On the shape of granular fronts down rough inclined planes

O. Pouliquen[a)]
LadHyX, Ecole Polytechnique, 91128 Palaiseau cedex, France

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The shape of fronts obtained when releasing a cohesionless granular material at the top of a rough inclined plane is experimentally investigated. The measurements are compared with theoretical predictions given by a model based on a depth averaged momentum equation. Good agreement is found when the empirical friction law proposed in a previous study [O. Pouliquen, Phys. Fluids **11**, 542 (1999)] is introduced in the model. © *1999 American Institute of Physics.* [S1070-6631(99)01807-3]

The flows of granular material down an inclined plane, being encountered in both geophysical and industrial applications, have been extensively studied.[1–6] Many investigations have been devoted to the uniform regime when the thickness of the flowing layer is constant. This configuration is simple enough to test some constitutive equations for granular flows. However, despite this simplicity no clear description emerges from the different theoretical[7,8] or experimental works.

Recently, new scalings have been observed[6,9] which relate the mean velocity of the flow, the thickness of the flowing layer and the inclination angle of the rough plane. In a previous study[9] we have shown that, based on these scaling properties, an empirical friction law can be proposed for the variation of the friction coefficient μ as a function of the mean velocity u and the thickness h of the granular layer. This law is written

$$\mu(u,h)=tg\,\theta_1+(tg\,\theta_2-tg\,\theta_1)\exp\left(-\frac{\beta h}{Ld}\frac{\sqrt{gh}}{u}\right), \qquad (1)$$

where θ_1, θ_2, L are characteristics of the material which can be measured from the deposit properties, d is the mean particle diameter, and β is a constant equal to 0.136 in the case of spherical particles. This relation has been derived from the measurement of the flow properties in the steady uniform regime for which the gravity is exactly balanced by the friction force. Our goal in this work was to check whether this empirical law can be relevant for predicting the evolution of granular flows in more complex configurations, i.e., when the thickness of the flow is no longer uniform.

The configuration investigated is the propagation of a front observed when a dry granular material is released at the top of the inclined plane through a controlled aperture. The experimental setup is described in our previous study.[9] A 2 m long and 70 cm wide surface is roughened by gluing one layer of particles. The bulk material is stored in a reservoir at the top of the plane and is suddenly released through an aperture whose opening can be precisely controled. A front of granular material then rushes down the slope. In the range of inclination and thickness studied[9] the front rapidly stabilizes and propagates with a steady shape [Fig. 1(a)] at a constant velocity [Fig. 1(b)]. The two control parameters are the inclination of the plane θ and the thickness h_∞ of the layer.

The method to precisely measure the front is the same as described in Ref. 9 and consists in lighting the system by a laser sheet arriving on the plane at a low incident angle [Fig. 2(a)]. For the granular material we have used the same four sets of beads as in our previous study[9] which correspond to different roughness conditions. We have performed experiments for the four systems of beads, for different inclinations and different openings of the gate. The aim of this study is to compare the measurements (Fig. 3) with theoretical predictions given by a simple model which takes into account the empirical friction law given by Eq. (1).

The theoretical model is based on depth averaged equations, the basic assumption being that the thickness of the flowing layer is small compared to the other length scales. This approach has been extensively developed in fluid mechanics[10,11] for film flow on inclined plane. It has been applied for granular flows by Takahashi,[12] Patton *et al.*[1] and more recently by Savage, Hutter and co-workers[13,14] who have shown that the motion of granular avalanches can be correctly predicted in some range of the parameters within this framework. Their model has been since applied to geophysical situation.[15,16]

Writing the depth averaged conservation equations in our case is straightforward: for a granular front propagating at a constant velocity u_∞ with a steady shape the mass conservation is trivially verified and the averaged momentum conservation reduces to the force balance between the gravity driven force, the friction force at the bed level, and a force related to the pressure difference between the right and left side of an elementary slice [Fig. 2(b)]:

$$\rho g h\sin(\theta)-\mu(h,u_\infty)\rho g h\cos(\theta)-\frac{\partial\int_0^h\sigma_{xx}dy}{\partial x}=0, \qquad (2)$$

where ρ is the material density assumed to be constant, θ the inclination of the plane, g the gravitational acceleration, σ_{xx} the normal stress in the x direction, and $\mu(h,u)$ is the friction coefficient describing the interaction between the flowing layer and the rough bed. An exact derivation of the depth

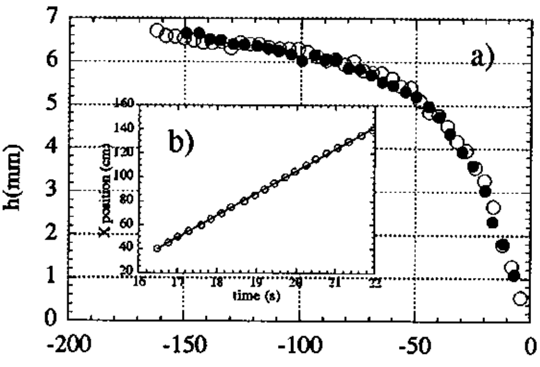

FIG. 1. (a) Front profiles measured 50 cm from the outlet (circles) and once it has propagated 1.5 m down the slope (filled circles). (b) Position of the front with time showing the constant propagation velocity. System of beads 1, $\theta=24.5°$, $h_\infty=6.7$ mm.

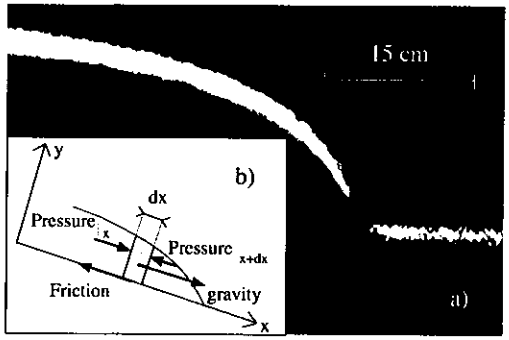

FIG. 2. (a) Picture of the front illuminated by the laser sheet for material 4, $\theta=21°$, $h_\infty=9.5$ mm. (b) Forces on an elementary material slice.

averaged equations for flow down inclined planes starting from the 3D conservation equations can be found in Ref. 13.

The last term in Eq. (2) is usually simplified as follows. Using the shallowness assumption, the vertical normal pressure is given by $\sigma_{yy}=\rho g y$. Assuming that the two normal stresses are proportional, i.e., $\sigma_{xx}=k\sigma_{yy}$, the integral in Eq. (2) can be computed and gives, after simplification, the following equation for the thickness:

$$\tan(\theta)-\mu(h,u_\infty)-k\frac{dh}{dx}=0. \qquad (3)$$

The first term in this equation represents the driven gravity force. The second term is the friction force exerted at the rough bed. This term is crucial as it contains the information about the rheological behavior of granular flows. Savage and Hutter[13] in their model have simply chosen for $\mu(h,u)$ a constant friction coefficient. This assumption seems to be valid for high inclination which indeed corresponds to their experimental conditions. However, for moderate inclination as in our problem, the constant friction coefficient approximation is no longer valid as it does not predict the existence of steady uniform flow. In order to describe the shape of the front in this regime we thus have introduced for $\mu(h,u)$ in Eq. (3) the empirical rheology given by Eq. (1).

The third term in Eq. (3) represents the spreading force related to the gradient of thickness. The coefficient of proportionality k between the two normal stresses is unfortunately unknown for the dense granular flow of interest here. In a very dilute regime corresponding to the kinetic regime, the pressure is isotropic as in fluids and k is equal to 1. Takahashi[12] and Patton *et al.*[1] have used this assumption. On the other hand, in the quasi static regime for very slow deformations, k can be related to the internal friction of the material through a Mohr-Coulomb plasticity theory as shown by Savage, Hutter and co-workers.[13,14] However, in the intermediate regime k is unknown. In this paper the two different values of k are tested when comparing the theoretical predictions with the experiments. We will successively use $k=1$, and $k=(1+\sin^2\theta_1)/(1-\sin^2\theta_1)$ corresponding to the

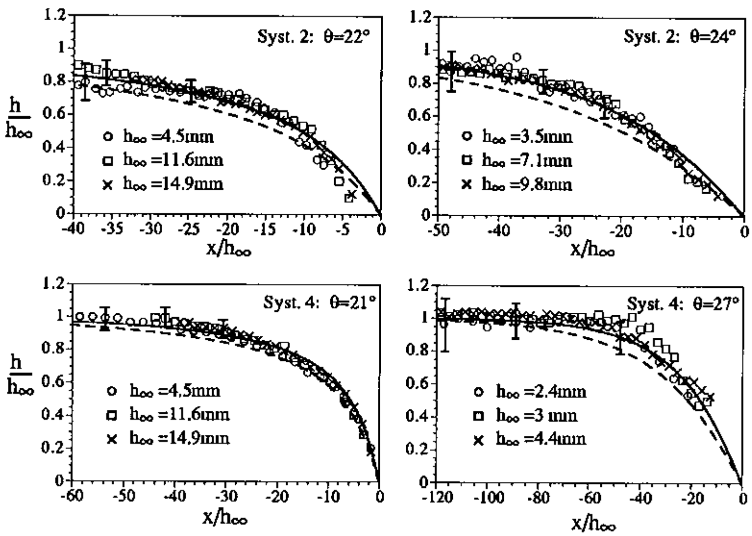

FIG. 3. Front profiles: comparison between experiments for three values of h_∞ (symbols) and theory [solid line: $k=1$; dashed line: $k=(1+\sin^2\theta_1)/(1-\sin^2\theta_1)$] for system of beads 2 and 4 (see Ref. 9 for the characteristics of the beads).

Mohr Coulomb prediction when the basal friction is equal to the internal friction coefficient taken equal to $\tan\theta_1$.[13]

A more elegant way of writing Eq. (3) can be found using the dimensionless variables $h'=h/h_\infty$ and $x'=x/h_\infty$ where h_∞ is the thickness of the layer far from the front where the flow is uniform. In order to do so, one has to substitute u_∞ in Eq. (3) by its expression as a function of θ and h_∞. The relation between u_∞, h_∞ and θ is simply written

$$\tan(\theta)=\mu(h_\infty,u_\infty). \tag{4}$$

From relations 1, 3 and 4, one get the dimensionless differential equation governing the thickness of the material:

$$\tan(\theta)-(\tan\theta_1+(\tan\theta_2-\tan\theta_1)\exp(\gamma h'^{3/2}))=k\frac{dh'}{dx'}, \tag{5}$$

where

$$\gamma=\ln\left(\frac{\tan\theta-\tan\theta_1}{\tan\theta_2-\tan\theta_1}\right).$$

Integrating numerically this differential equation leads to a quantitative prediction for the shape of the front in terms of $h'(x')$. It is important to note that for a given system of beads, i.e., once θ_1 and θ_2 are fixed, the only parameter in Eq. (5) is the inclination θ. This means that experiments carried out at the same inclination, but with different opening of the gate should exhibit the same front shape when expressed in terms of variables nondimensionalized by the thickness h_∞.

In Fig. 3 the results for system of beads 2 and 4 corresponding to two extreme roughness conditions are presented. For each system, the front is plotted for two extreme angles corresponding to the lower and higher inclination for which a steady uniform flow is observed. For each inclination we report the front shape obtained for three thicknesses h_∞. The other system of beads not shown in this paper gives similar results with the same good agreement between the theory and the experiments.

The first remark is that for a given system of beads and at a given inclination θ, the shape of the front in terms of the dimensionless variables is independent of the thickness h_∞ of the flow, as predicted by the depth averaged model. Yet, the shape of the front changes when increasing the inclination θ: the front has a tendency to stretch. This evolution is well predicted by the theory, as shown by the theoretical curves plotted in Fig. 3. These curves have been calculated from Eq. (5) using the values of θ_1 and θ_2 measured in our previous study.[9] Two theoretical curves are plotted in each graph, corresponding to the two different values of the pressure coefficient k. The difference between the two predictions is within the experimental error bars. It is then not possible to conclude which of the two values of k best fits the experiments.

In conclusion, by introducing the empirical friction law given by Eq. (1) in a depth averaged description, we have been able to *quantitatively* predict how the thickness of the avalanching layer goes to zero at the front for the whole range of inclination, thickness of the layer, and roughness we have studied. The agreement between the measurements and the theoretical predictions has been obtained without any fitting process, the parameters of the empirical friction law being determined by the deposit properties.[9]

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[a]Present address: IUSTI, 5 rue Enrico Fermi, 13453 Marseille cedex 13, France; electronic mail: olivier@iusti.univ-mrs.fr

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