3 + 1)/a_3} \\
&+ \sum_{n_1=0}^{p_1} \sum_{n_2=0}^{p_2} \sum_{n_3=0}^{p_3} \frac{\Delta(n_1, n_2, n_3) l_1! l_2! l_3! f_{n_1 n_2 n_3}^{x_1 x_2 x_3}(0, 0, 0)}{(l_1 + l_2 + l_3 + 2)! a_1^{l_1+1} a_2^{l_2+1} a_3^{l_3+1} n_1! n_2! n_3!} \\
&\times \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_1 + n_1 + 1)/a_1) - 1} g(u) [\log(\lambda^{-1} u)]^{(l_1 + l_2 + l_3 + 2)} \, \mathrm{d}u \right] \lambda^{-(\alpha_1 + n_1 + 1)/a_1},
\end{aligned}$$

and finally each remaining term $A_{em}[\log(\lambda)]$ takes into account the behaviour of function g at infinity and is defined as

$$A_{em}[\log(\lambda)] = -\sum_{n_1=0}^{p_1} \frac{\Delta^{\Lambda_e}(n_1)(-1)^m g_{em} m! l_1!}{(l_1+m+1)! a_1^{l_1+1} n_1!} \left[\boldsymbol{fp} \int_0^1 x_2^{\alpha_2 - a_2\Lambda_e} \log^{l_2}(x_2) \right.$$
$$\left. \times \left[\boldsymbol{fp} \int_0^1 x_3^{\alpha_3 - a_3\Lambda_e} \log^{l_3}(x_3) f_{x_1}^{n_1}(0, x_2, x_3) [\log(\lambda^{-1} x_2^{-a_2} x_3^{-a_3})]^{(l_1+m+1)} \, \mathrm{d}x_3 \right] \mathrm{d}x_2 \right]$$
$$-\sum_{n_2=0}^{p_2} \frac{\Delta^{\Lambda_e}(n_2)(-1)^m g_{em} m! l_2!}{(l_2+m+1)! a_2^{l_2+1} n_2!} \left[\boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - a_1\Lambda_e} \log^{l_1}(x_1) \right.$$
$$\left. \times \left[\boldsymbol{fp} \int_0^1 x_3^{\alpha_3 - a_3\Lambda_e} \log^{l_3}(x_3) f_{x_2}^{n_2}(x_1, 0, x_3) [\log(\lambda^{-1} x_1^{-a_1} x_3^{-a_3})]^{(l_2+m+1)} \, \mathrm{d}x_3 \right] \mathrm{d}x_1 \right]$$
$$-\sum_{n_3=0}^{p_3} \frac{\Delta^{\Lambda_e}(n_3)(-1)^m g_{em} m! l_3!}{(l_3+m+1)! a_3^{l_3+1} n_3!} \left[\boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - a_1\Lambda_e} \log^{l_1}(x_1) \right.$$
$$\left. \times \left[\boldsymbol{fp} \int_0^1 x_2^{\alpha_2 - a_2\Lambda_e} \log^{l_2}(x_2) f_{x_3}^{n_3}(x_1, x_2, 0) [\log(\lambda^{-1} x_1^{-a_1} x_2^{-a_2})]^{(l_3+m+1)} \, \mathrm{d}x_2 \right] \mathrm{d}x_1 \right]$$
$$+\sum_{n_1=0}^{p_1} \sum_{n_2=0}^{p_2} \frac{\Delta^{\Lambda_e}(n_1, n_2)(-1)^m g_{em} m! l_1! l_2!}{(l_1+l_2+m+2)! a_1^{l_1+1} a_2^{l_2+1} n_1! n_2!}$$
$$\times \left[\boldsymbol{fp} \int_0^1 x_3^{\alpha_3 - a_3\Lambda_e} \log^{l_3}(x_3) f_{x_1 x_2}^{n_1 n_2}(0, 0, x_3) [\log(\lambda^{-1} x_3^{-a_3})]^{(l_1+l_2+m+2)} \, \mathrm{d}x_3 \right]$$
$$+\sum_{n_1=0}^{p_1} \sum_{n_3=0}^{p_3} \frac{\Delta^{\Lambda_e}(n_1, n_3)(-1)^m g_{em} m! l_1! l_3!}{(l_1+l_3+m+2)! a_1^{l_1+1} a_3^{l_3+1} n_1! n_3!}$$
$$\times \left[\boldsymbol{fp} \int_0^1 x_2^{\alpha_2 - a_2\Lambda_e} \log^{l_2}(x_2) f_{x_1 x_3}^{n_1 n_3}(0, x_2, 0) [\log(\lambda^{-1} x_2^{-a_2})]^{(l_1+l_2+m+2)} \, \mathrm{d}x_2 \right]$$
$$+\sum_{n_2=0}^{p_2} \sum_{n_3=0}^{p_3} \frac{\Delta^{\Lambda_e}(n_2, n_3)(-1)^m g_{em} m! l_2! l_3!}{(l_2+l_3+m+2)! a_2^{l_2+1} a_3^{l_3+1} n_2! n_3!}$$
$$\times \left[\boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - a_1\Lambda_e} \log^{l_1}(x_1) f_{x_2 x_3}^{n_2 n_3}(x_1, 0, 0) [\log(\lambda^{-1} x_1^{-a_1})]^{(l_2+l_3+m+2)} \, \mathrm{d}x_1 \right]$$
$$-\sum_{n_1=0}^{p_1} \sum_{n_2=0}^{p_2} \sum_{n_3=0}^{p_3} \frac{\Delta^{\Lambda_e}(n_1, n_2, n_3)(-1)^m g_{em} m! l_1! l_2! l_3!}{(l_1+l_2+l_3+m+3)! a_1^{l_1+1} a_2^{l_2+1} a_3^{l_3+1} n_1! n_2! n_3!}$$
$$\times f_{x_1 x_2 x_3}^{n_1 n_2 n_3}(0, 0, 0) [\log(\lambda^{-1})]^{(l_1+l_2+l_3+m+3)}.$$

Example. Assume that $l \in \mathbb{N}$, $\alpha \in C \setminus \mathbb{N}$ with $\mathrm{Re}(\alpha) > -1$ and also that $r \geqslant \max[\mathrm{Re}(\alpha)+1, 1]$. Under these assumptions, previous results authorize us to give the next asymptotic expansion

$$\int_0^1 \int_0^1 \int_0^1 \frac{\log^l(x_1) u(x_2) v(x_3)}{x_1^{-\alpha}(1+\lambda x_1 x_2^2 x_3^3)} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = G_3(\lambda) + E_3(\lambda) + \sum_{e=0}^{[\![r-1]\!]} A_e[\log(\lambda)] + o(\lambda^{-r}),$$

as soon as $u \in \mathcal{D}^{q_2}(]0,1[)$, $v \in \mathcal{D}^{q_3}(]0,1[)$ if $q_2 = [\![2r]\!]$ and $q_3 = [\![3r]\!]$. By introducing for $\beta \in C$, $k \in \mathbb{N}$ and $s \in \mathcal{D}^q(]0,1[)$ for $q = \max([\![-\operatorname{Re}(\beta)]\!], 0)$ the following notations

$$T_k[\beta, s] := \boldsymbol{fp}\int_0^1 x^\beta \log^k(x) s(x)\,\mathrm{d}x, \quad G_k(\beta) := \boldsymbol{fp}\int_0^\infty \frac{u^{\beta-1}\log^k(u)}{1+u}\,\mathrm{d}u, \qquad (4.12)$$

and also function δ such that $\delta(a,b) := 0$ for complex values a and b except if $a = b$ then $\delta(a,a) := 1$, one indeed obtains

$$\begin{aligned} G_3(\lambda) &= \sum_{k=0}^{l}\sum_{m=0}^{k}\sum_{n=0}^{m} C_l^k C_k^m C_n^m (-1)^{l+k-m} 2^n 3^{m-n} T_n[-2(\alpha+1), u] \\ &\times T_{m-n}[-3(\alpha+1), v] G_{k-m}(\alpha) \log^{l-k}(\lambda)\lambda^{-(\alpha+1)} \\ &+ \sum_{n_2=0}^{p_2} \frac{u^{(n_2)}(0)}{2n_2!} T_l[\alpha - \tfrac{1}{2}(n_2+1), id] T_0[-\tfrac{3}{2}(n_2+1), v] G_0[\tfrac{1}{2}(n_2+1)]\lambda^{-(n_2+1)/2} \\ &+ \sum_{n_3=0}^{p_3} \frac{v^{(n_3)}(0)}{3n_3!} T_l[\alpha - \tfrac{1}{3}(n_3+1), id] T_0[-\tfrac{2}{3}(n_3+1), u] G_0[\tfrac{1}{3}(n_3+1)]\lambda^{-(n_3+1/3)} \\ &+ \sum_{e=0}^{[\![r-1]\!]} (-1)^e T_l[\alpha - (e+1), id] T_0[-2(e+1), u] T_0[-3(e+1), v]\lambda^{-(e+1)} \end{aligned}$$

and also, since α is not a positive integer,

$$\begin{aligned} E_3(\lambda) &= \sum_{n_2=0}^{p_2}\sum_{n_3=0}^{p_3} \frac{\delta(2n_2, 3n_3+1) u^{(n_2)}(0) v^{(n_3)}(0)}{6n_2! n_3!} \\ &\times \{T_l[\alpha - \tfrac{1}{2}(n_2+1), id] G_0[\tfrac{1}{2}(n_2+1)] - T[\alpha - \tfrac{1}{2}(n_2+1), id] G_1[\tfrac{1}{2}(n_2+1)] \\ &+ T_{l+1}[\alpha - \tfrac{1}{2}(n_2+1), id] G_0[\tfrac{1}{2}(n_2+1)]\}\lambda^{-(n_2+1/2)}, \end{aligned}$$

$$\begin{aligned} A_e[\log(\lambda)] &= \sum_{n_2=0}^{p_2} \frac{\delta(2e+1, n_2)(-1)^e}{2n_2!} u^{(n_2)}(0) \\ &\times \{T_l[\alpha - (e+1), id] T_0[-3(e+1), v] \log\lambda + T_{l+1}[\alpha - (e+1), id] T_0[-3(e+1), v] \\ &+ 3T_l[\alpha - (e+1), id] T_1[-3(e+1), v]\} \\ &+ \sum_{n_3=0}^{p_3} \frac{\delta(3e+2, n_3)(-1)^e}{3n_3!} v^{(n_3)}(0) \{T_l[\alpha - (e+1), id] T_0[-2(e+1), u] \log\lambda \\ &+ T_{l+1}[\alpha - (e+1), id] T_0[-2(e+1), u] + 2T_l[\alpha - (e+1), id] T_1[-2(e+1), u]\} \\ &+ \sum_{n_2=0}^{p_2}\sum_{n_3=0}^{p_3} \frac{\Delta^{e+1}(n_2, n_3) u^{(n_2)}(0) v^{(n_3)}(0)}{12n_2! n_3!} \{T_l[\alpha - (e+1), id] \log^2\lambda \\ &+ 2T_{l+1}[\alpha - (e+1), id] \log\lambda + T_{l+2}[\alpha - (e+1), id]\}, \end{aligned}$$

where id denotes the characteristic function on $]0,1[$.

Theorem 13 is established by induction on n. In fact we prove by induction a proposition which states a larger result.

Proposition 14. *For $n \geqslant 1$, real $r > 0$ and families (α_i), (a_i), (l_i), (q_i), (p_i) such that for any $i \in \{1, \ldots, n\}$: $\alpha_i \in C$ with* $\operatorname{Re}(\alpha_i) > -1$, $a_i \in \mathbb{R}_+^*$, $l_i \in \mathbb{N}$,

$q_i = \max([\![a_i r - \mathrm{Re}(\alpha_i)]\!], 0)$ *and* $p_i = [\![a_i r - \mathrm{Re}(\alpha_i)]\!] - 1$; *if* $f \in \mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$ *and* $g \in \mathcal{B}_r(]0,+\infty[)$, *then we have the following.*

Induction assumption 1. *As* λ *tends to infinity, expansion (4.1) holds. We note* $I_n^f(\lambda) = \langle T_n^g(\lambda), f\rangle = \sum_{i,\mathrm{Re}(\delta_i)\leqslant r}\sum_{j=0}^{J(i)}\langle D_n^{ij}, f\rangle \lambda^{-\delta_i}\log^j(\lambda) + o(\lambda^{-r})$.

Induction assumption 2. *For* $b>0$, $h\in\mathbb{N}$, β *complex such that* $br \geqslant \mathrm{Re}(\beta)+1>0$ *and* $\gamma := (\beta+1)/b$,

$$\begin{aligned}
\langle K_n, f\rangle &= \boldsymbol{fp}\int_0^\infty v^\beta \log^h(v)\bigg[\int_0^1\cdots\int_0^1 x_1^{\alpha_1}\dots x_n^{\alpha_n}\log^{l_1}(x_1)\dots\log^{l_n}(x_n)\\
&\quad\times f(x_1,\dots,x_n) g(v^b x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\bigg]\,\mathrm{d}v\\
&= \left(\frac{1}{b}\right)^{h+1}\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\gamma}\log^{l_1}(x_1)\cdots\bigg[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\gamma}\log^{l_n}(x_n)\\
&\quad\times\left\{\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\log^h(ux_1^{-a_1}\cdots x_n^{-a_n})g(u)\,\mathrm{d}u\right\}f(x_1,\dots,x_n)\,\mathrm{d}x_n\bigg]\cdots\mathrm{d}x_1\\
&\quad+\frac{h!}{b^{h+1}}\sum_{k=1}^n\frac{(-1)^k}{k!}\sum_{i_1,\dots,i_k}^n\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}\frac{\Delta^\gamma(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\dots n_{i_k}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\\
&\quad\times\frac{l_{i_1}!\dots l_{i_k}!}{(h+\sum_{j=1}^k l_{i_j}+k)!}\bigg\{\prod_{j;i_1,\dots,i_k}^n\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\gamma}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\dots\delta_{i_k}^n x)\\
&\quad\times\bigg[\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\bigg[\log(u\prod_{j;i_1,\dots,i_k}^n x_j^{-a_j})\bigg]^{(h+\sum_{j=1}^k l_{i_j}+k)}g(u)\,\mathrm{d}u\bigg]\,\mathrm{d}x_j\bigg\}\\
&\quad+\frac{h!}{b^{h+1}}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{m=0}^{M(e)}\frac{g_{em}(-1)^{m+1}m!}{(m+h+1)!}\bigg\{\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\Lambda_e}\log^{l_1}(x_1)\bigg[\cdots\\
&\quad\times\bigg[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\Lambda_e}\\
&\quad\times\log^{l_n}(x_n)f(x_1,\dots,x_n)\log^{m+h+1}(x_1^{-a_1}\cdots x_n^{-a_n})\,\mathrm{d}x_n\bigg]\cdots\bigg]\,\mathrm{d}x_1\bigg\}\\
&\quad+\frac{h!}{b^{h+1}}\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\dots,i_k}^n\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}\\
&\quad\times\frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\\
&\quad\times\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!\,l_{i_1}!\cdots l_{i_k}!}{(1+h+\sum_{j=1}^k l_{i_j}+m+k)!}\bigg\{\prod_{j;i_1,\dots,i_k}^n\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)\\
&\quad\times f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x)\bigg[\log(\prod_{j;i_1,\dots i_k}^n x_j^{-a_j})\bigg]^{(1+h+\sum_{j=1}^k l_{i_j}+m+k)}\mathrm{d}x_j\bigg\},
\end{aligned}\qquad(4.13)$$

where the involved sums or products are defined at the beginning of this section (see theorem 13).

Formula (4.13) gives the corrective terms arising when attempting to apply successively Fubini's theorem and change of variable $u := v^b x_1^{a_1} \cdots x_n^{a_n}$ to the initial integral $\langle K_n, f\rangle$. It is in fact an extension of lemma 12. If $n = 1$, remember that $\sum_{k=1}^{n-1} := 0$ and the result writes for $\mathrm{Re}(\alpha) > -1$, $l \in \mathbb{N}$, $a > 0$ and $\gamma = (\beta+1)/b$

$$\begin{aligned}\langle K_1, f\rangle &= \boldsymbol{fp}\int_0^{\infty} v^{\beta}\log^h(v)\left[\int_0^1 t^{\alpha}\log^l(t)f(t)g(v^b t^a)\,\mathrm{d}t\right]\mathrm{d}v\\ &= \left(\frac{1}{b}\right)^{h+1}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)f(t)\left[\boldsymbol{fp}\int_0^{\infty} u^{\gamma-1}\log^h(ut^{-a})g(u)\,\mathrm{d}u\right]\mathrm{d}t\\ &\quad-\sum_{\{i;\alpha_1+i+1=a\gamma\}}\frac{h!l!}{b^{h+1}(1+h+l)!}\left(\frac{1}{a}\right)^{l+1}\frac{f^i(0)}{i!}\left[\boldsymbol{fp}\int_0^{\infty}u^{\gamma-1}\log^{1+h+l}(u)g(u)\,\mathrm{d}u\right]\\ &\quad+\frac{h!}{b^{h+1}}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{m=0}^{M(e)}\frac{g_{em}(-1)^{m+1}m!}{(m+h+1)!}\boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e}\log^l(t)\log^{m+h+1}(t^{-a})f(t)\,\mathrm{d}t.\end{aligned}\tag{4.14}$$

By now, real $r > 0$ and function g are given with $g \in \mathcal{B}_r(]0,+\infty[)$. This formula (4.14) is proved in § 6. Because theorem 11 ensures induction assuption 1, proposition 14 is true for $n = 1$.

Assume that it remains true for $n \geqslant 1$ and consider $\alpha \in C$ with $\mathrm{Re}(\alpha) > -1$, $a > 0$, $l \in \mathbb{N}$, $q := \max([\![ar-\mathrm{Re}(\alpha)]\!], 0)$, $p := [\![ar-\mathrm{Re}(\alpha)]\!]-1$ and $f \in \mathcal{D}^{(q_1,\ldots,q_n,q)}(]0,1[^{n+1})$. With notation $x := (x_1,\ldots,x_n)$, we consider the two quantities

$$\begin{aligned}I_{n+1}^f(\lambda) = \int_0^1\cdots\int_0^1 & x_1^{\alpha_1}\cdots x_n^{\alpha_n}t^{\alpha}\log^{l_1}(x_1)\cdots\log^{l_n}(x_n)\log^l(t)f(x,t)\\ &\times g(\lambda x_1^{a_1}\cdots x_n^{a_n}t^a)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}t,\end{aligned}\tag{4.15}$$

$$\begin{aligned}\langle K_{n+1}, f\rangle = \boldsymbol{fp}\int_0^{\infty} v^{\beta}\log^h(v)\Bigg[\int_0^1\cdots\int_0^1 & t^{\alpha}\log^l(t)x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots\log^{l_n}(x_n)\\ &\times f(x,t)g(v^b t^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}t\Bigg]\mathrm{d}v,\end{aligned}\tag{4.16}$$

where β is complex and such that $br \geqslant \mathrm{Re}(\beta)+1 > 0$. Section 5 is devoted to the study of $I_{n+1}^f(\lambda)$ and § 6 will deal with $\langle K_{n+1}, f\rangle$.

5. Treatment of quantity $I_{n+1}^f(\lambda)$

Introduction of real $\lambda' := \lambda^{1/a}$ and of pseudo-function $K(t,u)$ defined by

$$\begin{aligned}\frac{K(t,u)}{t^{\alpha}\log^l(t)} = \int_0^1\cdots\int_0^1 & x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots\\ &\times\log^{l_n}(x_n)f(x,t)g(u^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\end{aligned}\tag{5.1}$$

leads (after a legitimate application of Fubini's theorem to $I^f_{n+1}(\lambda)$) to the useful form

$$I^f_{n+1}(\lambda) = \int_0^1 K(t, \lambda' t)\,\mathrm{d}t. \tag{5.2}$$

The expansion with respect to the large parameter λ' of this latter quantity consists of several steps and makes use of theorem 9.

Step 1. At this first stage it is shown that the pseudo-function $K(t,u)$ defined by (5.1) belongs to the set $\mathcal{F}^{ar}_{ar-1}(]0,1[,C)$. Clearly, $K(t,u) \in \mathcal{P}(]0,1[,C)$. Each property of definition 5 is now investigated and we set $r_1 := ar-1$, $r_2 := ar$. For $t \in]0,1[$, equality (5.1) rewrites

$$K(t,u) = t^\alpha \log^l(t) I^{h_t}_n(u^a) := t^\alpha \log^l(t) \langle < T^g_n(u^a), h_t\rangle, \tag{5.3}$$

where $h_t(x_1,\dots,x_n) := f(x_1,\dots,x_n,t)$. Since $a>0$, $u^a \to +\infty$ when $u \to +\infty$ and for given $t \in]0,1[$ induction assumption 1 (for n) provides (see (4.4)) the following expansion:

$$K(t,u) = \sum_{i=1}^{n} \sum_{n_i=0}^{p_i} \sum_{j=0}^{J} \langle T^{i,n_i,j}_n, t^\alpha \log^l(t) f(.,t)\rangle \log^j(u^a)[u^a]^{-(\alpha_i+n_i+1)/a_i}$$
$$+ \sum_{\mathrm{Re}(\Lambda_e)\leqslant r} \sum_{j=0}^{J(e)} \langle G^{e,j}_n, t^\alpha \log^l(t) f(.,t)\rangle \log^j(u^a)[u^a]^{-\Lambda_e} + o(u^{-ar}). \tag{5.4}$$

Expansion (5.4) shows that $K(t,u)$ obeys equality (3.1). Moreover, with these notations the integral $F = \int_\eta^1 t^{-s_2} G_{r_2}(t,\lambda' t)\,\mathrm{d}t$ reduces to

$$F = \int_\eta^1 \Bigg\{ K(t,\lambda' t) - \sum_{\mathrm{Re}(\Lambda_e)\leqslant r} \sum_{j=0}^{J(e)} \langle G^{e,j}_n, t^\alpha \log^l(t) f(.,t)\rangle \log^j[(\lambda' t)^a][\lambda' t]^{-a\Lambda_e}$$
$$- \sum_{i=1}^{n} \sum_{n_i=0}^{p_i} \sum_{j=0}^{J} \langle T^{i,n_i,j}_n, t^\alpha \log^l(t) f(.,t)\rangle \log^j[(\lambda' t)^a][\lambda' t]^{-a(\alpha_i+n_i+1)/a_i} \Bigg\}\,\mathrm{d}t. \tag{5.5}$$

Since $\eta > 0$, such an integral exists and is bounded. Consequently, inequality (3.2) is satisfied. If $D \in \mathcal{D}'^{(q_1,\dots,q_n)}(]0,1[^n)$, definitions of integer $p := [\![ar - \mathrm{Re}(\alpha)]\!] - 1$ and of the sum $\sum_{i=0}^{p} F(i)$ yield

$$\langle D, t^\alpha \log^l(t) f(.,t)\rangle = \sum_{N=0}^{p} d^{Nl} t^{\alpha+N} \log^l(t) + t^{s_1} L_D(t), \tag{5.6}$$

with $s_1 > ar-1$ and L_D bounded in a neighbourhood on the right of zero. More precisely, if $p<0$, $\sum_{N=0}^{p} := 0$ and since the assumption $[\![ar - \mathrm{Re}(\alpha)]\!] \leqslant 0$ ensures $\mathrm{Re}(\alpha) > ar-1$, we choose $s_1 := \frac{1}{2}[ar-1+\mathrm{Re}(\alpha)]$; $L_D(t) = \langle D, t^{\alpha-s_1}\log^l(t) f(.,t)\rangle$. If $p \geqslant 0$, $q = [\![ar - \mathrm{Re}(\alpha)]\!] = p+1 \geqslant 1$ and use of formula (2.17) for $f(.,t) \in \mathcal{D}^q(]0,1[)$ gives

$$d^{Nl} = \langle D, f^N_t(.,t)\rangle / N!,$$
$$t^{s_1} L_D(t) = \frac{t^{\alpha+p+1}\log^l(t)}{p!} \int_0^1 (1-v)^p \langle D, f^{p+1}_t(.,vt)\rangle\,\mathrm{d}v \tag{5.7}$$

and since $[\![ar-\mathrm{Re}(\alpha)]\!] > ar-\mathrm{Re}(\alpha)-1$, choice of $s_1 := \frac{1}{2}[\mathrm{Re}(\alpha)+[\![ar-\mathrm{Re}(\alpha)]\!]+ar-1]$ is possible. By applying (5.6) to each contribution on the right-hand side of expansion (5.4), one checks equality (3.3).

For given $u>0$ and thanks to (5.3), $K(t,u) = \langle T_n^g(u^a), t^\alpha \log^l(t) f(.,t)\rangle$, where $T_n^g(u^a) \in \mathcal{D}'^{(q_1,\dots,q_n)}(]0,1[^n)$. This remark gives the behaviour of $K(t,u)$ as $t\to 0$. Using (5.6), one obtains

$$K(t,u) = \sum_{N=0}^{p} h^{Nl}(u) t^{\alpha+N}\log^l(t) + t^{s_1} H_{r_1}(t,u), \quad s_1 > ar-1 \tag{5.8}$$

with: if $p<0$, $\sum_{N=0}^{p} := 0$ and $t^{s_1}H_{r_1}(t,u) := K(t,u)$ ($s_1 := \frac{1}{2}[ar-1+\mathrm{Re}(\alpha)]$) and if $p \geqslant 0$ the result, obtained by $\langle T_n^g(u^a), f(.,t)\rangle$ with respect to t, is

$$h^{Nl}(u) = \frac{1}{N!}\int_0^1 \cdots \int_0^1 x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots \times \log^{l_n}(x_n) f_t^N(x,0) g(u^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

$$t^{s_1}H_{r_1}(t,u) = \frac{t^{\alpha+p+1}\log^l(t)}{p!}\int_0^1 (1-v)^p \left[\int_0^1\cdots\int_0^1 x_1^{\alpha_1}\cdots x_n^{\alpha_n} \times \log^{l_1}(x_1)\cdots\log^{l_n}(x_n) f_t^{p+1}(x,vt) g(u^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right]\mathrm{d}v. \tag{5.9}$$

Thus, $K(t,u)$ obeys relation (3.4). The reader may easily check with the definition of $t^{s_1}H_{r_1}(t,u)$ that inequality (3.5) is true. Finally, induction assumption 1 allows us to expand each function $h^{Nl}(u)$ as $u\to+\infty$ and thereby provides equality (3.6).

Hence, $K(t,u)\in \mathcal{F}_{ar-1}^{ar}(]0,1[,C)$ and theorem 9 applies to (5.2) with respect to the parameter $\lambda' := \lambda^{1/a}$ and up to order $o(\lambda'^{-ar}) = o(\lambda^{-r})$. As $h^{Nl}(u)$ remains bounded when u tends to zero and for this theorem $\alpha_N := \alpha+N$ ($\mathrm{Re}(\alpha_N)\geqslant \mathrm{Re}(\alpha) > -1$), the third term on the right-hand side of (3.8) (the one which involves specific coefficients H_{pq}^{Nl} associated to the behaviour of function h^{Nl} near zero) is zero. Hence,

$$I_{n+1}^f(\lambda) = \int_0^1 K(t,\lambda' t)\,\mathrm{d}t = C_1(\lambda)+C_2(\lambda)+C_3(\lambda)+o(\lambda^{-r}),$$

where $C_1(\lambda)$, $C_2(\lambda)$ and $C_3(\lambda)$ designate the remaining contributions occurring in expansion (3.8)).

Step 2: treatment of contribution $C_1(\lambda) = C_1(\lambda')$. This term is the first sum arising on the right-hand side of (3.8). With the notation of theorem 9 (i.e. if $K(t,u) = \sum_{m,\mathrm{Re}(\gamma)\leqslant ar} K_{nm}(t)u^{-\gamma_n}\log^m(u) + u^{-s_2}G_{r_2}(t,u)$ as $u\to+\infty$), this term writes

$$C_1(\lambda') = \sum_{m,\mathrm{Re}(\gamma)\leqslant ar}\sum_{l=0}^{m} C_m^l\left[\boldsymbol{fp}\int_0^1 K_{nm}(t)t^{-\gamma_n}\log^{m-l}(t)\,\mathrm{d}t\right]\lambda'^{-\gamma_n}\log^l\lambda'$$
$$= \boldsymbol{fp}\int_0^1\left[\sum_{m,\mathrm{Re}(\gamma)\leqslant ar} K_{nm}(t)(\lambda' t)^{-\gamma_n}\log^m(\lambda' t)\right]\mathrm{d}t. \tag{5.10}$$

Observe that (5.10) is obtained by replacing $K(t,u)$ by its expansion in which $u :=$

$\lambda' t$. By using expansions (5.4), (4.1) and the link $\lambda' := \lambda^{1/a}$, this remark immediately leads to

$$C_1(\lambda) = \sum_{\mathrm{Re}(\Lambda_e)\leqslant r} \sum_{j=0}^{J(e)} \left[\boldsymbol{fp} \int_0^1 t^\alpha \log^l(t) \langle G_n^{e,j}, f(.,t)\rangle (\lambda' t)^{-a\Lambda_e} \log^j[(\lambda' t)^a] \, \mathrm{d}t \right]$$

$$+ \sum_{i=1}^{n} \sum_{n_i=0}^{p_i} \sum_{j=0}^{J} \left[\boldsymbol{fp} \int_0^1 t^\alpha \log^l(t) \langle T_n^{i,n_i,j}, f(.,t)\rangle (\lambda' t)^{-(a/a_i)(\alpha_i+n_i+1)} \log^j[(\lambda' t)^a] \, \mathrm{d}t \right]$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \frac{\Delta(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!}$$

$$\times \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \frac{l_{i_1}!\ldots l_{i_k}!}{(\sum_{j=1}^{k} l_{i_j} + k - 1)!}$$

$$\times \boldsymbol{fp} \int_0^1 t^{\alpha-(a/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)} \log^l(t) \Bigg\{ \prod_{j;i_1,\ldots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j-(a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}$$

$$\times \log^{l_j}(x_j) f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n \cdots \delta_{i_k}^n x') \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_{i_1}+n_{i_1}+1)/a_{i_1})-1} \right.$$

$$\left. \times \left[\log \left(\lambda^{-1} u t^{-a} \prod_{j;i_1,\ldots,i_k}^{n} x_j^{-a_j} \right) \right]^{\left(\sum_{j=1}^{k} l_{i_j}+k-1\right)} g(u)\, \mathrm{d}u \right] \mathrm{d}x_j \Bigg\} \, \mathrm{d}t$$

$$\times \lambda^{-(\alpha_{i_1}+n_{i_1}+1)/a_{i_1}}$$

$$+ \sum_{\mathrm{Re}(\Lambda_e)\leqslant r} \sum_{m=0}^{M(e)} g_{em}(-1)^m \Bigg\{ \boldsymbol{fp} \int_0^1 t^{\alpha-a\Lambda_e} \log^l(t) \left[\boldsymbol{fp} \int_0^1 x_1^{\alpha_1-a_1\Lambda_e} \log^{l_1}(x_1) \cdots \right.$$

$$\left. \times \left[\boldsymbol{fp} \int_0^1 x_n^{\alpha_n-a_n\Lambda_e} \log^{l_n}(x_n) f(x') \log^m \left(\lambda^{-1} t^{-a} \prod_{j=1}^{n} x_j^{-a_j} \right) \mathrm{d}x_n \right] \cdots \mathrm{d}x_1 \right] \mathrm{d}t \Bigg\} \lambda^{-\Lambda_e}$$

$$+ \sum_{\mathrm{Re}(\Lambda_e)\leqslant r} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}$$

$$\times \sum_{m=0}^{M(e)} \frac{(-1)^m g_{em} m! l_{i_1}! \cdots l_{i_k}!}{(\sum_{j=1}^{k} l_{i_j} + m + k)!} \boldsymbol{fp} \int_0^1 t^{\alpha-a\Lambda_e}$$

$$\times \log^l(t) \Bigg\{ \prod_{j;i_1,\ldots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j-a_j\Lambda_e} \log^{l_j}(x_j)$$

$$\times f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n \ldots \delta_{i_k}^n x') \left[\log \left(\lambda^{-1} t^{-a} \prod_{j;i_1,\ldots,i_k}^{n} x_j^{-a_j} \right) \right]^{\left(\sum_{j=1}^{k} l_{i_j}+m+k\right)} \mathrm{d}x_j \Bigg\} \, \mathrm{d}t \lambda^{-\Lambda_e}, \tag{5.11}$$

where it is recalled that $x' := (x_1, \ldots, x_n, t) = (x, t)$.

Step 3: treatment of contribution $C_2(\lambda) = C_2(\lambda')$. This contribution is the second sum on the right-hand side of (3.8). With our notations, this term is (since $\mathrm{Re}(\alpha) + N \leqslant ar - 1$ for $N \leqslant p$)

$$C_2(\lambda') = \sum_{N=0}^{p} \sum_{j=0}^{l} C_l^j (-1)^j \left[\boldsymbol{fp} \int_0^\infty h^{Nl}(v) v^{\alpha+N} \log^{l-j}(v)\, \mathrm{d}v \right] \lambda'^{-(\alpha+N+1)} \log^j(\lambda'). \tag{5.12}$$

If $p < 0$, $C_2(\lambda') = 0$. Else, definitions of λ' and of function h^{Nl} yield

$$C_2(\lambda) = \sum_{N=0}^{p} \sum_{j=0}^{l} C_l^j \log^j(\lambda^{-1/a}) \frac{\lambda^{-(\alpha+N+1)/a}}{N!} \left\{ \boldsymbol{fp} \int_0^\infty v^{\alpha+N} \log^{l-j}(v) \right.$$
$$\times \left[\int_0^1 \cdots \int_0^1 x_1^{\alpha_1} \cdots x_n^{\alpha_n} \log^{l_1}(x_1) \cdots \right.$$
$$\left. \left. \times \log^{l_n}(x_n) f_t^N(x,0) g(v^a x_1^{a_1} \cdots x_n^{a_n})\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right] \mathrm{d}v \right\}. \tag{5.13}$$

Induction assumption 2 allows us to cast $C_2(\lambda)$ in another form by using for positive integer L the relation

$$\sum_{j=0}^{l} \frac{l! \log^j(\lambda^{-1})}{j!(L+l-j)!} \log^{L+l-j}(B) = \frac{l!}{(L+l)!} \sum_{s=L}^{L+l} C_{L+l}^s \log^s(B) \log^{L+l-s}(\lambda^{-1}). \tag{5.14}$$

More precisely,

$$C_2(\lambda) = \sum_{N=0}^{p} \frac{1}{N!} \left(\frac{1}{a}\right)^{l+1} \left\{ \boldsymbol{fp} \int_0^1 x_1^{\alpha_1 - a_1\gamma_N} \log^{l_1}(x_1) \cdots \left[\boldsymbol{fp} \int_0^1 x_n^{\alpha_n - a_n\gamma_N} \log^{l_n}(x_n) \right. \right.$$
$$\left. \left. \times \left[\boldsymbol{fp} \int_0^\infty u^{\gamma_N - 1} \log^l(\lambda^{-1} u x_1^{-a_1} \ldots x_n^{-a_n}) g(u)\, \mathrm{d}u \right] f_t^N(x,0)\, \mathrm{d}x_n \right] \cdots \mathrm{d}x_1 \right\}$$
$$\times \lambda^{-(\alpha+N+1)/a}$$
$$+ \sum_{N=0}^{p} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \frac{\Delta^{\gamma_N}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}! \cdots n_{i_k}! N!}$$
$$\times \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \left(\frac{1}{a}\right)^{l+1} \frac{l_{i_1}! \cdots l_{i_k}! l!}{(l + \sum_{j=1}^k l_{i_j} + k)!}$$
$$\times \left\{ \prod_{j;i_1,\ldots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - a_j\gamma_N} \log^{l_j}(x_j) f_{x_{i_1} \cdots x_{i_k} t}^{n_{i_1} \cdots n_{i_k} N} (\delta_{i_1}^n \cdots \delta_{i_k}^n \delta x') \right.$$
$$\times \left[\boldsymbol{fp} \int_0^\infty u^{\gamma_N - 1} \left\{ \sum_{s=\sum_{j=1}^k l_{i_j}+k}^{l_{i_j} + \sum_{j=1}^k l_{i_j}+k} C_{l+\sum_{j=1}^k l_{i_j}+k}^s \log^s \left(u \prod_{j;i_1,\ldots,i_k}^{n} x_j^{-a_j} \right) \right. \right.$$
$$\left. \left. \left. \times [\log(\lambda^{-1})]^{\left(l+\sum_{j=1}^k l_{i_j}+k-s\right)} \right\} g(u)\, \mathrm{d}u \right] \mathrm{d}x_j \right\} \lambda^{-(\alpha+N+1)/a}$$
$$+ \sum_{N=0}^{p} \sum_{\{e;\Lambda_e=\gamma_N\}} \frac{1}{N!} \left(\frac{1}{a}\right)^{l+1} \sum_{m=0}^{M(e)} \frac{g_{em}(-1)^{m+1} m! l!}{(m+l+1)!}$$

$$
\times\left[\prod_{j=1}^{n} \boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a_j\Lambda_e}\log^{l_j}(x_j) f_t^N(x,0)\right.
$$

$$
\left.\times\left\{\sum_{s=m+1}^{m+l+1} C_{m+l+1}^s \log^s(x_1^{-a_1}\cdots x_n^{-a_n})[\log(\lambda^{-1})]^{(m+l+1-s)}\right\}\mathrm{d}x_j\right]
$$

$$
\times\lambda^{-(\alpha+N+1)/a}
$$

$$
+\sum_{N=0}^{p}\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\sum_{\{e;\Lambda_e=\gamma_N\}}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!N!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots
$$

$$
\times\left(\frac{1}{a}\right)^{l+1}\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\cdots l_{i_k}!l!}{(1+l+\sum_{j=1}^{k}l_{i_j}+m+k)!}
$$

$$
\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j) f_{x_{i_1}\cdots x_{i_k}t}^{n_{i_1}\cdots n_{i_k}N}(\delta_{i_1}^n\cdots\delta_{i_k}^n\delta x')\right.
$$

$$
\times\left\{\sum_{s=1+\sum_{j=1}^{k}l_{i_j}+m+k}^{1+l+\sum_{j=1}^{k}l_{i_j}+m+k} C_{1+l+\sum_{j=1}^{k}l_{i_j}+m+k}^{s}\right.
$$

$$
\left.\left.\times\log^s\left(\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\right)[\log(\lambda^{-1})]^{(1+l+\sum_{j=1}^{k}l_{i_j}+m+k-s)}\right\}\mathrm{d}x_j\right\}
$$

$$
\times\lambda^{-(\alpha+N+1)/a}, \tag{5.15}
$$

with this time $\gamma_N=\gamma_N(N):=(\alpha+N+1)/a$ and $\delta(x_1,\ldots,x_n,t):=(x_1,\ldots,x_n,0)$.

Step 4: treatment of contribution $C_3(\lambda)=C_3(\lambda')$. This contribution is the last term occurring on the right-hand side of (3.8). Thanks to the definition of h^{Nl}, as $u\to+\infty$,

$$
h^{Nl}(u)=\sum_{\mathrm{Re}(\delta_i)\leqslant r}\sum_{j=0}^{J(i)}\frac{\langle D_n^{ij},f_t^N(.,0)\rangle}{N!}[u^a]^{-\delta_i}\log^j[u^a]+o(u^{-ar}). \tag{5.16}
$$

This ensures

$$
C_3(\lambda')=\sum_{N=0}^{p}\sum_{s=0}^{l}C_l^s(-1)^s\sum_{\{i,a\delta_i=1+\alpha+N\}}\sum_{j=0}^{J(i)}\frac{a^j\langle D_n^{ij},f_t^N(.,0)\rangle}{N!(1+l+j-s)}
$$

$$
\times\log^{1+l+j}(\lambda')\lambda'^{-(\alpha+N+1)}. \tag{5.17}
$$

With relation

$$
\sum_{s=0}^{l}\frac{C_l^s(-1)^s}{(1+l+j-s)}=(-1)^l\frac{l!j!}{(1+j+l)!},
$$

$C_3(\lambda)$ rewrites

$$C_3(\lambda) = \sum_{N=0}^{p} \sum_{\{i, a\delta_i = 1+\alpha+N\}} \sum_{j=0}^{J(i)} \frac{(-1)^l l! j!}{(1+j+l)!} \frac{\langle D_n^{ij}, f_t^N(.,0)\rangle}{N! a^{l+1}} \log^{1+l+j}(\lambda) \lambda^{-(\alpha+N+1)/a}. \tag{5.18}$$

In fact, for each value of integer i (see equality (5.16)), expansion (4.1) shows that one has to deal (for $X := u^a$) with terms

$$e_{3L} := \frac{1}{L!} \log^L(X^{-1}B) = \sum_{j=0}^{L} \frac{(-1)^j \log^j(X) \log^{L-j}(B)}{j!(L-j)!}. \tag{5.19}$$

Each of these terms provides a contribution

$$\begin{aligned} c_{3L}(\lambda) = & \sum_{N=0}^{p} \sum_{\{i, a\delta_i = 1+\alpha+N\}} \frac{1}{N!} \left(\frac{1}{a}\right)^{l+1} \sum_{j=0}^{L} \frac{l! j! (-1)^{l+j} \log^{L-j}(B)}{j!(L-j)!(1+l+j)!} \\ & \times \log^{1+l+j}(\lambda) \lambda^{-(\alpha+N+1)/a} \\ = & -\sum_{N=0}^{p} \sum_{\{i, a\delta_i = 1+\alpha+N\}} \frac{1}{N!} \left(\frac{1}{a}\right)^{l+1} l! \sum_{s=0}^{L} \frac{C_{1+l+L}^s \log^s(B)}{(1+l+L)!} \\ & \times [\log(\lambda^{-1})]^{(1+l+L-s)} \lambda^{-(\alpha+N+1)/a}. \end{aligned}$$

After some algebra, one obtains

$$\begin{aligned} C_3(\lambda) = & \sum_{N=0}^{p} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \frac{\Delta(n_{i_1},\ldots,n_{i_k},N)}{n_{i_1}!\ldots n_{i_k}! N!} \\ & \times \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \left(\frac{1}{a}\right)^{l+1} \frac{l_{i_1}!\ldots l_{i_k}! l!}{(l+\sum_{j=1}^k l_{i_j}+k)!} \\ & \times \Bigg\{ \prod_{j;i_1,\ldots,i_k}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - a_j \gamma_N} \log^{l_j}(x_j) f_{x_{i_1}\cdots x_{i_k} t}^{n_{i_1}\cdots n_{i_k} N}(\delta_{i_1}^n \cdots \delta_{i_k}^n \delta x') \\ & \times \Bigg[\boldsymbol{fp} \int_0^\infty u^{\gamma_N - 1} \Bigg\{ \sum_{s=0}^{\sum_{j=1}^k l_{i_j}+k-1} C_{l+\sum_{j=1}^k l_{i_j}+k}^s \log^s \Bigg(u \prod_{j;i_1,\ldots,i_k}^{n} x_j^{-a_j} \Bigg) \\ & \times [\log(\lambda^{-1})]^{\left(l+\sum_{j=1}^k l_{i_j}+k-s\right)} \Bigg\} g(u)\,\mathrm{d}u \Bigg] \mathrm{d}x_j \Bigg\} \lambda^{-(\alpha+N+1)/a} \\ & + \sum_{N=0}^{p} \sum_{\{e; \Lambda_e = \gamma_N\}} \frac{1}{N!} \left(\frac{1}{a}\right)^{l+1} \sum_{m=0}^{M(e)} \frac{g_{em}(-1)^{m+1} m! l!}{(m+l+1)!} \\ & \times \Bigg[\prod_{j=1}^{n} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - a_j \Lambda_e} \log^{l_j}(x_j) f_t^N(x,0) \\ & \times \Bigg\{ \sum_{s=0}^{m} C_{m+l+1}^s \log^s(x_1^{-a_1} \ldots x_n^{-a_n}) [\log(\lambda^{-1})]^{(m+l+1-s)} \Bigg\} \mathrm{d}x_j \Bigg] \lambda^{-(\alpha+N+1)/a} \end{aligned}$$

$$+\sum_{N=0}^{p}\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\sum_{\{e;\Lambda_e=\gamma_N\}}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!N!}$$

$$\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\left(\frac{1}{a}\right)^{l+1}\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\ldots l_{i_k}!l!}{(1+l+\sum_{j=1}^{k}l_{i_j}+m+k)!}$$

$$\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)f^{n_{i_1}\ldots n_{i_k}N}_{x_{i_1}\cdots x_{i_k}t}(\delta^n_{i_1}\cdots\delta^n_{i_k}\delta x')\right.$$

$$\times\left\{\sum_{s=0}^{\sum_{j=1}^{k}l_{i_j}+m+k}C^s_{1+l+\sum_{j=1}^{k}l_{i_j}+m+k}\log^s\left(\prod_{j;i_1,\ldots i_k}^{n}x_j^{-a_j}\right)\right.$$

$$\left.\left.\times[\log(\lambda^{-1})]^{(1+l+\sum_{j=1}^{k}l_{i_j}+m+k-s)}\right\}\mathrm{d}x_j\right\}\lambda^{-(\alpha+N+1)/a}$$

$$+\frac{(-1)^n}{n!}\sum_{i_1,\ldots,i_n}^{n}\sum_{n_{i_1},\ldots,n_{i_n},N}^{p_{i_1},\ldots,p_{i_n},p}\frac{\Delta(n_{i_1},\ldots,n_{i_n},N)}{n_{i_1}!\ldots n_{i_n}!N!}$$

$$\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_n}}\right)^{l_{i_n}+1}\left(\frac{1}{a}\right)^{l+1}$$

$$\times\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\ldots l_{i_n}!l!}{(1+l+\sum_{j=1}^{n}l_{i_j}+m+n)!}$$

$$\times\left[f^{n_{i_1}\cdots n_{i_n}N}_{x_{i_1}\cdots x_{i_n}t}(\delta^n_{i_1}\ldots\delta^n_{i_n}\delta x')[\log(\lambda^{-1})]^{(1+l+\sum_{j=1}^{n}l_{i_j}+m+n)}\right]\lambda^{-(\alpha+N+1)/a},\tag{5.20}$$

where $\gamma_N=\gamma_N(N):=(\alpha+N+1)/a$ and the last term has been obtained by setting $k=n$ in the last contribution of (4.1).

Step 5: the final expansion of $I^f_{n+1}(\lambda)$. The expansion $I^f_{n+1}(\lambda)=C_1(\lambda)+C_2(\lambda)+C_3(\lambda)+o(\lambda^{-r})$ is found by adding the previous expansions of $C_1(\lambda)$, $C_2(\lambda)$ and of $C_3(\lambda)$. The relation

$$\sum_{s=0}^{L}C^s_{1+l+L}\log^s(A)\log^{1+l+L-s}(B)+\sum_{s=L+1}^{1+l+L}C^s_{1+l+L}\log^s(A)\log^{1+l+L-s}(B)$$

allows us to combine most of the terms of contribution $C_2(\lambda)$ with those of $C_3(\lambda)$. The final result takes the form $I^f_{n+1}(\lambda)=S(\lambda)+(\lambda)+G(\lambda)+o(\lambda^{-r})$ with, for $\gamma_N:=(\alpha+N+1)/a$,

$$S(\lambda)=\sum_{\mathrm{Re}(\Lambda_e)\leqslant r}\sum_{m=0}^{M(e)}g_{em}(-1)^m\left\{\boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e}\log^l(t)\left[\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\Lambda_e}\log^{l_1}(x_1)\cdots\right.\right.$$

$$\left.\left.\times\left[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\Lambda_e}\log^{l_n}(x_n)f(x')\log^m\left(\lambda^{-1}t^{-a}\prod_{j=1}^{n}x_j^{-a_j}\right)\mathrm{d}x_n\right]\cdots\mathrm{d}x_1\right]\mathrm{d}t\right\}\lambda^{-\Lambda_e},\tag{5.21}$$

$$O(\lambda) = \sum_{N=0}^{p} \frac{1}{N!}\left(\frac{1}{a}\right)^{l+1}\left\{\boldsymbol{fp}\int_0^1 x_1^{\alpha_1 - a_1\gamma_N}\log^{l_1}(x_1)\cdots\right.$$
$$\times\left[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n - a_n\gamma_N}\log^{l_n}(x_n)\right.$$
$$\times\left[\boldsymbol{fp}\int_0^\infty u^{\gamma_N - 1}\log^{l}(\lambda^{-1}ux_1^{-a_1}\ldots x_n^{-a_n})g(u)\,\mathrm{d}u\right]$$
$$\left.\left.\times f_t^N(x,0)\,\mathrm{d}x_n\right]\cdots\mathrm{d}x_1\right\}\lambda^{-(\alpha+N+1)/a}$$
$$+\sum_{N=0}^{p}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta(n_{i_1},\ldots,n_{i_k},N)}{n_{i_1}!\ldots n_{i_k}!N!}$$
$$\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\left(\frac{1}{a}\right)^{l+1}\frac{l_{i_1}!\cdots l_{i_k}!l!}{(l+\sum_{j=1}^{k}l_{i_j}+k+1-1)!}$$
$$\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a_j\gamma_N}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}t}^{n_{i_1}\cdots n_{i_k}N}(\delta_{i_1}^n\cdots\delta_{i_k}^n\delta x')\right.$$
$$\times\left[\boldsymbol{fp}\int_0^\infty u^{\gamma_N-1}\left\{\left[\log\left(\lambda^{-1}u\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\right)\right]^{\left(l+\sum_{j=1}^{k}l_{i_j}+k+1-1\right)}\right\}\right.$$
$$\left.\left.\times g(u)\,\mathrm{d}u\right]\mathrm{d}x_j\right\}\lambda^{-(\alpha+N+1)/a}$$
$$+\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!}$$
$$\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\frac{l_{i_1}!\ldots l_{i_k}!}{(\sum_{j=1}^{k}l_{i_j}+k-1)!}$$
$$\times\boldsymbol{fp}\int_0^1 t^{\alpha-(a/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}\log^{l}(t)$$
$$\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-(a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x')\right.$$
$$\times\left[\boldsymbol{fp}\int_0^\infty u^{((\alpha_{i_1}+n_{i_1}+1)/a_{i_1})-1}\right.$$
$$\times\left[\log\left(\lambda^{-1}ut^{-a}\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\right)\right]^{\left(\sum_{j=1}^{k}l_{i_j}+k-1\right)}$$
$$\left.\left.\times g(u)\,\mathrm{d}u\right]\mathrm{d}x_j\right\}\mathrm{d}t\lambda^{-(\alpha_{i_1}+n_{i_1}+1)/a_{i_1}}, \tag{5.22}$$

and also

$$
\begin{aligned}
G(\lambda) = &-\sum_{N=0}^{p} \sum_{\{e;a\Lambda_e=\alpha+N+1\}} \frac{1}{N!}\left(\frac{1}{a}\right)^{l+1} \sum_{m=0}^{M(e)} \frac{g_{em}(-1)^m m!l!}{(m+l+1)!} \\
&\times\left[\prod_{j=1}^{n} \boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e} \right.\\
&\left.\times \log^{l_j}(x_j) f_t^N(x,0)\log^{m+l+1}(\lambda^{-1}x_1^{-a_1}\cdots x_n^{-a_n})\,\mathrm{d}x_j\right]\lambda^{-(\alpha+N+1)/a} \\
&+\sum_{N=0}^{p}\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k!}\sum_{\{e;\Lambda_e=\gamma_N\}}\sum_{i_1,\dots,i_k}^{n}\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}\frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\dots n_{i_k}!N!} \\
&\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\left(\frac{1}{a}\right)^{l+1} \\
&\times\sum_{m=0}^{M(e)}\frac{(-1)^m g_{em} m! l_{i_1}!\cdots l_{i_k}! l!}{(l+\sum_{j=1}^{k} l_{i_j}+m+k+1)!} \\
&\times\left\{\prod_{j;i_1,\dots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j) f_{x_{i_1}\cdots x_{i_k}t}^{n_{i_1}\dots n_{i_k}N}(\delta_{i_1}^n\cdots\delta_{i_k}^n\delta x')\right.\\
&\left.\times\left[\log\left(\lambda^{-1}\prod_{j;i_1,\dots i_k}^{n}x_j^{-a_j}\right)\right]^{(l+\sum_{j=1}^{k}l_{i_j}+m+k+1)}\mathrm{d}x_j\right\}\lambda^{-(\alpha+N+1)/a} \\
&+\frac{(-1)^n}{n!}\sum_{i_1,\dots,i_n}^{n}\sum_{n_{i_1},\dots,n_{i_n},N}^{p_{i_1},\dots,p_{i_n},p}\frac{\Delta(n_{i_1},\dots,n_{i_n},N)}{n_{i_1}!\dots n_{i_n}!N!} \\
&\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_n}}\right)^{l_{i_n}+1}\left(\frac{1}{a}\right)^{l+1}\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\cdots l_{i_n}!l!}{(1+l+\sum_{j=1}^{n}l_{i_j}+m+n)!} \\
&\times\left[f_{x_{i_1}\cdots x_{i_n}t}^{n_{i_1}\cdots n_{i_n}N}(\delta_{i_1}^n\cdots\delta_{i_n}^n\delta x')[\log(\lambda^{-1})]^{(1+l+\sum_{j=1}^{n}l_{i_j}+m+n)}\right]\lambda^{-(\alpha+N+1)/a} \\
&+\sum_{\mathrm{Re}(\Lambda_e)\leqslant r}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{i_1,\dots,i_k}^{n}\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}\frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \\
&\times\sum_{m=0}^{M(e)}\frac{(-1)^m g_{em}m!l_{i_1}!\dots l_{i_k}!}{(\sum_{j=1}^{k}l_{i_j}+m+k)!}\boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e}\log^l(t) \\
&\times\left\{\prod_{j;i_1,\dots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)\right.\\
&\left.\times f_{x_{i_1}\dots x_{i_k}}^{n_{i_1}\dots n_{i_k}}(\delta_{i_1}^n\dots\delta_{i_k}^n x')\left[\log\left(\lambda^{-1}t^{-a}\prod_{j;i_1,\dots,i_k}^{n}x_j^{-a_j}\right)\right]^{\left(\sum_{j=1}^{k}l_{i_j}+m+k\right)}\mathrm{d}x_j\right\}\mathrm{d}t\lambda^{-\Lambda_e},
\end{aligned}
\tag{5.23}
$$

Note that $S(\lambda)$ and $G(\lambda)$ involve the behaviour of function g at infinity. For convenience both $O(\lambda)$ and $G(\lambda)$ are rewritten as $O(\lambda) = O_1(\lambda) + O_2(\lambda) + O_3(\lambda)$ and $G(\lambda) = G_1(\lambda) + G_2(\lambda) + G_3(\lambda) + G_4(\lambda)$, where each sum $O_i(\lambda)$ or $G_i(\lambda)$ is associated in the right order to the occurring one in the decompositions of $O(\lambda)$ or $G(\lambda)$. For instance, $G_3(\lambda)$ designates the third sum arising on the right-hand side of (5.23). Using the notations $\alpha_{n+1} := \alpha$, $l_{n+1} := l$, $a_{n+1} := a$, $x_{n+1} := t$, $x' := (x_1, \ldots, x_n, x_{n+1})$, $n_{n+1} := N$ and also $p_{n+1} := [\![ar - \mathrm{Re}(\alpha)]\!] - 1$, formula (4.1) states that $I^f_{n+1}(\lambda) = S(\lambda) + O_{n+1}(\lambda) + G_{n+1}(\lambda) + o(\lambda^{-r})$, where if $A \in \{O, G\}$

$$A_{n+1}(\lambda) := \sum_{k'=1}^{n+1} \frac{(-1)^{k'+1}}{k'!} \sum_{i_1,\ldots,i'_k}^{n+1} A^\lambda_{n+1}(i_1, \ldots, i_{k'}), \tag{5.24}$$

with

$$\begin{aligned} O^\lambda_{n+1}(i_1, \ldots, i_{k'}) = & \sum_{n_{i_1},\ldots,n_{i_{k'}}}^{p_{i_1},\ldots,p_{i_{k'}}} \frac{\Delta(n_{i_1}, \ldots, n_{i_{k'}})}{n_{i_1}! \cdots n_{i_{k'}}!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_{k'}}}\right)^{l_{i_{k'}}+1} \\ & \times \frac{l_{i_1}! \cdots l_{i_{k'}}!}{(\sum_{j=1}^{k'} l_{i_j} + k' - 1)!} \Bigg\{ \prod_{j;i_1,\ldots,i_{k'}}^{n+1} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - (a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)} \\ & \times \log^{l_j}(x_j) f^{n_{i_1}\cdots n_{i_{k'}}}_{x_{i_1}\ldots x_{i_{k'}}} (\delta^{n+1}_{i_1} \cdots \delta^{n+1}_{i_{k'}} x') \left[\boldsymbol{fp} \int_0^\infty u^{((\alpha_{i_1}+n_{i_1}+1)/a_{i_1})-1} \right. \\ & \times \left. \left[\log \left(\lambda^{-1} u \prod_{j;i_1,\ldots,i_{k'}}^{n+1} x_j^{-a_j} \right) \right]^{(\sum_{j=1}^{k'} l_{i_j} + k' - 1)} g(u) \, \mathrm{d}u \right] \mathrm{d}x_j \Bigg\} \lambda^{-(\alpha_{i_1}+n_{i_1}+1)/a_{i_1}} \end{aligned} \tag{5.25}$$

and also

$$\begin{aligned} G^\lambda_{n+1}(i_1, \ldots, i_{k'}) = & \sum_{\mathrm{Re}(\Lambda_e) \leqslant r} \sum_{n_{i_1},\ldots,n_{i_{k'}}}^{p_{i_1},\ldots,p_{i_{k'}}} \frac{\Delta^{\Lambda_e}(n_{i_1}, \ldots, n_{i_{k'}})}{n_{i_1}! \ldots n_{i_{k'}}!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_{k'}}}\right)^{l_{i_{k'}}+1} \\ & \times \sum_{m=0}^{M(e)} \frac{(-1)^m g_{em} m! l_{i_1}! \cdots l_{i_{k'}}!}{(\sum_{j=1}^{k'} l_{i_j} + m + k')!} \Bigg\{ \prod_{j;i_1,\ldots,i_{k'}}^{n+1} \boldsymbol{fp} \int_0^1 x_j^{\alpha_j - a_j \Lambda_e} \log^{l_j}(x_j) \\ & \times f^{n_{i_1}\cdots n_{i_{k'}}}_{x_{i_1}\cdots x_{i_{k'}}} (\delta^{n+1}_{i_1} \cdots \delta^{n+1}_{i_{k'}} x') \left[\log \left(\lambda^{-1} \prod_{j;i_1,\ldots,i'_k}^{n+1} x_j^{-a_j} \right) \right]^{(\sum_{j=1}^{k'} l_{i_j} + m + k')} \mathrm{d}x_j \Bigg\} \lambda^{-\Lambda_e}. \end{aligned} \tag{5.26}$$

For instance, consider the contribution $O_{n+1}(\lambda)$. Three cases arise.

(i) If $k' = 1$, then

$$\sum_{i_1=1}^{n+1} O^\lambda_{n+1}(i_1) = O^\lambda_{n+1}(n+1) + \sum_{i_1=1}^{n} O^\lambda_{n+1}(i_1).$$

Clearly (with $\Delta(i_1) = 1$), $O^\lambda_{n+1}(n+1) = O_1(\lambda)$ and also

$$\sum_{i_1=1}^{n} O^\lambda_{n+1}(i_1) = O^1_3(\lambda)$$

if $O^1_3(\lambda)$ denotes the part of $O_3(\lambda)$ associated with $k = 1$.

(ii) If $k' \in \{2, \ldots, n\}$, observe that

$$\sum_{i_1,\ldots,i_{k'}}^{n+1} O^\lambda_{n+1}(i_1, \ldots, i_{k'}) = \sum_{i_1,\ldots,i_{k'}}^{n} O^\lambda_{n+1}(i_1, \ldots, i_{k'}) + \sum_{i_1,\ldots,i_{k'}}^{n+1,a} O^\lambda_{n+1}(i_1, \ldots, i_{k'})$$

if this latter sum means a sum over all different integers i_j (with $1 \leqslant j \leqslant k'$) such that $n+1 \in \{i_1, \ldots, i_{k'}\}$. By setting $k' = k$ one finds that

$$O_3(\lambda) - O^1_3(\lambda) = \sum_{k'=2}^{n} \sum_{i_1,\ldots,i_{k'}}^{n} (k'!)^{-1}(-1)^{k'+1} O^\lambda_{n+1}(i_1, \ldots, i_{k'})$$

because in such circumstances

$$\prod_{j;i_1,\ldots,i_{k'}}^{n+1} A_j = A_{n+1}\left[\prod_{j;i_1,\ldots,i_{k'}}^{n} A_j\right].$$

Moreover, according to (5.25),

$$O^\lambda_{n+1}(i_1, \ldots, i_{k'}) = \sum_{n_{i_1},\ldots,n_{i_{k'}}}^{p_{i_1},\ldots,p_{i_{k'}}} F^{n_{i_1},\ldots,n_{i_{k'}}}_{i_1,\ldots,i_{k'}}(\lambda)$$

with

$$F^{n_{i_1},\ldots,n_{i_{k'}}}_{i_1,\ldots,i_{k'}}(\lambda) = F^{n_{i_{\sigma(1)}},\ldots,n_{i_{\sigma(k')}}}_{i_{\sigma(1)},\ldots,i_{\sigma(k')}}(\lambda)$$

for any permutation σ of the set $\{i_1, \ldots, i_{k'}\}$ (where the integers i_j are all different) thanks to the proposition 7 and definition of $\prod^{n+1}_{j;i_1,\ldots,i_{k'}} A_j$. By using the meaning of the sum $\sum^{p_{i_1},\ldots,p_{i_j}}_{n_{i_1},\ldots,n_{i_j}}$ and keeping the notations $p := p_{n+1}$ and $N := n_{n+1}$, one gets the important relation

$$\sum_{N=0}^{p} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} F^{n_{i_1},\ldots,n_{i_k},N}_{i_1,\ldots,i_k,n+1}(\lambda) = \frac{1}{k+1} \sum_{i_1,\ldots,i_{k+1}}^{n+1,a} \sum_{n_{i_1},\ldots,n_{i_{k+1}}}^{p_{i_1},\ldots,p_{i_{k+1}}} F^{n_{i_1},\ldots,n_{i_{k+1}}}_{i_1,\ldots,i_{k+1}}(\lambda). \tag{5.27}$$

Observe that if $n+1 \in \{i_1, \ldots, i_{k'}\}$ then $\prod^{n+1}_{j;i_1,\ldots,i_{k'}} A_j = \prod^{n}_{j;i_1,\ldots,i_{k'}} A_j$. Thus, if we set $k' = k+1$ the sum

$$\sum_{k'=2}^{n} \sum_{i_1,\ldots,i_{k'}}^{n+1,a} (k'!)^{-1}(-1)^{k'+1} O^\lambda_{n+1}(i_1, \ldots, i_{k'})$$

turns out to be $O_2(\lambda) - O^n_2(\lambda)$ if $O^n_2(\lambda)$ denotes the contribution to the sum $O_2(\lambda)$ associated with the case $k = n$.

(iii) If $k' = n+1$, then

$$\sum_{i_1,\ldots,i_{n+1}}^{n+1} O^\lambda_{n+1}(i_1, \ldots, i_{n+1}) = \sum_{i_1,\ldots,i_{n+1}}^{n+1,a} O^\lambda_{n+1}(i_1, \ldots, i_{n+1})$$

and equality (5.27) leads to

$$\frac{(-1)^{n+1+1}}{(n+1)!}\sum_{i_1,\dots,i_{n+1}}^{n+1,a} O^{\lambda}_{n+1}(i_1,\dots,i_{n+1}) = \frac{(n+1)(-1)^n}{(n+1)!}\sum_{N=0}^{p}\sum_{i_1,\dots,i_n}^{n} O^{\lambda}_{n+1}(i_1,\dots,i_{n+1})$$
$$= O^n_2(\lambda). \tag{5.28}$$

Hence, it has been shown that $O_{n+1}(\lambda) = O(\lambda)$. The reader may check that such a method also ensures that $G_{n+1}(\lambda) = G(\lambda)$.

6. Induction assumption 2 for $n+1$

It is recalled that for $\mathrm{Re}(\beta) > -1$, $br \geqslant \mathrm{Re}(\beta)+1$ and $n \geqslant 1$

$$\langle K_{n+1}, f\rangle = \boldsymbol{fp}\int_0^\infty v^\beta \log^h(v) \times\left[\int_0^1\cdots\int_0^1 t^\alpha \log^l(t) x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots \times \log^{l_n}(x_n) f(x,t) g(v^b t^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}t\right]\mathrm{d}v. \tag{6.1}$$

Introduction of integer $M := \inf\{i$ with i integer such that $b[\mathrm{Re}(\alpha)+i+1] > a[\mathrm{Re}(\beta)+1]\}$ and of functions $T[f]$, $R[f]$ such that $T[f] := 0$ if $M \leqslant 0$, else $T[f](x,t) := \sum_{m=0}^{M-1} f_t^m(x,0)t^m/m!$, and $R[f] := f - T[f]$ leads to $\langle K_{n+1}, f\rangle = K' + K''$ with $K' := \langle K_{n+1}, R[f]\rangle$ and $K'' := \langle K_{n+1}, T[f]\rangle$. Note that $[\mathrm{Re}(\alpha)+M]/a \leqslant [\mathrm{Re}(\beta)+1]/b \leqslant r$, i.e. $M \leqslant [\![ar - \mathrm{Re}(\alpha)]\!]$. Consequently, if $M > 0$, $q = [\![ar - \mathrm{Re}(\alpha)]\!] \geqslant M$ and the assumption $f \in \mathcal{D}^{(q_1,\dots,q_n,q)}(]0,1[^{n+1})$ allows to define $T[f]$.

Step 1: treatment of contribution K'. For a given real $A > 0$, K' rewrites

$$K' = \int_0^A v^\beta \log^h(v) W(v)\,\mathrm{d}v + \boldsymbol{fp}\int_A^\infty v^\beta \log^h(v) W(v)\,\mathrm{d}v := I_1 + I_2, \tag{6.2}$$

where the new function $W(v)$ obeys

$$W(v) = \int_0^1 t^\alpha \log^l(t)\left[\int_0^1\cdots\int_0^1 x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots\log^{l_n}(x_n) \times R[f](x,t) g(v^b t^a x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right]\mathrm{d}t. \tag{6.3}$$

Because $\mathrm{Re}(\beta) > -1$ and W is obviously bounded on $[0, A]$, the first integral I_1 appearing on the right-hand side of (6.2) exists. Moreover, definition (6.3) of function W associated with the assumptions bearing both on functions f and g and with inequality $br \geqslant \mathrm{Re}(\beta)+1$ authorizes us to expand $W(v)$, as $v \to +\infty$, by using the previous results (i.e. the expansion valid for $I^{R[f]}_{n+1}(v^b)$). More precisely, if $R := [\mathrm{Re}(\beta)+1]/b$ one obtains

$$W(v) = \sum_{j,\mathrm{Re}(\delta)\leqslant R} b^j \langle D^{ij}_{n+1}, R[f]\rangle v^{-b\delta_i}\log^j v + o(v^{-[\mathrm{Re}(\beta)+1]}). \tag{6.4}$$

Consequently, the function H defined by $H(v) := v^\beta \log^h(v) W(v)$ belongs to the set

$\mathcal{P}(]A,+\infty[,C)$ and $I_2 := \boldsymbol{fp}\int_A^\infty H(v)\,\mathrm{d}v$ exists. Thanks to this expansion (6.4), I_2 also takes the form

$$I_2 = \int_A^\infty v^\beta \log^h(v)\Big[W(v) - \sum_{j,\mathrm{Re}(\delta)\leqslant R} \langle D_{n+1}^{ij}, R[f]\rangle v^{-b\delta_i}\log^j(v^b)\Big]\,\mathrm{d}v$$
$$+ \sum_{j,\mathrm{Re}(\delta)\leqslant R} b^j\langle D_{n+1}^{ij}, R[f]\rangle \boldsymbol{fp}\int_A^\infty v^{\beta-b\delta_i}\log^{h+j}(v)\,\mathrm{d}v. \tag{6.5}$$

If $\mathcal{L}$ designates the first integral on the right-hand side of this latter equality, the introduction of functions F_t and Q such that

$$W(v) = \int_0^1 t^\alpha \log^l(t) F_t(vt^{a/b})\,\mathrm{d}t$$

and

$$Q(t,v) := v^\beta\log^h(v)t^\alpha\log^l(t)F_t(vt^{a/b}) - \sum_{j,\mathrm{Re}(\delta)\leqslant R} b^j\langle D_{n+1}^{ij}, R[f]\rangle v^{\beta-b\delta_i}\log^{h+j}(v)$$

gives $\mathcal{L} = \int_A^\infty [\int_0^1 Q(t,v)\,\mathrm{d}t]\,\mathrm{d}v$ and also

$$\mathcal{L} = \int_0^1\Big[\int_A^\infty Q(t,v)\,\mathrm{d}v\Big]\,\mathrm{d}t = \int_0^1 t^\alpha\log^l(t)\Big[\boldsymbol{fp}\int_A^\infty v^\beta\log^h(v)F_t(vt^{a/b})\,\mathrm{d}v\Big]\,\mathrm{d}t$$
$$- \sum_{j,\mathrm{Re}(\delta)\leqslant R} b^j\langle D_{n+1}^{ij}, R[f]\rangle \boldsymbol{fp}\int_A^\infty v^{\beta-b\delta_i}\log^{h+j}(v)\,\mathrm{d}v. \tag{6.6}$$

Observe that Fubini's theorem has been applied to $\mathcal{L}$. Definitions of integral I_1 and of function F_t also yield

$$I_1 = \int_0^1 t^\alpha\log^l(t)\Big[\int_A^\infty v^\beta\log^h(v)F_t(vt^{a/b})\,\mathrm{d}v\Big]\,\mathrm{d}t. \tag{6.7}$$

Combination of results (6.5)–(6.7) show in fact that it is allowed to invert the integrations bearing on variables t and v for the integral K'; more precisely

$$K' = \int_0^1 t^\alpha\log^l(t)[\boldsymbol{fp}\int_0^\infty v^\beta\log^h(v)F_t(vt^{a/b})\,\mathrm{d}v]\,\mathrm{d}t$$

with a function F_t such that

$$F_t(u) := \langle T_n^g(u^b), R[f](.,t)\rangle = \int_0^1\cdots\int_0^1 x_1^{\alpha_1}\cdots x_n^{\alpha_n}\log^{l_1}(x_1)\cdots\log^{l_n}(x_n)$$
$$\times R[f](x,t)g(u^b x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n. \tag{6.8}$$

This new form of K' is now treated. For given $t\in]0,1[$, observe that $R[f](.,t)\in\mathcal{D}^{(q_1,\dots,q_n)}(]0,1[^n)$ and consequently (see given explanations for function W) the function F_t admits, as $u\to+\infty$, the following expansion (with $R := [\mathrm{Re}(\beta)+1]/b$)

$$F_t(u) = \sum_{j,\mathrm{Re}(\delta)\leqslant R} b^j\langle D_n^{ij}, R[f](.,t)\rangle u^{-b\delta_i}\log^j u + o(u^{-[\mathrm{Re}(\beta)+1]}). \tag{6.9}$$

This result justifies the existence of $J := \boldsymbol{fp}\int_0^\infty v^\beta\log^h(v)F_t(vt^{a/b})\,\mathrm{d}v$ since application of lemma 6 (for change of scale $u := vt^{a/b}$ with $t^{a/b}>0$) leads with expansion

(6.9) and after some algebra to

$$J = t^{-(a/b)(\beta+1)} \sum_{m=0}^{h} C_h^m \log^{h-m}(t^{-a/b}) \boldsymbol{fp} \int_0^\infty [vt^{a/b}]^\beta \log^m[vt^{a/b}] F_t(vt^{a/b})\, \mathrm{d}(vt^{a/b})$$

$$= t^{-(a/b)(\beta+1)} \sum_{m=0}^{h} C_h^m \log^{h-m}(t^{-a/b}) \Big\{ \boldsymbol{fp} \int_0^\infty u^\beta \log^m(u) F_t(u)\, \mathrm{d}u$$

$$+ \sum_{\{i;\, b\delta_i=\beta+1\}} \sum_{j=0}^{J(i)} \frac{b^j \langle D_n^{ij}, R[f](.,t)\rangle}{m+j+1} \log^{m+j+1}(t^{a/b}) \Big\}. \tag{6.10}$$

Consequently, keeping the notation $I_n^f(\lambda) = \langle T_n^g(\lambda), f\rangle$, the integral K' rewrites

$$K' = \int_0^1 t^{\alpha-(a/b)(\beta+1)} \log^l(t) \Big[\boldsymbol{fp} \int_0^\infty u^\beta \log^h(ut^{-a/b}) \langle T_n^g(u^b), R[f](.,t)\rangle\, \mathrm{d}u \Big] \mathrm{d}t + T, \tag{6.11}$$

where the corrective term is $T = \langle C, f\rangle - \langle C, T[f]\rangle$ if the complex $\langle C, \phi\rangle$ is defined (see (6.10)) by

$$\langle C, \phi\rangle = \sum_{\{i; b\delta_i=\beta+1\}} \sum_{j=0}^{J(i)} \sum_{m=0}^{h} \frac{C_h^m (-1)^{j+m+1} b^j}{m+j+1}$$

$$\times \Big[\boldsymbol{fp} \int_0^1 t^{\alpha-(a/b)(\beta+1)} \log^l(t) \log^{h+j+1}(t^{-a/b}) \langle D_n^{ij}, \phi(.,t)\rangle\, \mathrm{d}t \Big]. \tag{6.12}$$

Using definition of function $T[f](.,t)$ and another change of variable $u := vt^{a/b}$ for one integration, one obtains

$$K' = \boldsymbol{fp} \int_0^1 t^{\alpha-(a/b)(\beta+1)} \log^l(t) \Big[\boldsymbol{fp} \int_0^\infty u^\beta \log^h(ut^{-a/b}) \langle T_n^g(u^b), f(.,t)\rangle\, \mathrm{d}u \Big] \mathrm{d}t$$

$$+ \langle C, f\rangle - \sum_{m=0}^{M-1} (m!)^{-1} \boldsymbol{fp} \int_0^1 t^{\alpha+m} \log^l(t) \Big[\boldsymbol{fp} \int_0^\infty v^\beta \log^h(v) G_m(v^b t^a)\, \mathrm{d}v \Big] \mathrm{d}t \tag{6.13}$$

with functions G_m defined by

$$G_m(u) = \int_0^1 \cdots \int_0^1 x_1^{\alpha_1} \cdots x_n^{\alpha_n} \log^{l_1}(x_1) \cdots$$

$$\times \log^{l_n}(x_n) f_t^m(x, 0) g(u x_1^{a_1} \cdots x_n^{a_n})\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{6.14}$$

Step 2: treatment of contribution K''. Definitions of functions $T[f]$ and G_m ensure (with $\sum_{m=0}^{M-1} := 0$, if $M < 1$)

$$K'' = \sum_{m=0}^{M-1} (m!)^{-1} \boldsymbol{fp} \int_0^\infty v^\beta \log^h(v) \Big[\int_0^1 t^{\alpha+m} \log^l(t) G_m(v^b t^a)\, \mathrm{d}t \Big] \mathrm{d}v. \tag{6.15}$$

For each m, the function $G_m = \langle T_n^g(u), F_t^m(.,0)\rangle$ is bounded on $\mathbb{R}_+$ and admits (see induction assumption 1) an expansion near infinity up to order $O(u^{-r})$ and thereby belongs to the set $\mathcal{B}_r(]0,\infty[) \subset \mathcal{L}_r(]0,\infty[)$. Consequently, lemma 12 applies to each of the above contributions to K'' and leads to

$$\boldsymbol{fp}\int_0^\infty v^\beta \log^h(v)\left[\int_0^1 t^{\alpha+m}\log^l(t)G_m(v^b t^a)\,\mathrm{d}t\right]\mathrm{d}v$$
$$= \boldsymbol{fp}\int_0^1 t^{\alpha+m}\log^l(t)\left[\boldsymbol{fp}\int_0^\infty v^\beta \log^h(v)G_m(v^b t^a)\,\mathrm{d}v\right]\mathrm{d}t + \Delta$$

with $\Delta := 0$ except if $a(\beta+1) = b(\alpha+m+1)$, i.e. $\gamma := (\beta+1)/b = (\alpha+m+1)/a$. For $M := \inf\{i$, with i integer such that $b[\mathrm{Re}(\alpha)+i+1] > a[\mathrm{Re}(\beta)+1]\}$, it may happen that $m_1 := M-1$ satisfies such a relation, with the associated value of Δ

$$\Delta := -\frac{h!l!}{(h+l+1)!}\left(\frac{b}{a}\right)^{l+1}\left[\boldsymbol{fp}\int_0^\infty v^\beta \log^{h+l+1}(v)G_{m_1}(v^b)\,\mathrm{d}v\right]. \tag{6.16}$$

Taking into account these expressions of K', K'', the integral $\langle K_{n+1}, f\rangle$ rewrites $\langle K_{n+1}, f\rangle = E_1 + E_2 + E_3$ with $E_2 := \langle C, f\rangle$ and

$$E_1 = \boldsymbol{fp}\int_0^1 t^{\alpha-(a/b)(\beta+1)}\log^l(t)\left[\boldsymbol{fp}\int_0^\infty u^\beta \log^h(ut^{-a/b})\langle T_n^g(u^b), f(.,t)\rangle\,\mathrm{d}u\right]\mathrm{d}t,$$

$$E_3 = -\sum_{\{m;a\gamma=\alpha+m+1\}}\frac{h!l!}{(h+l+1)!}\left(\frac{b}{a}\right)^{l+1}\left[\boldsymbol{fp}\int_0^\infty v^\beta \log^{h+l+1}(v)\frac{G_m(v^b)}{m!}\,\mathrm{d}v\right]. \tag{6.17}$$

At this stage, the reader may check that such a method also allows us to treat the case of

$$\langle K_1, f\rangle := \boldsymbol{fp}\int_0^\infty v^\beta \log^h(v)\left[\int_0^1 t^\alpha \log^l(t)f(t)g(v^b t^a)\,\mathrm{d}t\right]\mathrm{d}v$$

by choosing

$$R[f] = R[f](t) = f(t) - \sum_{m=0}^{M-1}\frac{f^m(0)t^m}{m!},\quad F_t(u) = R[f](t)g(u^b) = \langle T_0^g(u^b), R[f](t)\rangle$$

with

$$F_t(u) = \sum_{i=0}^{E}\sum_{j=0}^{M(i)} R[f](t)b^j g_{ij}u^{-b\Lambda_i}\log^j u + o(u^{-[\mathrm{Re}(\beta)+1]}),\quad \mathrm{Re}(\Lambda_i) \leqslant R. \tag{6.18}$$

Hence, $\langle D^{ij}, \phi(t)\rangle = g_{ij}\phi(t)$ and also $G_m(u) = f^m(0)g(u)$. Keeping the notation $\gamma = (\beta+1)/b$ this yields

$$E_1 = \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)\left[\boldsymbol{fp}\int_0^\infty u^\beta \log^h(ut^{-a/b})f(t)g(u^b)\,\mathrm{d}u\right]\mathrm{d}t,$$

$$E_3 = -\sum_{\{m;\, a\gamma=\alpha+m_1\}}\frac{h!l!f^m(0)}{m!(h+l+1)!}\left(\frac{b}{a}\right)^{l+1}\boldsymbol{fp}\int_0^\infty v^\beta \log^{h+l+1}(v)g(v^b)\,\mathrm{d}v,$$

and also

$$E_2 = \sum_{\{i,\, b\Lambda_i=\beta+1\}}\sum_{j=0}^{M(i)}\left[\sum_{m=0}^{h}\frac{C_h^m(-1)^m}{m+j+1}\right]\frac{g_{ij}}{b^{h+1}}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)\log^{j+h+1}(t^{-a})f(t)\,\mathrm{d}t. \tag{6.19}$$

Change of variable $w = u^b$ or $w = v^b$ for the integrals arising in the expressions of E_1 and E_3 and also relation

$$\sum_{m=0}^{h} \frac{C_h^m(-1)^m}{(m+j+1)} = (-1)^h \sum_{l=0}^{h} \frac{C_h^l(-1)^l}{(1+j+h-l)} = \frac{j!h!}{(j+h+1)!}$$

lead to formula (4.14).

Step 3: treatment of contribution E_1. Use of the relation

$$\log^h(ut^{-a/b}) = \sum_{s=0}^{h} C_h^s b^{s-h} \log^{h-s}(t^{-a}) \log^s(u)$$

and of induction assumption 2 (see expansion (5.1)) in order to treat each contribution

$$\boldsymbol{fp}\int_0^\infty u^\beta \log^s(u)\langle T_n^g(u^b), f(.,t)\rangle\,\mathrm{d}u,$$

makes it possible to cast the term E_1 in the following form

$$\begin{aligned}
E_1 = \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)\Bigg[\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\gamma}\log^{l_1}(x_1)\cdots \\
\times\Bigg[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\gamma}\log^{l_1}(x_n) \\
\times\Bigg\{\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\Bigg[\sum_{s=0}^{h} C_h^s\frac{b^{s-h}}{b^{s+1}}\log^{h-s}(t^{-a})\log^s(ux_1^{-a_1}\cdots x_n^{-a_n})\Bigg] \\
\times g(u)\,\mathrm{d}u\Bigg\} f(x')\,\mathrm{d}x_n\Bigg]\cdots\mathrm{d}x_1\Bigg]\,\mathrm{d}t \\
+\sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta^\gamma(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!} \\
\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t) \\
\times\Bigg\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-(a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\ldots\delta_{i_k}^n x') \\
\times\Bigg[\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\sum_{s=0}^{h}\frac{b^{s-h}}{b^{s+1}}\frac{C_h^s\log^{h-s}(t^{-a})l_{i_1}!\cdots l_{i_k}!s!}{(s+\sum_{j=1}^k l_{i_j}+k)!} \\
\times\Bigg[\log\Bigg(u\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\Bigg)\Bigg]^{(s+\sum_{j=1}^k l_{i_j}+k)} g(u)\,\mathrm{d}u\Bigg]\,\mathrm{d}x_j\Bigg\}\,\mathrm{d}t \\
+\sum_{\{e;\Lambda_e=\gamma\}}\sum_{m=0}^{M(e)} g_{em}(-1)^{m+1}m!\,\boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e}\log^l(t) \\
\times\Bigg[\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\Lambda_e}\log^{l_1}(x_1)\cdots\Bigg[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\Lambda_e}\log^{l_1}(x_n)f(x')
\end{aligned}$$

$$\times\left\{\sum_{s=0}^{h} C_h^s \frac{b^{s-h}}{b^{s+1}} \frac{s!}{(m+s+1)!} \log^{h-s}(t^{-a})\right.$$

$$\left.\left.\left.\times \log^{m+s+1}(x_1^{-a_1}\cdots x_n^{-a_n})\right\} \mathrm{d}x_n\right]\cdots \mathrm{d}x_1\right] \mathrm{d}t$$

$$+\sum_{k=1}^{n-1}\frac{(-1)^{k+1}}{k!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\dots,i_k}^{n}\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}$$

$$\times \frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}$$

$$\times \sum_{m=0}^{M(e)} g_{em}(-1)^m m! \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)$$

$$\times\left\{\prod_{j;i_1,\dots i_k}^{n} \boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j) f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x')\right.$$

$$\times\sum_{s=0}^{h}\frac{b^{s-h}}{b^{s+1}}\frac{C_h^s\log^{h-s}(t^{-a})!s!l_{i_1}!\dots l_{i_k}!}{(1+s+\sum_{j=1}^{k} l_{i_j}+m+k)!}$$

$$\left.\times\left[\log\left(\prod_{j;i_1,\dots,i_k}^{n} x_j^{-a_j}\right)\right]^{(1+s+\sum_{j=1}^{k} l_{i_j}+m+k)} \mathrm{d}x_j\right\}\mathrm{d}t, \tag{6.20}$$

where it is recalled that $\gamma := (\beta+1)/b$ and $x' := (x_1,\dots,x_n,t) = (x,t)$. The first integral on the right-hand side of (6.20), noted E, turns out to be

$$E = \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)\left[\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\gamma}\log^{l_1}(x_1)\cdots\left[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\gamma}\log^{l_1}(x_n)\right.\right.$$

$$\left.\left.\times\left\{\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\frac{\log^h(ux_1^{-a_1}\dots x_n^{-a_n}t^{-a})}{b^{h+1}}g(u)\,\mathrm{d}u\right\}f(x')\,\mathrm{d}x_n\right]\cdots\mathrm{d}x_1\right]\mathrm{d}t, \tag{6.21}$$

i.e. it reduces to the integral obtained by applying without any caution Fubini's theorem and change of variable $u := v^b x_1^{a_1}\cdots x_n^{a_n} t^a$.

Step 4: treatment of contribution E_2. Using (see result (3.15))

$$\sum_{m=0}^{h}\frac{C_h^m(-1)^m}{(m+j+1)} = (-1)^h\sum_{k=0}^{h}\frac{C_h^k(-1)^k}{(1+h+j-k)} = \frac{h!j!}{(h+j+1)!},$$

$E_2 := \langle C, f\rangle$ becomes (see equality (6.12))

$$E_2 = \frac{h!}{b^{h+1}}\sum_{\{i;\delta_i=\gamma\}}\sum_{j=0}^{J(i)}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)\log^{h+j+1}(t^{-a})\frac{j!(-1)^{j+1}}{(h+j+1)!}\langle D_n^{ij}, f(.,t)\rangle\,\mathrm{d}t, \tag{6.22}$$

with

$$\langle T_n^g(v), f(.,t)\rangle = \sum_{\mathrm{Re}(\delta_i)\leqslant\gamma}\sum_{j=0}^{J(i)}\langle D_n^{ij}, f(.,t)\rangle v^{-\delta_i}\log^j v + o(v^{-\gamma}),$$

for $v \to \infty$. Formula (4.1) for $\lambda := v$ gives this expansion of $\langle T_n^g(v), f(.,t)\rangle$ and thereafter provides the functions $\langle D_n^{ij}, f(.,t)\rangle$ of variable t. This requires to deal with a sum of terms

$$A_L := \log^L(v^{-1}H)v^{-\delta_i} = \sum_{j=0}^{L} C_L^j(-1)^j v^{-\delta_i}\log^{L-j}(H)\log^j(v) \tag{6.23}$$

where $L \in \mathbb{N}$, $\delta_i = \Lambda_e$ or $\delta_i = (\alpha_l + n_l + 1)/a_l$, $l \in \{1,\ldots,n\}$, $n_l \in \{0,\ldots,q_l\}$ and the function H depends on $(x_1,\ldots,x_n,u)$. To each contribution is associated a corrective term e_{2L} such that

$$\begin{aligned} e_{2L} &= -\frac{h!}{b^{h+1}}\sum_{\{i;\delta_i=\gamma\}} \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t) \\ &\quad\times\left[\sum_{j=0}^{L}\frac{L!j!}{j!(L-j)!}\frac{\log^{L-j}(H)}{(1+h+j)!}\log^{1+h+j}(t^{-a})\,\mathrm{d}t\right] \\ &= -\frac{h!}{b^{h+1}}\sum_{\{i;\delta_i=\gamma\}} \boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t) \\ &\quad\times\left[\sum_{s'=1+h}^{1+h+L}\frac{L!C_{1+h+L}^{s'}}{(1+h+L)!}\log^{s'}(t^{-a})\log^{1+L+h-s'}(H)\right]\mathrm{d}t. \end{aligned}$$

This result leads after some algebra to the expression of $E_2 := \int_0^1 t^{\alpha-a\gamma}\log^l(t)\xi(t)\,\mathrm{d}t$. One obtains the following expression for $\xi(t)$:

$$\begin{aligned} \xi(t) &= \frac{h!}{b^{h+1}}\sum_{\{\Lambda_e=\gamma\}}\sum_{m=0}^{M(e)}\frac{g_{em}(-1)^{m+1}m!}{(1+h+m)!}\left[\prod_{j=1}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)\right. \\ &\quad\left.\times f(x')\left\{\sum_{s'=h+1}^{h+m+1}C_{1+h+m}^{s'}\log^{s'}(t^{-a})\left[\log\left(\prod_{j=1}^{n}x_j^{-a_j}\right)\right]^{(1+h+m-s')}\right\}\mathrm{d}x_j\right] \\ &\quad+\frac{h!}{b^{h+1}}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}\frac{\Delta^\gamma(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\ldots n_{i_k}!} \\ &\quad\times\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}\frac{l_{i_1}!\cdots l_{i_k}!}{(h+\sum_{j=1}^{k}l_{i_j}+k)!} \\ &\quad\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-(a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}\right. \\ &\quad\times\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x')\boldsymbol{fp}\int_0^\infty u^{\gamma-1} \\ &\quad\times\left[\sum_{s'=h+1}^{h+\sum_{j=1}^{k}l_{i_j}+k}C_{h+\sum_{j=1}^{k}l_{i_j}+k}^{s'}\log^{s'}(t^{-a})\right. \end{aligned}$$

$$\times\left[\log\left(u\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\right)\right]^{(h+\sum_{j=1}^{k}l_{i_j}+k-s')}g(u)\,\mathrm{d}u\Bigg]\,\mathrm{d}x_j$$

$$+\frac{h!}{b^{h+1}}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\ldots,i_k}^{n}\sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}}$$

$$\times\frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1}$$

$$\times\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\cdots l_{i_k}!}{(1+h+\sum_{j=1}^{k}l_{i_j}+m+k)!}$$

$$\times\left\{\prod_{j;i_1,\ldots,i_k}^{n}\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^{n}\cdots\delta_{i_k}^{n}x')\right.$$

$$\times\left\{\sum_{s'=h+1}^{1+h+\sum_{j=1}^{k}l_{i_j}+m+k}C_{1+h+\sum_{j=1}^{k}l_{i_j}+m+k}^{s'}\log^{s'}(t^{-a})\right.$$

$$\times\left.\left.\left[\log\left(\prod_{j;i_1,\ldots,i_k}^{n}x_j^{-a_j}\right)\right]^{(1+h+\sum_{j=1}^{k}l_{i_j}+m+k-s')}\right\}\mathrm{d}x_j\right\}. \tag{6.24}$$

Observe that the choice of $k=n$ for the last sum occurring on the right-hand side of (6.24) provides the following contribution:

$$C_n=\frac{h!}{b^{h+1}}\frac{(-1)^n}{n!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\ldots,i_n}^{n}\sum_{n_{i_1},\ldots,n_{i_n}}^{p_{i_1},\ldots,p_{i_n}}$$

$$\times\frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_n})}{n_{i_1}!\cdots n_{i_n}!}\left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1}\cdots\left(\frac{1}{a_{i_n}}\right)^{l_{i_n}+1}$$

$$\times\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\ldots l_{i_n}!}{(1+h+\sum_{j=1}^{n}l_{i_j}+m+n)!}f_{x_{i_1}\cdots x_{i_n}}^{n_{i_1}\cdots n_{i_n}}(0,\ldots,0,t)$$

$$\times[\log(t^{-a})]^{(1+h+\sum_{j=1}^{n}l_{i_j}+m+n)},$$

because $\{i_1,\ldots,i_n\}=\{1,\ldots,n\}$ and in such circumstances

$$\prod_{j;i_1,\ldots,i_n}^{n}x_j^{-a_j}:=1.$$

Taking into account in equality (6.20) the relation

$$\sum_{s=0}^{h}C_h^s\frac{b^{s-h}}{b^{s+1}}\frac{s!\log^{h-s}(t^{-a})}{(1+s+L)!}\log^{1+s+L}(H)$$

$$=\frac{h!}{b^{h+1}}\sum_{s'=0}^{h}C_{1+h+L}^{s'}\frac{\log^{s'}(t^{-a})[\log(H)]^{1+h+L-s'}}{(1+h+L)!}.$$

It is possible to write $\mathcal{E} := E_1 + E_2 - E$ as

$$\mathcal{E} = \frac{h!}{b^{h+1}} \sum_{\{\Lambda_e=\gamma\}} \sum_{m=0}^{M(e)} \frac{g_{em}(-1)^{m+1}m!}{(1+h+m)!} \boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e} \log^l(t)\Big[\boldsymbol{fp}\int_0^1 x_1^{\alpha_1-a_1\Lambda_e}\log^{l_1}(x_1)\cdots$$
$$\times\Big[\boldsymbol{fp}\int_0^1 x_n^{\alpha_n-a_n\Lambda_e}\log^{l_n}(x_n)f(x')\log^{1+h+m}\Big(t^{-a}\prod_{j=1}^n x_j^{-a_j}\Big)\,\mathrm{d}x_n\Big]\cdots\mathrm{d}x_1\Big]\,\mathrm{d}t$$
$$+\frac{h!}{b^{h+1}}\sum_{k=1}^n\frac{(-1)^k}{k!}\sum_{i_1,\dots,i_k}^n\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}\frac{\Delta^\gamma(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}$$
$$\times\Big(\frac{1}{a_{i_1}}\Big)^{l_{i_1}+1}\cdots\Big(\frac{1}{a_{i_k}}\Big)^{l_{i_k}+1}\frac{l_{i_1}!\cdots l_{i_k}!}{(h+\sum_{j=1}^k l_{i_j}+k)!}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)$$
$$\times\Big\{\prod_{j;i_1,\dots,i_k}^n\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-(a_j/a_{i_1})(\alpha_{i_1}+n_{i_1}+1)}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x')$$
$$\times\Big[\boldsymbol{fp}\int_0^\infty u^{\gamma-1}\Big[\log\Big(ut^{-a}\prod_{j;i_1,\dots,i_k}^n x_j^{-a_j}\Big)\Big]^{\left(h+\sum_{j=1}^k l_{i_j}+k\right)}g(u)\,\mathrm{d}u\Big]\,\mathrm{d}x_j\Big\}\,\mathrm{d}t$$
$$+\frac{h!}{b^{h+1}}\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\dots,i_k}^n\sum_{n_{i_1},\dots,n_{i_k}}^{p_{i_1},\dots,p_{i_k}}$$
$$\times\frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!}\Big(\frac{1}{a_{i_1}}\Big)^{l_{i_1}+1}\cdots\Big(\frac{1}{a_{i_k}}\Big)^{l_{i_k}+1}$$
$$\times\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\cdots l_{i_k}!}{(1+h+\sum_{j=1}^k l_{i_j}+m+k)!}\boldsymbol{fp}\int_0^1 t^{\alpha-a\Lambda_e}$$
$$\times\log^l(t)\Big\{\prod_{j;i_1,\dots,i_k}^n\boldsymbol{fp}\int_0^1 x_j^{\alpha_j-a_j\Lambda_e}\log^{l_j}(x_j)f_{x_{i_1}\cdots x_{i_k}}^{n_{i_1}\cdots n_{i_k}}(\delta_{i_1}^n\cdots\delta_{i_k}^n x')$$
$$\times\Big[\log\Big(t^{-a}\prod_{j;i_1,\dots,i_k}^n x_j^{-a_j}\Big)\Big]^{\left(1+h+\sum_{j=1}^k l_{i_j}+m+k\right)}\mathrm{d}x_j\Big\}\,\mathrm{d}t+f, \tag{6.25}$$

with

$$f=\frac{h!}{b^{h+1}}\frac{(-1)^n}{n!}\sum_{\{e;\Lambda_e=\gamma\}}\sum_{i_1,\dots,i_n}^n\sum_{n_{i_1},\dots,n_{i_n}}^{p_{i_1},\dots,p_{i_n}}$$
$$\times\frac{\Delta^{\Lambda_e}(n_{i_1},\dots,n_{i_n})}{n_{i_1}!\dots n_{i_n}!}\sum_{m=0}^{M(e)}\frac{(-1)^{m+1}g_{em}m!l_{i_1}!\dots l_{i_n}!}{(1+h+\sum_{j=1}^n l_{i_j}+m+n)!}$$
$$\times\Big(\frac{1}{a_{i_1}}\Big)^{l_{i_1}+1}\cdots\Big(\frac{1}{a_{i_n}}\Big)^{l_{i_n}+1}\boldsymbol{fp}\int_0^1 t^{\alpha-a\gamma}\log^l(t)f_{x_{i_1}\cdots x_{i_n}}^{n_{i_1}\cdots n_{i_n}}(0,\dots,0,t)$$
$$\times[\log(t^{-a})]^{(1+h+\sum_{j=1}^n l_{i_j}+m+n)}\,\mathrm{d}t.$$

Step 5: treatment of contribution E_3. According to definitions (6.14) and (6.17), this term is

$$E_3 = -\frac{h!l!}{(h+l+1)!}\left(\frac{b}{a}\right)^{l+1} \sum_{\{m;a\gamma=\alpha+m+1\}} \left\{ \boldsymbol{fp}\int_0^\infty v^\beta \log^{h+l+1}(v) \right.$$
$$\left. \times \left[\int_0^1 \cdots \int_0^1 x_1^{\alpha_1}\cdots x_n^{\alpha_n} \log^{l_1}(x_1)\cdots\log^{l_n}(x_n) \frac{f_t^m(x,0)}{m!} g(v^b x_1^{a_1}\cdots x_n^{a_n})\,\mathrm{d}x_1\cdots\mathrm{d}x_n \right] \right\}. \tag{6.26}$$

Induction assumption 2 is used for n to deal with this form of E_3. After some algebra, this yields

$$\begin{aligned}
E_3 = &-\frac{h!}{b^{h+1}} \sum_{\{N;a\gamma=\alpha+N+1\}} \frac{1}{N!}\left(\frac{1}{a}\right)^{l+1} \frac{l!}{(1+h+l)!} \left\{ \prod_{j=1}^n \boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a\gamma} \log^{l_j}(x_j) \right. \\
&\left. \times \left[\boldsymbol{fp}\int_0^\infty u^{\gamma-1} \log^{h+l+1}\left(u\prod_{j=1}^n x_j^{-a_j}\right) g(u)\,\mathrm{d}u \right] f_t^N(x,0)\,\mathrm{d}x_j \right\} \\
&+\frac{h!}{b^{h+1}} \sum_{k=1}^n \frac{(-1)^{k+1}}{k!} \sum_{\{N;a\gamma=\alpha+N+1\}} \sum_{i_1,\ldots,i_k}^n \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \\
&\times \frac{\Delta^\gamma(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!N!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \left(\frac{1}{a}\right)^{l+1} \frac{l_{i_1}!\cdots l_{i_k}!l!}{(1+h+l+\sum_{j=1}^k l_{i_j}+k)!} \\
&\times \left\{ \prod_{j;i_1,\ldots,i_k}^n \boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a_j\gamma} \log^{l_j}(x_j) f_{x_{i_1},\ldots,x_{i_k}t}^{n_{i_1},\ldots,n_{i_k}N}(\delta_{i_1}^n\cdots\delta_{i_k}^n \delta x') \right. \\
&\left. \times \left[\boldsymbol{fp}\int_0^\infty u^{\gamma-1} \left[\log\left(u \prod_{j;i_1,\ldots,i_k}^n x_j^{-a_j}\right)\right]^{(1+h+l+\sum_{j=1}^k l_{i_j}+k)} g(u)\,\mathrm{d}u \right] \mathrm{d}x_j \right\} \\
&-\frac{h!}{b^{h+1}} \sum_{\{N;a\gamma=\alpha+N+1\}} \frac{1}{N!}\left(\frac{1}{a}\right)^{l+1} \sum_{\{e;\Lambda_e=\gamma\}} \sum_{m=0}^{M(e)} \frac{g_{em}(-1)^{m+1}m!l!}{(1+h+l+m+1)!} \\
&\times \left\{ \prod_{j=1}^n \boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a_j\Lambda_e} \log^{l_j}(x_j) f_t^N(x,0)\,\mathrm{d}x_j \left[\log\left(\prod_{j=1}^n x_j^{-a_j}\right)\right]^{(1+h+l+m+1)} \right\} \\
&-\frac{h!}{b^{h+1}} \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \sum_{\{N;a\gamma=\alpha+N+1\}} \sum_{e;\Lambda_e=\gamma} \sum_{i_1,\ldots,i_k}^n \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \\
&\times \frac{\Delta^{\Lambda_e}(n_{i_1},\ldots,n_{i_k})}{n_{i_1}!\cdots n_{i_k}!N!} \left(\frac{1}{a_{i_1}}\right)^{l_{i_1}+1} \cdots \left(\frac{1}{a_{i_k}}\right)^{l_{i_k}+1} \left(\frac{1}{a}\right)^{l+1} \\
&\times \sum_{m=0}^{M(e)} \frac{(-1)^{m+1} g_{em} m! l_{i_1}!\cdots l_{i_k}!l!}{(1+h+l+\sum_{j=1}^k l_{i_j}+m+k+1)!} \left\{ \prod_{j;i_1,\ldots,i_k}^n \boldsymbol{fp}\int_0^1 x_j^{\alpha_j - a_j\Lambda_e} \right. \\
&\times \log^{l_j}(x_j) f_{x_{i_1},\ldots,x_{i_k}t}^{n_{i_1},\ldots,n_{i_k}N}(\delta_{i_1}^n\ldots\delta_{i_k}^n \delta x') \\
&\left. \times \left[\log\left(\prod_{j;i_1,\ldots,i_k}^n x_j^{-a_j}\right)\right]^{(1+h+l+\sum_{j=1}^k l_{i_j}+m+k+1)} \mathrm{d}x_j \right\},
\end{aligned} \tag{6.27}$$

with $\delta(x_1, \ldots, x_n, t) := (x_1, \ldots, x_n, 0)$.

By using the proposed method in the previous section for $I^f_{n+1}(\lambda)$, the reader may check that by adding the contributions $\mathcal{E}$, E and E_3 one obtains the proposed formula (4.13) for $\langle K_{n+1}, f\rangle$. In fact, it is worth noting that for $k \geqslant 1$

$$\sum_{\{N;a\gamma=\alpha+N+1\}} \sum_{i_1,\ldots,i_k}^{n} \sum_{n_{i_1},\ldots,n_{i_k}}^{p_{i_1},\ldots,p_{i_k}} \Delta^\gamma(n_{i_1},\ldots,n_{i_k}) F^{n_{i_1},\ldots,n_{i_k},N}_{i_1,\ldots,i_k,n+1}$$
$$= \frac{1}{k+1} \sum_{i_1,\ldots,i_{k+1}}^{n+1,a} \sum_{n_{i_1},\ldots,n_{i_{k+1}}}^{p_{i_1},\ldots,p_{i_{k+1}}} \Delta^\gamma(n_{i_1},\ldots,n_{i_{k+1}}) F^{n_{i_1},\ldots,n_{i_{k+1}}}_{i_1,\ldots,i_{k+1}}.$$

References

Barlet, D. 1982 Développement asymptotique des fonctions obtenues sur les fibres. *Invent. Math.* **68**, 129–174.

Bleistein, N. & Handelsman, R. A. 1975 *Asymptotic expansions of integrals.* New York: Holt, Rinehart and Winston.

Bruning, J. & Heintze, E. 1984 The asymptotic expansion of Minakschisundaram–Pleijel in the equivariant case. *Duke. Math. Jl* **51**, 959–980.

Bruning, J. 1984 On the asymptotic expansion of some integrals. *Arch. Math.* **42**, 253–259.

Bruning, J. & Seeley, R. 1985 Regular singular asymptotics. *Adv. Math.* **58**, 133–148.

Estrada, R. & Kanwal, R. P. 1990 A distributional theory for asymptotic expansions. *Proc. R. Soc. Lond.* A **428**, 399–430.

Estrada, R. & Kanwal, R. P. 1992 The asymptotic expansion of certain multi-dimensional generalized functions. *J. Math. Analyt. Appl.* **163**, 264–283.

Estrada, R. & Kanwal, R. P. 1994 *Asymptotic analysis: a distributional approach.* Boston, MA: Birkhauser.

Hadamard, J. 1932 *Lecture on Cauchy's problem in linear differential equations.* New York: Dover.

Lesch, M. 1993 *Uber eine Klasse singulärer Differentialoperatoren und asymptotische Methoden.* Habilitationssschrift.

McClure, J. P. & Wong, R. 1987 Asymptotic expansion of a multiple integral. *SIAM. J. Math. Analysis* **18**, 1630–1637.

Schwartz, L. 1966 *Théorie des distributions.* Paris: Hermann.

Sellier, A. 1994 Asymptotic expansions of a class of integrals. *Proc. R. Soc. Lond.* A **445**, 693–710.

Wong, R. 1989 *Asymptotic approximations of integrals.* Boston, MA: Academic.

Received 10 November 1994; revised 4 July 1995; accepted 2 November 1995