

Subcritical dynamos in shear flows

F. Rincon^{1,2,*}, G.I. Ogilvie², M.R.E. Proctor², and C. Cossu³

¹ Laboratoire d'Astrophysique de Toulouse-Tarbes, Université de Toulouse, CNRS, 14 avenue Edouard Belin, F-31400 Toulouse, France

² Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

³ Laboratoire d'hydrodynamique (LadHyX), CNRS – École polytechnique, 91128 Palaiseau cedex, France

Received xxx, accepted xxx

Published online later

Key words accretion, accretion disks – dynamo theory – magnetohydrodynamics (MHD) – instabilities

Identifying generic physical mechanisms responsible for the generation of magnetic fields and turbulence in differentially rotating flows is fundamental to understand the dynamics of astrophysical objects such as accretion disks and stars. In this paper, we discuss the concept of subcritical dynamo action and its hydrodynamic analogue exemplified by the process of nonlinear transition to turbulence in non-rotating wall-bounded shear flows. To illustrate this idea, we describe some recent results on nonlinear hydrodynamic transition to turbulence and nonlinear dynamo action in rotating shear flows pertaining to the problem of turbulent angular momentum transport in accretion disks. We argue that this concept is very generic and should be applicable to many astrophysical problems involving a shear flow and non-axisymmetric instabilities of toroidal velocity or magnetic fields such as Kelvin-Helmoltz, MRI, Tayler or global magnetoshear instabilities. In the light of several recent numerical results, we finally suggest that, similarly to a standard linear instability, subcritical MHD dynamo processes in high-Reynolds number shear flows could drive turbulent flows that would in turn generate an independent small-scale dynamo.

© xxxx WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 Introduction

1.1 Differential rotation, non-normality and subcriticality

Differential rotation is a major ingredient of the physics of astrophysical objects and plays a central role in dynamo theory. From a local point of view (zooming in on a narrow region of the flow), it can be decomposed into a global rotation (Coriolis acceleration) and a shear flow (velocity gradient). Both of these components are individually important for dynamo theory. Let us focus on the shear component of differential rotation and neglect global rotation for a moment. An important consequence of the presence of shear is that it can induce a growth of velocity field fluctuations (referred to in the hydrodynamic transition community as the lift-up effect, see Sect. 2) or magnetic field fluctuations (referred to in dynamo theory as the Ω effect for magnetic fluctuations having no spatial dependence along the direction of the shear velocity). These amplification processes, even though they result from linear terms in the equations, do not lead to an exponential growth of fluctuations on long timescales. Instead, velocity/magnetic perturbations grow algebraically for a viscous/resistive timescale and then decay if no other mechanism is present to sustain them. The short-time dynamics associated with linear shear flow oper-

ators is related mathematically to their non-normality (Trefethen 1993 ; Schmid & Henningson 2000), which makes it possible to combine several individually decaying linear eigenvectors into a transiently growing structure that ultimately has to decay viscously or resistively. The fact that transient growth is possible in shear flows, combined with the observation that linearly stable shear flows become nevertheless turbulent at moderate values of the Reynolds number, has given rise to the concept of subcritical, or bypass transition in hydrodynamics. The word subcriticality here relates to the fact that transition in a linearly stable shear flow is possible at finite values of the Reynolds number, while the critical Reynolds number for the “linear bifurcation” is infinity in such a flow. In this subcritical scenario, fluctuations that are transiently amplified to finite amplitudes can become linearly unstable, leading to sustained turbulence (or some form of complex nonlinearity in general, see Baggett, Driscoll & Trefethen 1995) as a consequence of the nonlinear saturation of the instability modes. Whether bypass transition is possible or not is not completely obvious, however, because one has to ensure that the nonlinearities that must come into play to obtain sustained activity on long timescales can actually play such a role (Waleffe 1995a). It was shown by Hamilton, Kim & Waleffe (1995) that subcritical transition is possible for non-rotating wall-bounded shear flows.

In this paper, we aim at making use of the current knowledge on subcriticality in shear flows to understand some as-

* Corresponding author: e-mail: rincon@ast.obs-mip.fr

pects of dynamo action in differentially rotating astrophysical flows, thus we will also consider some effects due to global rotation. In the framework of this particular study, subcriticality will refer to the observation that the sustenance of a given field relies on the presence of nonlinearity in the system, i.e. that if the system was to be linearized around its background *steady* state, there would be no linearly unstable mode that could generate turbulence or magnetic fields permanently. This definition has the advantage of making it clear that subcriticality is related to the dynamo problem.

1.2 Subcriticality in accretion disks

In the first part of the paper (Sect. 2-3), we will mostly concentrate on two problems pertaining to the issue of turbulent transport in accretion disks to illustrate subcriticality in rotating shear flows, which will enable us to make connexions with different types of dynamo problems involving differential rotation (Sect. 4).

The first of these two problems is to understand the origin of turbulence in non-magnetized disks. There is currently no known hydrodynamic instability, either linear or nonlinear, in the Keplerian shear flow regime representative of the orbital dynamics of thin accretion disks, that could maintain the vigorous turbulent state required for accretion to take place. The second problem is to understand the statistical properties of turbulence in magnetized disks where a natural candidate for the generation of turbulence is the magneto-rotational instability (MRI, Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991). The existence of a developed MRI turbulence state relies on the presence of a sustained magnetic field within the disk. If the central accreting object cannot provide an external field threading the whole disk or if the disk resistivity is large enough (or equivalently the disk cold enough) for fossil fields to decay on short timescales in comparison to the disk lifetime, the difficult question of the origin of turbulence in disks directly translates into the equally difficult question of the generation of magnetic fields - dynamo action - in these objects, and its links with the MRI.

Let us first introduce the problem of transport in *non-magnetized disks* or at least in cold and shielded regions of disks where the fluid is not coupled to magnetic fields, as in protoplanetary disks (e.g. Fromang, Terquem & Balbus 2002). The Rayleigh criterion tells us that a simple Keplerian shear flow is linearly stable to axisymmetric perturbations from the hydrodynamical point of view, i.e. that turbulence cannot be generated by a linear axisymmetric hydrodynamic instability in this flow. Angular momentum transport in a non-magnetized disk therefore has to rely on a *fundamentally nonlinear hydrodynamic process*. Hence, the existence of a subcritical transition to turbulence in linearly stable rotating shear flows (particularly anticyclonic ones) has been invoked for many years to explain the origin of turbulence in this context. The main argument in favour of such a transition finds its roots in the well-known experimental

evidence for transition to turbulence in non-rotating wall-bounded shear flows (such as pipe flow) that are also known to be linearly stable. This argument is further qualitatively strengthened by the observed high sensitivity of shear flow stability to initial conditions, which has to do with the previously mentioned non-normality of shear flow operators. There is a long ongoing debate on whether or not subcritical nonlinear transition is possible in Rayleigh-stable shear flows. A flavour of the experimental debate can be found in Tillmark & Alfredsson (1996), Richard & Zahn (1999) and Ji et al. (2006), while on the numerical and theoretical sides, we refer the reader to Hawley, Balbus & Winters (1999), Longaretti (2002) and Lesur & Longaretti (2005). A more exhaustive recent review of this problem can be found in Rincon, Ogilvie & Cossu (2007a). One of the purposes of the present paper is simply to point out the important differences between the physics of subcritical transition in rotating and non-rotating shear flows in terms of transient growth and nonlinear interactions.

The problem of *magnetized disks* is a priori a completely different one because of the existence of the MRI. In the presence of a mean field threading the disk, the MRI grows velocity and magnetic field perturbations with optimally correlated radial and azimuthal components, providing a natural mechanism for angular momentum transport even in three-dimensional saturated regimes (see Lesur & Longaretti (2007) for a study of the efficiency of the process in different magnetic Prandtl number regimes, though). The problem here is that there are large uncertainties regarding the physical conditions that pertain to different types and even different regions of disks. It is in particular not obvious that all disks are threaded by a mean field originating from the central accreting object or that a preexisting field can stay in the disk for a period of time comparable to a disk lifetime and sustain the MRI on this timescale. Is it possible to obtain a sustained turbulent magnetized state in that case? The recent discovery by Donati et al. (2005) that protoplanetary disks can host their own magnetic field, at least in their inner regions, indicates that this is a natural question to ask. As mentioned earlier, answering this question requires us to understand if (MHD) dynamo action is possible in a Keplerian shear flow, which is a rather involved problem in view of the current knowledge in dynamo theory. Let us try to make this point more evident, and assume that there is some undetermined form of dynamo action in a disk, which makes magnetic field perturbations grow in time. Such fluctuations are very likely to trigger the MRI very quickly (the instability develops in weak-field regimes), showing the imbrication of the dynamo process and the MRI. Besides, the process by which MRI works is the magnetic braking of orbiting fluid particles, a fundamentally dynamical process that is not present in kinematic dynamos. Therefore, if dynamo action is present in the system, it must be a *fundamentally nonlinear dynamo*. To cite Hawley & Balbus (1992), who first noticed that point, "the use of a kinematic dynamo model is inappropriate for an accretion disk [...] The tur-

bulence is driven by the very forces [Lorentz forces] the kinematic dynamo excludes from the outset". Even though this kind of dynamo looks fairly unusual, Brandenburg et al. (1995), Stone et al. (1996) and Hawley, Gammie & Balbus (1996) demonstrated its existence. In particular, Hawley et al. (1996) compare the results of experiments in two and three dimensions, with or without a Lorentz force term, to illustrate very clearly the previous points. Some important limitations of these simulations have however recently been pointed out (Fromang & Papaloizou 2007a). If they do not challenge the existence of the MRI dynamo process, at least at large magnetic Prandtl numbers (Fromang et al. 2007b), they do however question its efficiency and demonstrate that how the MRI dynamo works remains a largely open question that deserves further investigation.

1.3 Outline

The first goal of the present paper is to provide connexions between the two forementioned hydrodynamic and MHD problems, which are individually discussed in detail in Rincon et al. (2007a) and Rincon, Ogilvie & Proctor (2007b). As will be shown in Sect. 2 and Sect. 3, this comparative approach is motivated by several important similarities between the two problems, namely hardwired nonlinearity and linear non-normality. In Sect. 2, we first discuss the dynamical principles of transition in linearly stable shear flows and show that this problem can be viewed as a nonlinear "hydrodynamic dynamo" problem analogous to an MHD dynamo problem. This allows us to point out the limits of the analogies between nonlinear hydrodynamic instability in anticyclonic Rayleigh-stable shear flows and non-rotating wall-bounded shear flows and to question the existence of a subcritical transition in Keplerian flows. We then make use of similar methods to address the MRI dynamo problem and to illustrate the concept of the nonlinear self-sustaining MHD process (Sect. 3). We then move to the second major goal of this paper, which is to show that the subcritical dynamo concept is a very generic one that applies to many types of shear flow problems. This is done in Sect. 4, where we also discuss the relations between this dynamo model and more standard mean-field models such as the $\alpha\Omega$ dynamo. Finally, based on these results and recent numerical results, notably simulations of MRI turbulence in the dynamo regime, we suggest a possible scenario for the MRI dynamo at large Re and Rm involving small-scale dynamo action (Sect. 5). The paper ends with a brief conclusion.

2 Hydrodynamic transition in shear flows

In this Section, we start by setting up a very simple hydrodynamic model problem, namely incompressible rotating plane Couette flow, to show that subcritical hydrodynamic transition to turbulence in some particular differentially rotating flows can be analysed using dynamo arguments, i.e. in terms of dynamical exchanges between "poloidal" and

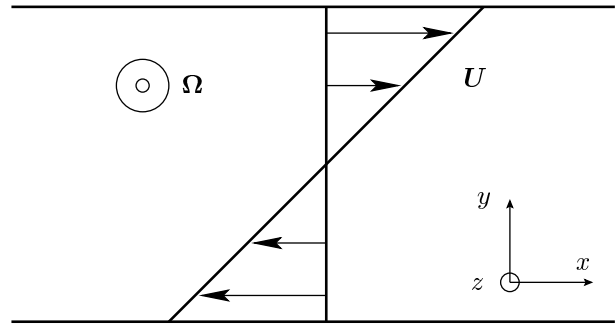


Fig. 1 Geometry of rotating plane Couette flow.

"toroidal" components of velocity field fluctuations with vanishing spatial average. We then describe how transition to turbulence is actually thought to work in non-rotating shear flows that are linearly stable for all finite values of the Reynolds numbers using this dynamo analogy¹. We finally build on these results to discuss the possibility of a similar hydrodynamic transition in linearly stable rotating shear flows such as a Keplerian flow. The analysis performed here also sets the scene for the MHD part of the paper and the following discussion.

2.1 Incompressible rotating plane Couette flow

Incompressible rotating plane Couette flow, represented in Fig. 1, is a local approximation of differentially rotating flows such as those encountered in accretion disks. The flow is characterized by a background velocity field $\mathbf{U} = Sy \mathbf{e}_x$ with linear shear S driven by countermoving walls situated at $y = \pm d$ and by a global, uniform in space, rotation vector $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$. The Reynolds number for this flow is usually defined in the transition literature as

$$Re = \frac{Sd^2}{\nu} \quad (1)$$

where ν is the kinematic viscosity. We can also define a rotation number,

$$R_\Omega = -\frac{2\Omega}{S}, \quad (2)$$

which is positive for cyclonic flows (shear flow vorticity parallel to $\boldsymbol{\Omega}$) and negative for anticyclonic flows (shear flow vorticity anti-parallel to $\boldsymbol{\Omega}$) and is related to a parameter commonly used in accretion disk theory to characterize differential rotation, $q = -d \ln \Omega / d \ln r$, according to $R_\Omega = -2/q$. A non-rotating flow has $R_\Omega = 0$, a flow on the Rayleigh line has $R_\Omega = -1$ and a Keplerian flow has $R_\Omega = -4/3$. The flow encounters an axisymmetric centrifugal instability for $-1 < R_\Omega < 0$. Our aim is to highlight the generic dynamo nature of this problem for $R_\Omega = 0$ and $R_\Omega = -1$ and its potential relevance to accretion disks

¹ The term "dynamo" has already been used in a hydrodynamic context to describe the generation of large-scale vorticity fluctuations by small-scale helical turbulence (the AKA effect, see for instance Moiseev et al. 1988), which is a distinct phenomenon from what we describe in this paper.

and stars, so we choose to refer to the streamwise direction x as being the toroidal direction and to the (y, z) plane as being the poloidal plane. We further use the word axisymmetric to qualify x -independent perturbations (in the accretion disk terminology, x corresponds to the azimuthal direction, $-y$ to the radial direction, and z to the vertical direction). Denoting by an overbar an average over x , any velocity field perturbation of the background flow is separated into an axisymmetric part $\bar{\mathbf{u}}$ and a non-axisymmetric “wave” part \mathbf{u}' ,

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \quad \text{with} \quad \nabla \cdot \mathbf{u} = \nabla \cdot \bar{\mathbf{u}} = \nabla \cdot \mathbf{u}' = 0. \quad (3)$$

Note that both \mathbf{u} and \mathbf{u}' are chosen to have zero volume average, so that we have a form of dynamo problem for the perturbations of the background Couette flow in the centrifugally stable regimes. Using d and $1/S$ as space and time units, the momentum equation for the axisymmetric toroidal \bar{u}_x and poloidal $\bar{\mathbf{u}}_p$ components of the velocity field read

$$(\partial_t + \bar{\mathbf{u}}_p \cdot \nabla) \bar{u}_x + (R_\Omega + 1)\bar{u}_y = \frac{1}{Re} \Delta \bar{u}_x + F_x, \quad (4)$$

$$(\partial_t + \bar{\mathbf{u}}_p \cdot \nabla) \bar{\mathbf{u}}_p - R_\Omega \bar{u}_x \mathbf{e}_y = -\nabla_p \bar{p} + \frac{1}{Re} \Delta \bar{\mathbf{u}}_p + \mathbf{F}_p, \quad (5)$$

where

$$\mathbf{F} = -\overline{\mathbf{u}' \cdot \nabla \mathbf{u}'} \quad (6)$$

is the mean force associated with the non-axisymmetric part of the flow and p is the pressure divided by the constant density. $\bar{\mathbf{u}}_p$ is a solenoidal two-dimensional velocity field that can be written in terms of a streamfunction $\psi(y, z, t)$ with vanishing volume average:

$$\bar{\mathbf{u}}_p = \nabla \times (\psi \mathbf{e}_x). \quad (7)$$

The associated toroidal vorticity reads

$$\omega_x = -\Delta \psi. \quad (8)$$

Taking the curl of Eq. (5) to eliminate \bar{p} , one obtains

$$\partial_t \bar{u}_x - \frac{\partial(\psi, \bar{u}_x)}{\partial(y, z)} = -(R_\Omega + 1)\bar{u}_y + \frac{1}{Re} \Delta \bar{u}_x + F_x, \quad (9)$$

$$\partial_t \omega_x - \frac{\partial(\psi, \omega_x)}{\partial(y, z)} = -R_\Omega \partial_z \bar{u}_x + \frac{1}{Re} \Delta \omega_x + \mathbf{e}_x \cdot \nabla \times \mathbf{F} \quad (10)$$

where $\partial(\cdot, \cdot)/\partial(\cdot, \cdot)$ denotes the Jacobian. It can be seen that both \bar{u}_x and ω_x have linear source or sink terms proportional to \bar{u}_y and $\partial_z \bar{u}_x$, and a fully nonlinear source or sink term associated with the axisymmetric part of the advection term \mathbf{F} . This observation also stand for shear profiles different from that of plane Couette flow.

2.2 Subcritical transition in non-rotating shear flows

Let us now specialize to interesting particular cases and first concentrate on the non-rotating case $R_\Omega = 0$. The phenomenology of transition in non-rotating shear flows has been a mystery since the end of the 19th century and the experiments by O. Reynolds (1883). For a long time, the experimental observation that wall-bounded shear flows encounter transition to turbulence at modest Re values could not be reconciled with theory, which could only predict the absence of linear instability in many of these flows. There is now some significant evidence that the transition is related to a nonlinear mechanism christened the self-sustaining process (SSP), which was first described in detail by Hamilton et al. (1995). The actual complexity of this transition and its sensitivity to initial conditions depends on the background shear profile (e.g. plane Couette flow, plane Poiseuille flow, pipe Hagen-Poiseuille flow or Blasius boundary layer) and on the structure of the phase space of the corresponding dynamical system (i.e. the presence of fixed points, homoclinic or heteroclinic orbits, etc.). Transition is currently best understood for plane Couette flow, which encounters a simple saddle-node bifurcation (Rincon et al. 2007a; Wang, Gibson & Waleffe 2007; Viswanath 2007), while pipe Hagen-Poiseuille flow, for instance, seems to encounter a complex chaotic transition whose limits in parameter space are now referred to as “the edge of chaos” (Schneider, Eckhardt & Yorke 2006, 2007).

A nice way of describing the SSP is to think of it in terms of subcritical “hydrodynamic dynamo”. We first note that for $R_\Omega = 0$, only \bar{u}_x has a linear “source” term. This term is an hydrodynamic analogue of the Ω -effect of dynamo theory. It is actually called the lift-up effect in the transition literature (Ellingsen & Palm 1975; Landahl 1980) and is of course a purely axisymmetric effect which leads to algebraic linear amplification of the toroidal velocity field. However, as for the Ω -effect, this is only a transient effect, for in the absence of a poloidal source term to regenerate the poloidal velocity field, viscous damping ultimately kills both poloidal and toroidal velocity fields on a viscous time scale. As can be seen in Eq. (10) there is no linear poloidal velocity source term in the absence of rotation, so that the only way poloidal motions can be sustained is via the nonlinear interaction term $\nabla \times \mathbf{F}$, which is only non-vanishing if the total flow as a non-axisymmetric part. This is clearly a form of antidynamo theorem for non-rotating linearly stable hydrodynamic shear flows. Therefore, the dynamo question is to ask how non-axisymmetry can emerge in such a system, leading to a self-sustaining solution via nonlinear feedback on poloidal motions. To answer this question, one has to look at what the lift-up effect actually produces. Starting with a $\mathcal{O}(1/Re)$ poloidal velocity field depending on y and z , the lift-up effect has the ability to generate on a $\mathcal{O}(Re)$ timescale a toroidal velocity field also depending on y and z with an $\mathcal{O}(1)$ amplitude, comparable to that of the background shear flow. There is a lot of experimental evidence for this so-called “streaks” field (Hamilton

et al. 1995). As this field is modulated in z , it is actually a finite-amplitude shear flow that contains inflexion points in that direction, and is therefore unstable to non-axisymmetric Kelvin-Helmholtz type instabilities. Waleffe (1995b, 1997, 1998) showed that the three-dimensional velocity field associated with these instabilities leads to a \mathbf{F} term that can regenerate the weak poloidal velocity field. In other words, Reynolds stresses generated by the secondary instability in its weakly nonlinear regime can close the dynamo loop. For a given Re value, it is possible to show that there is a continuous family of self-sustained solutions with different aspect ratio, i.e. different periodicity in x and z .

A very important requirement for the process to work is that there must be a very good spatial overlap (correlation) between the nonlinear interaction term \mathbf{F} and the axisymmetric poloidal velocity. The spatial shape of the feedback is notably very sensitive to the geometry of the flow and to the wavelength of the instability mode. A second important remark is that the process is fully nonlinear and subcritical for two reasons. First, for a given Re , a finite amplitude $\mathcal{O}(1/Re)$ poloidal field is required to initiate the process. Such an amplitude is of course extremely small at very large Re but can never be taken infinitesimally small as would be the case for a linear instability (Nagata (1990) described the corresponding branch of nonlinear solutions - which had not been described in terms of a SSP at that time - as a “bifurcation from infinity”). The second reason is that the feedback term is a fully nonlinear interaction term due to the non-axisymmetric instability mode. To summarize the process in terms of an initial value problem, let us enumerate the three elements that are required to obtain a subcritical “hydrodynamic dynamo” in a non-rotating shear flow:

1. linear transient amplification up to $\mathcal{O}(1)$ amplitudes of an axisymmetric toroidal velocity field, from a seed but finite $\mathcal{O}(1/Re)$ axisymmetric poloidal velocity field,
2. non-axisymmetric linear shear instability of the finite-amplitude, z -modulated axisymmetric toroidal velocity field,
3. regeneration of the weak axisymmetric poloidal velocity field by nonlinear self-interactions of the non-axisymmetric instability mode.

An equivalent description of the SSP can be given in terms of a coherent structure (Rincon et al. 2007a) which, for plane Couette flow, is a saddle-node fixed point in phase space. The initial value problem formulation described above then consists in following phase space trajectories sticking as much as possible to the stable manifold of the fixed point and approaching it closely before being ejected along its unstable manifold, towards a turbulent attractor. A very nice graphical representation of this process is given by Gibson, Halcrow and Cvitanović (2008).

2.3 Subcritical transition in rotating shear flows

Is a subcritical hydrodynamic transition possible in linearly stable *rotating* shear flows? As mentioned in the introduction, the observation that their non-rotating linearly stable counterparts become turbulent at modest values of Re has been used as an argument in favour of such a scenario for a long time. Now that transition in non-rotating flows is better understood in terms of the SSP, a natural question to ask is whether a similar process can occur in rotating shear flows. This question was addressed by Rincon et al. (2007a). Here, we summarize the main points of this study using the subcritical dynamo phenomenology. The starting point is that the linearized non-rotating case has an interesting linearly stable rotating counterpart, the Rayleigh-line regime $R_\Omega = -1$ (we note in passing that the Rayleigh-line regime is similar to the Keplerian regime in the sense that both regimes are anticyclonic and linearly stable. The Rayleigh line is of course closer to the centrifugal instability region). For $R_\Omega = -1$, the linear source term in Eq. (9) is exactly vanishing, but there is still a linear Coriolis acceleration term in Eq. (10). From the linear point of view, this means that the lift-up effect has an exact analog on the Rayleigh line. This effect, christened the anti lift-up effect by Antkowiak & Brancher (2007), has the ability to transiently generate a finite-amplitude axisymmetric poloidal velocity field from a seed $\mathcal{O}(1/Re)$ axisymmetric toroidal velocity field. In other words, streaks now generate rolls. The fact that such an axisymmetric linear transient amplification effect is present on the Rayleigh line is certainly encouraging for the prospect of a rotating SSP, but the full picture requires nonlinearity to come into play. In that respect, the Rayleigh line regime appears to be far less favourable than the non-rotating case, for two important reasons. The first one is that the presence of a finite-amplitude axisymmetric poloidal flow on the Rayleigh line, unlike that of the axisymmetric toroidal flow in the non-rotating case, causes a strong poloidal nonlinear advection of the total axisymmetric flow via the nonlinear terms after ∂_t in Eqs. (4)-(5) as soon as the poloidal flow reaches a significant amplitude by means of the anti lift-up effect. This renders the axisymmetric part of the flow much more complex than in the non-rotating case. The second reason is that the non-axisymmetric instabilities of the resulting nonlinear axisymmetric flow do not generate a nonlinear feedback that would show a good spatial correlation with the seed streaks field. These instabilities are actually quite different from those encountered in the non-rotating case for now the axisymmetric flow is dominated by its poloidal component instead of its toroidal component which was unstable to shearing instabilities in the non-rotating regime. Overall, we found it impossible to proceed as in the non-rotating case to obtain fully nonlinear three-dimensional steady solutions of the fluid equations.

The current status regarding stability with respect to R_Ω in rotating plane Couette flow is depicted in Fig. 2. Several comments on this figure are in order before we close

the hydrodynamic Section. First, it is important to note that subcritical solutions obtained in the non-rotating case can be continued nonlinearly into the anticyclonic linearly unstable regime, where they take on the form of nonlinear wavy Taylor vortices (instabilities of the Taylor vortices that result from the centrifugal instability in this regime). They can also be continued into the cyclonic linearly stable regime which is therefore nonlinearly unstable. A similar behaviour cannot be found in the neighbourhood of the Rayleigh line, demonstrating that a symmetry between the cyclonic regime and its anticyclonic counterpart beyond the Rayleigh line does not exist in three dimensions (the absence of this symmetry was also demonstrated by Lesur & Longaretti (2005) using direct numerical simulations). The second comment is that in the case of a general differential rotation profile for which R_Ω is different from 0 or -1 , there is no lift-up or anti lift-up effect anymore. In the Keplerian case, for instance, only non-axisymmetric transient growth (also sometimes called swing amplification) is possible, which makes it pretty much useless to think in terms of a poloidal and toroidal description in that case. The only way the Keplerian regime could be related to the previous type of subcritical hydrodynamic dynamo would be to make a nonlinear connexion with a SSP on the Rayleigh line which, as we showed, probably does not exist. It therefore appears that subcritical transition in anticyclonic shear flows, if any, has only very little to do with subcritical transition in non-rotating or cyclonic shear flows. The possibility of a subcritical dynamo mechanism relying on instabilities of transiently amplified non-axisymmetric structures remains open but recent high-resolution simulations have not been able to isolate such a physical process either (Shen, Stone & Gardiner 2006).

3 Subcritical MRI dynamo in Keplerian flow

We now move to the MHD problem of uncovering the physical processes that give rise to what is called MRI dynamo action in Keplerian shear flows. For this purpose, we keep our rotating plane Couette flow description, setting $R_\Omega = -4/3$. This problem has much in common with the non-rotating hydrodynamic problem, which makes it appealing to attempt to apply the SSP phenomenology again. First, as mentioned in the introduction, any MRI dynamo must be intrinsically nonlinear, because the Lorentz force is mandatory for the MRI. Besides, since magnetic fields feel shear but not Coriolis acceleration, it is possible to algebraically (and transiently) amplify axisymmetric toroidal fields from poloidal fields through the Ω effect in the same manner as the amplification of streaks from rolls in the non-rotating hydrodynamic problem (see Livermore & Jackson (2004) for a discussion on non-normality in MHD). One might then ask if the toroidal magnetic field encounters non-axisymmetric instabilities such as an MRI and if the nonlinear interactions of such instabilities have the ability to close the

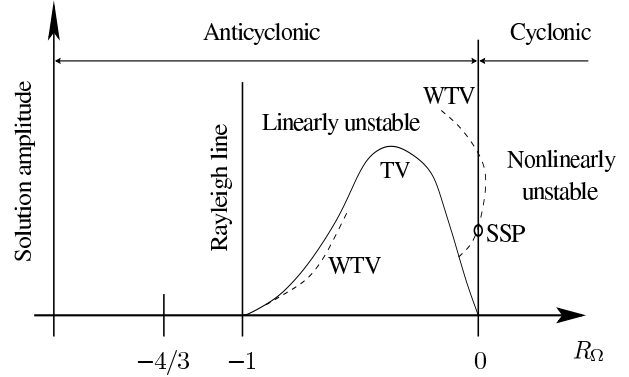


Fig. 2 Stability diagram for rotating plane Couette flow for a value of Re in the domain of existence of the self-sustaining process. The amplitude of the solutions in this diagram is somewhat arbitrary but is intended to approximate the behaviour of the shearing rate at the flow boundaries. The TV full line curve is for nonlinear axisymmetric Taylor Vortices arising in the linearly unstable regime, the dashed WTV lines is for the three-dimensional Wavy Taylor Vortices bifurcating from the axisymmetric TVs, and SSP stands for the fixed point associated with the self-sustaining process at $R_\Omega = 0$. This point can be obtained by direct continuation of WTVs to $R_\Omega = 0$ and the corresponding branch of solutions can be continued in the linearly stable cyclonic region (see Rincon et al. (2007a) for further details and references).

dynamo loop by feeding back on the axisymmetric poloidal field.

It is straightforward to show that the induction part of the MHD rotating plane Couette flow problem takes on a form very similar to the momentum equation for the non-rotating hydrodynamic shear flow problem. For this purpose, we introduce the magnetic Reynolds number

$$Rm = \frac{Sd^2}{\eta}, \quad (11)$$

where η is the magnetic diffusivity, and we decompose the magnetic field \mathbf{b} in terms of an axisymmetric part $\bar{\mathbf{b}}$ and a non-axisymmetric wave part \mathbf{b}'

$$\mathbf{b} = \bar{\mathbf{b}} + \mathbf{b}' \quad \text{with} \quad \nabla \cdot \mathbf{b} = \nabla \cdot \bar{\mathbf{b}} = \nabla \cdot \mathbf{b}' = 0. \quad (12)$$

Note that both $\bar{\mathbf{b}}$ and \mathbf{b}' are chosen to have zero volume average, so that we are dealing with a genuine dynamo problem (no net magnetic flux is threading the domain). Similarly to the hydrodynamic problem, we introduce a flux function $\chi(y, z, t)$ to describe the axisymmetric part of the poloidal magnetic field $\bar{\mathbf{b}}_p$

$$\bar{\mathbf{b}}_p = \nabla \times (\chi \mathbf{e}_x). \quad (13)$$

The induction equation for χ and for the axisymmetric toroidal component of the magnetic field \bar{b}_x reads

$$\partial_t \bar{b}_x - \mathbf{e}_x \cdot \nabla \times (\bar{\mathbf{u}} \times \bar{\mathbf{b}}) = \bar{b}_y + \mathbf{e}_x \cdot \nabla \times \mathbf{E} + \frac{1}{Rm} \Delta \bar{b}_x, \quad (14)$$

$$\partial_t \chi - \frac{\partial(\psi, \chi)}{\partial(y, z)} = E_x + \frac{1}{Rm} \Delta \chi, \quad (15)$$

where ψ has been defined in Eq. (7), the first linear term on the right hand side of Eq. (14) stands for magnetic induction by the shear flow and

$$\mathbf{E} = \overline{\mathbf{v}' \times \mathbf{b}'} \quad (16)$$

is the axisymmetric part of the electromotive force (EMF) generated by the nonlinear interaction between the wave parts of the velocity and magnetic fields. It is straightforward to notice that Eqs. (14)-(15) share the most interesting characteristics of Eqs. (9)-(10), namely linear non-normality in the toroidal part and nonlinear feedback of three-dimensional structures in the poloidal part. Taking once again the point of view of an initial value formulation and starting from an axisymmetric $\mathcal{O}(1/Rm)$ poloidal magnetic field (expressed in terms of an Alfvén velocity), an axisymmetric $\mathcal{O}(1)$ toroidal field can be transiently generated on a timescale $\mathcal{O}(Rm)$ by the Ω effect. Rincon et al. (2007b) showed (solving the full MHD equations, including a Lorentz force in the momentum equation) that such a toroidal field encounters a non-axisymmetric instability. There are several instability modes with different symmetry properties which can be predicted from the symmetries imposed on the axisymmetric fields. The important point is that for some of these modes, nonlinear interactions give rise to a toroidal EMF which axisymmetric projection has the ability to regenerate the initial poloidal seed field, thereby leading to dynamo action. We interpreted these instabilities in terms of an MRI of the axisymmetric toroidal field: the axisymmetric poloidal field here is too weak to be MRI-unstable (only very large poloidal wavenumbers would be unstable, and they are damped by viscous and resistive terms in our dissipative set-up). Another indication is that the instabilities are purely non-axisymmetric, which is in line with the MRI instability growth rate γ obtained from the local dispersion relation $\gamma \sim \mathbf{k} \cdot \mathbf{V}_A$ where the Alfvén velocity \mathbf{V}_A would be dominated by the toroidal field. Also, the modes are spatially centred and symmetric or antisymmetric with respect to the local extrema of the toroidal field in the poloidal plane. Finally, we noticed that the presence of the instability required an MRI unstable rotation to be present.

Overall, we found it possible to obtain nonlinear fixed points in the same way as in the hydrodynamic problem, but only for a very restricted range of low values of Re . In this regime, non-axisymmetric modes have real eigenvalues corresponding to purely imaginary frequencies (the modes arise from a steady pitchfork bifurcation). The full three-dimensional nonlinear solution that can be computed from these modes therefore takes the form of a steady solution (equivalently a fixed point). For larger values of Re , we observed collisions between pairs of real eigenvalues of the instability modes, which turn into complex conjugate pairs (the corresponding modes subsequently arise from a Hopf bifurcation). In such a situation, we expect nonlinear travelling waves to be present instead of steady solutions, similarly to what happens in the hydrodynamic problem for Hagen-Poiseuille flow (Wedin & Kerswell 2004)

or plane Poiseuille flow (Waleffe 2001). These travelling waves can in principle be captured numerically exactly like steady solutions (by performing a Galilean transformation and adding the phase speed as an extra unknown in the problem). In the present MHD problem, however, such a numerical solution cannot be obtained easily because a symmetry used to decrease the computing costs is lost when the mode eigenvalues turn into complex conjugate pairs. Seeking coherent time-dependent solutions to the subcritical MRI dynamo problem using direct time-stepping techniques therefore looks more promising than using continuation techniques such as those described in Rincon et al. (2007a,2007b). Such an investigation is currently underway (Lesur & Ogilvie 2008).

Like the hydrodynamic self-sustaining process, the process described above is genuinely nonlinear. First, a small but finite-amplitude seed axisymmetric poloidal field is required to obtain a sufficiently large axisymmetric toroidal field to trigger non-axisymmetric instabilities (but sufficiently weak at the same time for an MRI to be possible). Besides, two nonlinear terms, a Lorentz force and a fluctuating EMF, are needed for the instability to develop and for the feedback on the axisymmetric poloidal field to be possible. Note that the Lorentz force is a nonlinear term in that context because the axisymmetric toroidal field that becomes MRI-unstable is part of the total magnetic field perturbation (it has zero net-flux and would decay on a resistive timescale without the three-dimensional instability feedback). To summarize this self-sustaining MHD process by means of an initial value description, let us restate the three elements that are required to obtain a subcritical MRI dynamo in a Keplerian shear flow:

1. linear transient amplification of an axisymmetric toroidal magnetic field by the Ω effect acting on a $\mathcal{O}(1/Rm)$ seed axisymmetric poloidal magnetic field,
2. non-axisymmetric linear instability (MRI) of the finite-amplitude axisymmetric toroidal magnetic field,
3. regeneration of the seed axisymmetric poloidal magnetic field by nonlinear self-interactions of the non-axisymmetric instability mode.

As discussed just before, an equivalent description of this SSP can be done in terms of steady or travelling coherent structures (Rincon et al. 2007b). A tentative interpretation of the role of such coherent structures in simulations of MRI turbulence in zero net-flux set-ups is suggested in Sect. 5.

4 Connexions with other dynamo models

The phenomenology of the subcritical shear dynamo scenario presented above is obviously not specific to Keplerian shear flows. Dynamos models relying on non-axisymmetric hydromagnetic instabilities in differentially rotating flows have been thought for in the context of the geodynamo for

instance (Fearn & Proctor 1983, 1984, 1987). In this Section, we show that several recent dynamo models introduced by Cline, Brummell & Cattaneo (2003), Spruit (2002) and Miesch (2007a) that are relevant to astrophysics fit perfectly into the subcritical dynamo concept. We then attempt to compare the subcritical scenario with more standard dynamo models such as the $\alpha\Omega$ dynamo and to provide some connexions between these two views. Finally, we comment briefly on another form of subcritical dynamo recently discovered by Ponty et al. (2007).

4.1 Dynamos driven by non-axisymmetric instabilities

A fundamental ingredient in the subcritical dynamo loop presented in the previous sections is the possibility of triggering non-axisymmetric instabilities in the system. Working in the Keplerian regime allows for one such instability, the MRI, to exist, while the subcritical hydrodynamic transition in non-rotating shear flows relies on a non-axisymmetric instability triggered by a spatially modulated shear flow profile. The interesting point is that there are many other systems where non-axisymmetric MHD or hydrodynamic instabilities are present. Here, we will focus on several dynamo models that have recently been discussed in the astrophysical literature (we will come back to the previously mentioned geodynamo model of Fearn & Proctor (1984, 1987) in the next paragraph). An interesting example is that of the shear-buoyancy dynamo discovered by Cline et al. (2003) using direct numerical simulations (a typical initial value problem), which can be summarized in the following way:

1. linear transient amplification of an axisymmetric toroidal magnetic field by the Ω effect acting on a seed axisymmetric poloidal magnetic field,
2. axisymmetric buoyancy instability of the toroidal field inducing an axisymmetric poloidal flow,
3. linear transient amplification of an axisymmetric toroidal velocity field from the axisymmetric poloidal flow (thereby steepening the original shear),
4. non-axisymmetric linear instability (Kelvin-Helmholtz) of the steeper axisymmetric toroidal velocity field,
5. regeneration of the seed axisymmetric poloidal magnetic field by nonlinear self-interactions of the non-axisymmetric instability mode.

This loop looks very similar to the subcritical dynamo loops described earlier and indeed Cline et al. (2003) clearly noticed that their dynamo was intrinsically nonlinear. Several other elements in their paper further demonstrate that these dynamos proceed in the same way. First, Cline et al. (2003) mention a poloidal magnetic field threshold depending on Rm under which no dynamo action is possible, very much

like in the Keplerian problem (in which the threshold amplitude scales like $1/Rm$) and in the non-rotating hydrodynamic problem (a $1/Re$ threshold amplitude is possible in that case but this point is still a matter of debate though, see Kreiss, Lundbladh & Henningson (1993), Baggett & Trefethen (1997) and the experiments by Peixinho & Mullin (2007)). Since the threshold is determined by the physics of algebraic growth via the Ω effect, which is present in all these problems, we suspect that an axisymmetric poloidal magnetic field amplitude scaling like $1/Rm$ is also required in the shear-buoyancy case for the dynamo to be triggered. Cline et al. (2003) also report both steady and cyclic regimes depending on their parameters, which is in line with our previous argument regarding the possibility of having nonlinear travelling waves in the large Re regime of the Keplerian problem. We finally note that an important reason why the Cline et al. (2003) dynamo works is because they deal with a non-rotating system, meaning that the poloidal velocity field perturbations associated with the magnetic buoyancy instability are getting wound up into strong toroidal velocity field perturbations by the lift-up effect (step 3 of their process). It is that toroidal velocity field that renders the total shear profile unstable to non-axisymmetric Kelvin-Helmholtz instabilities, which in turn produce the feedback. As mentioned in Sect. 2, it is not possible anymore to produce a strong axisymmetric toroidal velocity field and the associated Kelvin-Helmholtz instability in the presence of a global rotation rate comparable to the shearing rate. In such a rotating regime, their dynamo should therefore disappear unless an extra non-axisymmetric physical mechanism kicks in.

Another problem which takes on the same form is that of dynamo action in stellar radiation zones. Spruit (2002) suggested the existence of a dynamo relying on the non-axisymmetric Tayler (1973) - Pitts & Tayler (1985) instability of toroidal magnetic fields generated by the Ω effect. This scenario seemed to be confirmed by numerical simulations performed by Braithwaite (2006), but this view has recently been challenged by Zahn, Brun & Mathis (2007) and Gellert, Rüdiger & Elstner (2008) on the basis of direct numerical simulations. Zahn et al. (2007) do observe the amplification of an axisymmetric toroidal field from a seed poloidal field via the Ω effect and the subsequent occurrence of a non-axisymmetric instability, exactly like in the previous dynamo problems, but they fail to obtain a significant feedback of the instability on the axisymmetric poloidal field, whose fate is therefore resistive decay. The critical point in the scenario is therefore how the feedback is produced. Spruit (2002) and Braithwaite (2006) argued that the feedback is done directly by an energy exchange between the non-axisymmetric instability mode and either the axisymmetric poloidal or toroidal field. This argument has been criticized by Zahn et al. (2007), who pointed out that the only way to obtain a feedback from an $m = 1$ component into an $m = 0$ component was through nonlinear interactions. So, the current problem to understand whether this dynamo can operate is to determine whether

such a regenerating nonlinear feedback is possible in the system. Our experience with the MRI problem and the hydrodynamic problem is that this is extremely sensitive to the spatial correlation between the nonlinear feedback term and the original axisymmetric poloidal field. This correlation is quite sensitive to the aspect ratio of the instability mode, which depends on the poloidal localization of the mode and its non-axisymmetric wavenumber. In the simulations by Zahn et al. (2007), the instability is of course localized where the toroidal field is strong, i.e. in a very narrow zone of large differential rotation, which covers a small poloidal area compared to the total poloidal area covered by the poloidal magnetic field introduced originally. A more favourable situation for the Spruit dynamo to be found (but not necessarily a more realistic one from the astrophysical point of view !) probably requires a large poloidal overlap between the differential rotation zone and the axisymmetric poloidal field. This way, the resulting non-axisymmetric instability of the toroidal field generated through the Ω effect would probably cover a larger poloidal area and feedback more coherently on the axisymmetric poloidal magnetic field.

We finally briefly mention that another astrophysical situation in which a subcritical shear dynamo could be operating is that of the solar tachocline. In the model of Miesch (2007a), the imposition of a latitudinal shear and the occurrence of non-axisymmetric global magnetoshear instabilities (Miesch, Gilman & Dikpati 2007b) like the clamshell instability provide all the necessary ingredients for such a dynamo.

4.2 $\alpha\Omega$ and kinematic dynamos

There are clearly major differences between the subcritical dynamo scenario and a mean-field $\alpha\Omega$ dynamo scenario. The first one is the absence of kinematic regime in the subcritical case (hence its name). The second one, which is more qualitative, is that subcritical dynamos discovered so far take on the form of three-dimensional structures that are extremely coherent in both time and space, as shown spectacularly by Cline et al. (2003), while mean-field theory relies on a statistical description of the dynamo process. Specific differences between the $\alpha\Omega$ dynamo and the MRI dynamo have also been pointed out by Hawley & Balbus (1992) and Brandenburg et al. (1995).

So, is there a way to reconcile both views somehow ? One can of course attempt to envision the nonlinear feedback of non-axisymmetric instability modes as some form of mean-field back reaction on the poloidal magnetic field (see for instance Zahn et al. (2007) and Gellert et al. (2008) for the case of the Taylor instability). Cline et al. (2003) however showed that the feedback process in their dynamo loop could not be cast in the simple mathematical form of an α term, so it is not clear currently whether one can actually construct a standard mean-field model from a subcritical dynamo in general. A possible route towards a unified description is given by Moffatt (1970), who attempted to compute

an α from an ensemble of inertial waves. In his model, each individual wave gives rise nonlinearly to a small average EMF. A random ensemble of such waves can generate a net α effect provided that the velocity field associated with the waves lacks reflexional symmetry. Similarly, one can attempt to calculate an α effect or at least a toroidal EMF from a collection of non-axisymmetric instabilities, or to solve simultaneously for an axisymmetric mean-field model and a single-mode non-axisymmetric instability, taking the EMF produced by the non-axisymmetric instability as an input for the mean-field model. Such a so-called $2\frac{1}{2}$ D model was introduced in the context of the geodynamo by Fearn & Proctor (1984, 1987). In their case, the non-axisymmetric instability is a convective instability in a differentially rotating sphere, so once again all the ingredients for a subcritical shear dynamo are present in this problem. They failed to obtain dynamically consistent steady dynamo solutions using an iterative solver but Jones, Longbottom & Hollerbach (1995), following the same idea, found time-dependent solutions to the same problem using direct time-stepping.

We finally note that depending on the symmetries of the problem, it is possible that some modes exert some destructive feedback instead of a regenerating one, so that the subcritical dynamo effect could disappear on average in some cases. A way to avoid this kind of cancellation on average is to impose some form of global symmetry breaking in the system, like a global rotation of the system. We note however that a coherent dynamo process consisting of individual events (like the buoyant rise of individual magnetic flux tubes) does not require this kind of ingredient. For instance, the recurrent but well-separated bursting events during which turbulence gets generated all of a sudden in otherwise quiescent shear flows (some form of time-dependent “hydrodynamic dynamo”) have often been associated with the hydrodynamic SSP (Waleffe 1997; Jimenez & Pinelli 1999; Jimenez & Simens 2001) which does not rely on any form of imposed symmetry-breaking.

4.3 Subcritical dynamo in the Taylor-Green flow

Before we close this Section, we briefly discuss another type of subcritical dynamo discovered recently by Ponty et al. (2007) using a three-dimensional Taylor-Green forcing for the velocity field. The fundamental difference between their problem and the problem discussed here is that we do impose a global shear while they only have local velocity gradients in their flow. An important consequence of imposing a global shear in the system is that the subcritical dynamo branches bifurcate “from infinity” (Nagata 1990) while in their problem, there is a real linear dynamo bifurcation with a well-defined critical value for Rm . Their dynamo is subcritical in the sense that finite-amplitude MHD solutions exist for Rm below its finite critical value. They associate this hysteresis with a nonlinear Lorentz force effect, which in this respect is quite similar to what occurs in the MRI dynamo problem. It is possible that the presence of strong local shear in their model is responsible for subcriticality,

however the two types of subcritical dynamos look quite distinct at the moment. To make another (far less rigorous) analogy with hydrodynamic transition problems, their bifurcation diagram resembles that of the subcritical bifurcation of Tollmien-Schlichting waves in plane Poiseuille flow (Zahn et al. 1974 ; Herbert 1976 ; Orszag & Patera 1980), which is a completely distinct phenomenon from the self-sustaining process in the same flow (Waleffe 2001).

5 Subcritical shear dynamos and small-scale dynamo action

A final point that is worth discussing is to what extent the self-sustaining coherent dynamo structures described in this paper are important to understand dynamo action in a highly turbulent medium. It has been shown that the hydrodynamic self-sustaining process is a cornerstone of transition to turbulence in linearly stable shear flows and that this process leaves an imprint on the statistical quantities (e.g. transport) associated with the turbulent flow after the transition. As discussed by Lesur et al. (2005), the hydrodynamic self-sustaining process in a shearing box is a fundamentally large-scale process that continuously extracts energy from the shear (see their Fig. 9 and the corresponding text). In other words, the SSP acts in the same way as a standard linear instability from the turbulence point of view, by forcing the system at large scales. That nonlinear instabilities in shear flows extract energy from the shear in the same way as linear instabilities extract energy from a general free energy source is further supported by inspection of the energy budgets and of the behaviour of structure functions (including those related to the forcing) in a turbulent flow driven by a linear instability like turbulent convection (Rincon 2006) and in a turbulent shear flow with no walls where the transition process is fundamentally subcritical (Casciola et al. 2003). The energy cascade clearly proceeds in a very similar way for both types of forcing.

We conjecture that self-sustaining MHD processes generated by subcritical shear dynamos are also confined to large scales in the limit of large Re , and that their main role is to drive MHD turbulence continuously by extracting energy from the shear. If this was to be true, then an important consequence would be that small-scale dynamo action should take place exactly in the same way in MRI turbulence with zero net-flux and in turbulence driven by other means (artificial forcing, thermal convection, MRI with net-flux), provided that there is a sufficient scale separation between the forcing scales of the turbulence and the small-scale dynamo scales. There are now some clear numerical indications - including MRI dynamo simulations - that some universality with respect to the forcing process exists for the small-scale dynamo at $Pm > 1$. In this regime, the fore-mentioned scale separation is easy to obtain, because the small-scale dynamo relies on the viscous scale eddies (see Zel'dovich et al. (1984) for theory and Schekochihin et al. (2004) for an exhaustive numerical study). The numerical

results obtained by Schekochihin et al. (2005) for idealized large-scale random forcing, by Christensen, Olson & Glatzmaier (1999) and Cattaneo (2003) for convection and by Fromang et al. (2007b) for a MRI dynamo set-up all show a similar behaviour for the dynamo threshold in the $Re - Rm$ plane. The results presented by Fausto Cattaneo at the recent Catania workshop on MHD also show that snapshots taken from turbulent convection simulations and MRI dynamo simulations in a numerical Taylor-Couette experiment at large Re and Rm are almost indistinguishable. The magnetic field maps at $Pm = 1$ of Schekochihin et al. (2004) and the ones by Fromang et al. (2007b) at $Pm = 2$ (in the isotropic plane of their simulation labeled (x, z) in their notation, corresponding to $(-y, z)$ here) also look very similar. Overall, these new results tend to support our conjecture that there is a large-scale subcritical process involving the MRI that drives turbulence, and that this turbulence in turn operates as an independent small-scale dynamo at moderate to large Pm . We note in passing that the situation at low Pm is more tricky since it is currently unknown whether the small-scale dynamo in that regime has something to do with the forcing scales of the turbulence or if it is universal with respect to the forcing mechanism (Schekochihin et al. 2007).

An important final remark regarding the MRI dynamo problem is that the estimate for MRI growth rates $\gamma \sim \mathbf{V}_A \cdot \mathbf{k}$ predicts that even extremely small scales should be unstable to the MRI in the presence of very weak fields, casting some doubt on the argument that the MRI dynamo could be forced mostly at large scales in the limit of large Re and Rm . We note that the local MRI analysis, as any local analysis, is only valid when the scale of the background field is far larger than that of the instability. In this respect, the previous growth rate estimate does not strictly apply at scales $1/k$ comparable to those of the strongly tangled fields observed in MRI turbulence at moderate to large Pm , thus there might well be some cut-off scale in the MRI dynamo problem below which forcing by the MRI becomes dynamically negligible. Fig. 4 of Fromang et al. (2007b) shows that the forcing of poloidal magnetic fields in their simulations is fairly large-scale and falls off before the viscous scales. The numerics are unfortunately not yet asymptotic and there is no published work on the MRI dynamo so far in which an appreciable scale separation between forcing and dissipation exists. It is therefore likely that testing our conjecture numerically and discriminating between different scenarios will take a few more years.

6 Conclusions

In this paper, we discussed the concept of subcritical dynamo action in shear flows and applied it to the problems of subcritical hydrodynamic transition and MRI dynamo action in accretion disks. We further showed that the subcritical dynamo scenario is relevant to many hydrodynamic and magnetohydrodynamic problems that involve two basic in-

redients, namely shear and non-axisymmetric instabilities, and pointed out that the coherence of the process contrasts with the statistical description on which standard mean-field theory is based. We finally conjectured that coherent structures generated by subcritical dynamo action could be a backbone of MHD turbulence in shear flows in the sense that their main role would be to extract energy from the shear to drive turbulence at large scales, thereby leaving some room in wavenumber space for an independent small-scale dynamo to proceed.

The whole picture is obviously not complete yet. There might be a way to unify the statistical mean-field kinematic picture and the coherent subcritical picture. There is a need to understand further which role SSPs play in astrophysical objects such as accretion disks and stars. To this end, there is a lot of work to do to relate the initial value problem description of these processes to the phase space structure of the associated dynamical systems. We have shown that a description of subcritical MHD dynamos in terms of fixed points is helpful to understand simple configurations. However, numerical evidence (Cline et al. 2003) suggests that a fully chaotic behaviour can be obtained easily for subcritical MHD dynamos in more complex configurations. Therefore, it might be necessary to describe these dynamo processes in terms of more complicated phase space structures than fixed points and to attempt to identify transition regions in parameter space similar to the hydrodynamic “edge of chaos” (Schneider et al. 2006, 2007). From what we have learned so far, it is worth emphasizing that creating new connexions between the shear flow and transition community and the dynamo community would undoubtedly prove extremely helpful to make some important progress on these matters.

Acknowledgements. We thank Sébastien Fromang, Geoffroy Lesur, John Papaloizou, Tobias Heinemann, Tarek Yousef, Alexander Schekochihin, Tomo Tatsuono, Arnaud Antkowiak, Jean-François Pinton, Dan Lathrop, Hantao Ji, Fausto Cattaneo, Steve Tobias and Sacha Brun for many interesting discussions on this topic. This work has been partially supported by the Leverhulme Trust and the Isaac Newton Trust.

References

- Antkowiak, A., Brancher, P.: 2007, *JFM*, 578, 295
 Baggett, J. S., Driscoll, T. A., Trefethen, L. N.: 1995, *PhFl*, 7, 833
 Baggett, J. S., Trefethen, L. N.: 1997, *PhFl*, 9, 1043
 Balbus, S. A., Hawley, J. F.: 1991, *ApJ*, 376, 214
 Braithwaite, J.: 2006, *A&A*, 449, 451
 Brandenburg, A., Nordlund, A., Stein, R. F., Torkelsson, U.: 1995, *ApJ*, 446, 741
 Casciola, C. M., Gualtieri, P., Benzi, R., Piva, R.: 2003, *JFM*, 476, 105
 Cattaneo, F.: 2003, in: N. E. Piskunov, W. W. Weiss, D. F. Gray (eds.) *Modeling of Stellar Atmospheres*, IAU Symp. 210
 Chandrasekhar, S.: 1960, *PNAS*, 46, 253
 Christensen, U., Olson, P., Glatzmaier, G. A.: 1999, *GeoJI*, 138, 393
 Cline, K. S., Brummell, N. H., Cattaneo, F.: 2003, *ApJ*, 599, 1449
 Donati, J.-F., Paletou, F., Bouvier, J., Ferreira, J.: 2005, *Nature*, 438, 466
 Ellingsen, T., Palm, E.: 1975, *PhFl*, 18, 487
 Fearn, D. R., Proctor, M. R. E.: 1983, *JFM*, 128, 1
 Fearn, D. R., Proctor, M. R. E.: 1984, *PEPI*, 36, 78
 Fearn, D. R., Proctor, M. R. E.: 1987, *GApFD*, 38, 293
 Fromang, S., Papaloizou, J.: 2007a, *A&A*, 476, 1113
 Fromang, S., Papaloizou, J., Lesur, G., Heinemann, T.: 2007b, *A&A*, 476, 1123
 Fromang, S., Terquem, C., Balbus, S. A.: 2002, *MNRAS*, 329, 18
 Gellert, M., Rüdiger, G., Elstner, D.: 2008, *A&A*, 479, L33
 Gibson, J. F., Halcrow, J., Cvitanović, P.: 2008, accepted in *JFM* [arXiv:0705.3957]
 Hamilton, J. M., Kim, J., Waleffe, F.: 1995, *JFM*, 287, 317
 Hawley, J. F., Balbus, S. A.: 1992, *ApJ*, 400, 595
 Hawley, J. F., Balbus, S. A., Winters, W. F.: 1999, *ApJ*, 518, 394
 Hawley, J. F., Gammie, C. F., Balbus, S. A.: 1996, *ApJ*, 464, 690
 Herbert, T.: 1976, in: A. I. van de Vooren, P. J. Zandbergen (eds.), *Proc. Int. Conf. Numer. Methods Fluid Dyn.*, 235
 Ji, H., Burin, M. J., Scharfman, E., Goodman, J.: 2006, *Nature*, 444, 343
 Jimenez, J., Pinelli, A.: 1999, *JFM*, 389, 335
 Jimenez, J., Simens, M. P.: 2001, *JFM*, 2001, 81
 Jones, C. A., Longbottom, A. W., Hollerbach, R.: 1995, *PEPI*, 92, 119
 Kreiss, G., Lundbladh, A., Henningson, D. S.: 1993, *JFM*, 270, 175
 Landahl, M. T.: 1980, *JFM*, 98, 243
 Lesur, G., Longaretti, P.-Y.: 2005, *A&A*, 444, 25
 Lesur, G., Longaretti, P.-Y.: 2007, *MNRAS*, 378, 1471
 Lesur, G., Ogilvie, G.-I.: 2008, in preparation
 Livermore, P., Jackson, A.: 2004, *PRSLA*, 460, 1453
 Longaretti, P.-Y.: 2002, *ApJ*, 576, 587
 Miesch, M. S.: 2007a, *ApJ*, 658, L131
 Miesch, M. S., Gilman, P. A., Dikpati, M.: 2007b, *ApJS*, 168, 337
 Moiseev, S. S., Rutkevich, P. B., Tur, A. V., Yanovskii, V. V.: 1988, *SJETP*, 67, 294
 Nagata, M.: 1990, *JFM*, 217, 519
 Orszag, S. A., Patera, A. T.: 1980, *PhRvL*, 45, 989
 Parker, E. N.: 1955, *ApJ*, 122, 293
 Peixinho, J., Mullin, T.: 2007, *JFM*, 582, 169
 Pitts, E., Tayler, R. J.: 1985, *MNRAS*, 216, 139
 Ponty, Y., Laval, J. P., Dubrulle, B., Daviaud, F., Pinton, J. F.: 2007, *PhRvL*, 99, 224501
 Reynolds, O.: 1883, *PhTrRS*, 174, 935
 Richard, D., Zahn, J. P.: 1999, *A&A*, 347, 734
 Rincon, F.: 2006, *JFM*, 563, 43
 Rincon, F., Ogilvie, G. I., Cossu, C.: 2007a, *A&A*, 463, 817
 Rincon, F., Ogilvie, G. I., Proctor, M. R. E.: 2007b, *PhRvL*, 98, 254502
 Schekochihin, A. A., Cowley, S. C., Taylor, S. F., Maron, J. L., McWilliams, J. C.: 2004, *ApJ*, 612, 276
 Schekochihin, A. A., Haugen, N. E. L., Brandenburg, A., Cowley, S. C., Maron, J. L., McWilliams, J. C.: 2005, *ApJ*, 625, L115
 Schekochihin, A. A., Iskakov, A. B., Cowley, S. C., McWilliams, J. C., Proctor, M. R. E., Yousef, T. A.: 2007, *NJPh*, 9, 300
 Schmid, P. J., Henningson, D. S.: 2000, *Stability and transition in shear flows* (Springer-Verlag, Berlin)
 Schneider, T., Eckhardt, B., Yorke, J. A.: 2006, *Chaos*, 16, 041103
 Schneider, T., Eckhardt, B., Yorke, J. A.: 2007, *PhRvL*, 99, 034502
 Shen, Y., Stone, J. M., Gardiner, T. A.: 2006, *ApJ*, 653, 513
 Spruit, H. C.: 2002, *A&A*, 381, 923
 Stone, J. M., Hawley, J. F., Gammie, C. F., Balbus, S. A.: 1996, *ApJ*, 463, 656

- Taylor, R. J.: 1973, MNRAS, 161, 365
- Tillmark, N., Alfredsson, P. H.: 1996, in: S. Gavrilakis, L. Machiels, P. A. Monkewitz (eds.), *Advances in Turbulence VI*, Kluwer, Dordrecht
- Trefethen, L. N., Trefethen, A. E., Reddy, S. C., Driscoll, T. A.: 1993, Sci, 261, 578
- Velikhov, E. P.: 1959, SJETP, 36, 1398
- Viswanath, D.: 2007, submitted
[arXiv:physics/0701337]
- Waleffe, F.: 1995a, PhFl, 7, 3060
- Waleffe, F.: 1995b, SAM, 95, 319
- Waleffe, F.: 1997, PhFl, 9, 883
- Waleffe, F.: 1998, PhRvL, 81, 4140
- Waleffe, F.: 2001, JFM, 435, 93
- Wang, J., Gibson, J., Waleffe, F.: 2007, PhRvL, 98, 204501
- Wedin, H., Kerswell, R. R.: 2004, JFM, 508, 333
- Zahn, J.-P., Brun, A. S., Mathis, S.: 2007, A&A, 474, 145
- Zel'dovich, Y. B., Ruzmaikin, A. A., Molchanov, S. A., Sokolov, D. D.: 1984, JFM, 144, 1