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Fully nonlinear global modes in spatially developing media

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Abstract

Global modes on a doubly infinite one-dimensional domain $-\infty < x < +\infty$ are studied in the context of the complex Ginzburg–Landau equation with slowly spatially varying coefficients, i.e., coefficients depending only on a slow streamwise coordinate $X = \epsilon x$, where ϵ is a small parameter. A fully nonlinear frequency selection criterion is derived for global mode solutions under the assumption of weak inhomogeneity of the medium. The global mode is found to be governed by the fully nonlinear equations in a region of finite size, and by the linearized equations in the vicinity of $X = \pm\infty$. Asymptotic matching techniques are used to relate the WKBJ approximations in the linear and nonlinear regions through appropriate transition layers. The *real* global frequency is determined by requiring that spatial branches issuing from $X = -\infty$ and $X = +\infty$ be continuously connected at a saddle point of the local nonlinear dispersion relation $\omega = \Omega^{nl}(k, R^2, X)$ between the frequency ω , the wave number k and amplitude R at a given station X . The results constitute a fully nonlinear generalization of the linear frequency selection criteria previously obtained by Chomaz et al. (1991), Monkewitz et al. (1993), and Le Dizès et al. (1996).

1. Introduction

Our present understanding of hydrodynamic instabilities relies, to a great extent, on the scientific achievements of Fritz Busse and his colleagues. For instance, an intimate knowledge of the Busse balloon [3] has indeed become prerequisite reading to any serious student of Rayleigh–Bénard convection in a fluid layer heated from below, one of the most widely studied closed flow systems. This is just one example of Fritz Busse's fundamental and lasting contributions to hydrodynamics. Fritz has devoted a relatively small portion of his boundless energy to the study of open shear flows, which may explain why our comprehension of instabilities in open flows may have not reached the same fully mature stage. This contribution, which is dedicated to Fritz Busse in celebration of his 60th birthday, illustrates one possible approach which could be successful in spatially developing open flows exhibiting a global resonance.

Spatially developing flows may, under certain conditions, develop self-sustained oscillations. Fluctuations saturate at a finite amplitude in the locally unstable regions of the flow and become tuned at an overall frequency. The intrinsic frequency and the associated spatial distribution of fluctuations define a *global mode*. Relatively little is known regarding the nonlinear evolution stage of such objects. Chomaz [4] and Couairon and Chomaz [7] have

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recently introduced generalized definitions of absolute, convective and global instability applicable to the fully nonlinear case and demonstrated that linear absolute instability is not always synonymous with global instability. Furthermore, these authors have brought to light the close relationship existing between the dynamical systems point of view and the more classical description in terms of a dispersion relation. The goal of the present study is the derivation of a *fully nonlinear* global mode selection criterion in the context of the one-dimensional complex Ginzburg–Landau equation with slowly varying coefficients.

The main stability concepts pertaining to spatially developing open flows have been reviewed by Huerre and Monkewitz [11]. Recent developments have been discussed in Le Dizès et al. [16]. It is here sufficient to recall that specific classes of spatially developing open flows such as bluff-body wakes [9,13,18,24], low density jets [19,23] and counterflow mixing layers [25] may sustain in specific parameter ranges synchronized periodic oscillations over an extended region of the flow. In other words, such systems undergo a Hopf bifurcation as some control parameter is continuously varied. Global mode concepts have precisely been introduced to account for this peculiar behavior of open flows.

In most basic flows of interest, the velocity profile is nonuniform in the streamwise direction, and in order to distinguish between local and global properties it is essential to assume that streamwise variations are slow over a typical instability wavelength. Under this hypothesis, local stability properties are obtained within the parallel flow approximation by freezing the slow streamwise coordinate X . At the local level of description, linear stability properties of normal modes $e^{i(kx-\omega t)}$ are then characterized by a local dispersion relation $D(k, \omega, X) = 0$ between complex wave number k and frequency ω . The goal of a global analysis is to construct in a self-consistent manner a global spatial distribution made up of saturated instability waves governed by the local properties of the flow. The global frequency ω_g is then considered as an eigenvalue to be determined by requiring that perturbations decay at $X = \pm\infty$. The resulting global mode is viewed as a theoretical analog of the self-sustained oscillatory structures that are observed experimentally.

The analysis by Chomaz et al. [6] of the linearized Ginzburg–Landau equation with slowly varying coefficients has demonstrated that a region of absolute instability is a necessary condition for linearly unstable global modes to exist. The dominant complex global frequency ω_g is obtained, at leading order in the WKBJ approximation, for a saddle point of the local linear dispersion relation $\omega = \Omega^\ell(k, X)$, i.e.,

$$\omega_g \sim \omega_s = \Omega^\ell(k_s, X_s), \quad (1)$$

where the complex pair (k_s, X_s) satisfies

$$\frac{\partial \Omega^\ell}{\partial k}(k_s, X_s) = \frac{\partial \Omega^\ell}{\partial X}(k_s, X_s) = 0. \quad (2)$$

The same linear frequency selection criterion has been derived by Monkewitz et al. [20] in the framework of the Navier–Stokes equations linearized around an arbitrary weakly nonparallel basic flow. More recently, the frequency selection criterion pertaining to the Ginzburg–Landau equation over a doubly infinite domain has been examined anew by resorting to a comparison equation method [16]. It has been shown that global modes are either of type 1, with two simple turning points in the complex X -plane, or type 2, with a double turning point as defined in (2). The validity of the selection criterion (2) has further been confirmed by the exact Green function calculations of Hunt and Crighton [12].

The extension of these concepts to the nonlinear régime has until recently appeared somewhat problematic. It seems, at first sight, natural to perform a weakly nonlinear analysis in the vicinity of the global mode onset. Such an approach is straightforward for nonparallel effects of *order unity*, as shown by Chomaz et al. [5] in the case of the Ginzburg–Landau equation with a linearly or quadratically varying control parameter. Paradoxically, the introduction of a weakly nonparallel assumption in addition to the weakly nonlinear one leads to unexpected

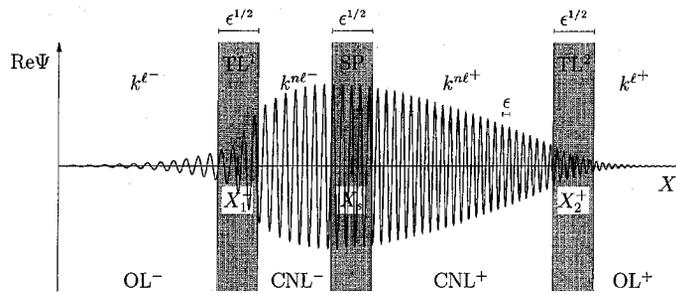


Fig. 1. Fully nonlinear global mode spatial structure: OL^\pm , outer semi-infinite linear regions close to $X = \pm\infty$; $TL^{1,2}$, transition layers of thickness $O(\epsilon^{1/2})$ connecting linear and nonlinear regions; CNL^\pm , central nonlinear regions of size $O(1)$; SP, saddle point region of thickness $O(\epsilon^{1/2})$.

difficulties [16]: in the limit of zero nonparallelism, the Landau constant does not converge to a well-defined limit of *constant* sign. In other words, the Landau equation formalism which purportedly describes a Hopf bifurcation near onset does not apply unless drastic conditions are met. In order to circumvent these difficulties, we choose in the present study to jump to the fully nonlinear régime while keeping the weakly nonparallel assumption. Thus we implement a nonlinear WKBJ approximation scheme which remains valid for amplitudes of order unity, as encountered in the phase dynamics of patterns [8,15,17,21] or in the description of slowly varying fully nonlinear wavetrains [2,10].

The main objective of the paper is the generalization of the linear frequency selection criterion (1), (2) to fully nonlinear global modes provided that the local nonlinear dispersion relation of the weakly inhomogeneous flow is given. We should emphasize that, in contrast to the linear criterion, the present formulation will only involve purely real values of the wave number, frequency and local streamwise variable. In the present study there is no need to extend analytically the dispersion relation in the complex k -plane and X -plane, a feature which considerably simplifies the analysis.

The inhomogeneities of the medium are characterized by a small parameter

$$\epsilon \sim \lambda/L \ll 1, \quad (3)$$

where L is a typical evolution length scale of the basic flow and λ a typical wavelength of the instability. Under the assumption $\epsilon \ll 1$, global mode solutions depend on two different length scales: a fast space variable x related to the rapid spatial oscillations of the carrier wave and a slow variable $X = \epsilon x$ describing the slow spatial evolution of the “basic flow”. Fluctuations $\Psi(x, t)$ around a given basic state are governed by a partial differential equation, the coefficients of which depend on the slow variable X . The local properties of the flow are then recovered by freezing X . This separation of scales calls for an analysis in terms of nonlinear WKBJ approximations: the slowly evolving local characteristics may be exploited to construct global mode solutions composed of slowly varying nonlinear wavetrains.

In the course of the study, we are led to distinguish different regions characterizing the global mode spatial structure, as sketched in Fig.1. In central nonlinear regions of finite size (CNL^\pm) the wave has finite amplitude and satisfies the fully nonlinear governing equation of the problem. These central nonlinear regions are surrounded by two semi-infinite outer linear regions (OL^\pm) where the global mode satisfies the linearized equation and is exponentially decaying towards infinity, as required by the boundary conditions. Transition layers ($TL^{1,2}$) of size $\epsilon^{1/2}$ (measured in units of X) allow a smooth crossover from the central nonlinear regions (CNL^\pm) into the outer linear regions (OL^\pm). As shown later, the existence of a nonlinear global mode requires two wave number branches to meet in the core of the nonlinear regions, at a saddle point of the local nonlinear dispersion relation. An additional

internal layer (SP) of size $\epsilon^{1/2}$ around this saddle point needs to be introduced in order to generate the nonlinear frequency selection criterion at arbitrary order. The selection mechanism is therefore entirely determined by the saddle point structure. Thus this inner transition layer (SP) is surrounded by nonlinear regions of finite extent (CNL $^{\pm}$) which in turn are connected through transition layers (TL 1,2) to the outer linear regions (OL $^{\pm}$) extending towards infinity.

The organization of the present paper reflects the global mode structure. The global mode problem is stated in Section 2 and the linear WKBJ approximations pertaining to the outer regions (OL $^{\pm}$) are recalled in Section 3. Corresponding nonlinear WKBJ solutions relevant to the central nonlinear regions (CNL $^{\pm}$) are derived in Section 4. In Section 5 the leading order nonlinear frequency selection criterion is obtained in the following form: *a global mode can only be constructed if two wave number branches meet at a saddle point in the bulk of the nonlinear regions*. The structure of the transition layers connecting linear and nonlinear regions is analyzed in Section 6. In closing we summarize the main conclusions and address some future developments.

2. Nonlinear eigenvalue problem

Throughout the study, complex scalar fluctuations $\Psi(x, t)$ around a given basic state are assumed to be governed by a nonlinear equation of the form

$$i \frac{\partial \Psi}{\partial t} = \Omega^{nl} \left(-i \frac{\partial}{\partial x}, |\Psi|^2, X \right) \Psi, \tag{4}$$

where $X = \epsilon x$ is a slow space variable which describes the inhomogeneities of the medium and ϵ is the WKBJ parameter introduced in (3). The operator Ω^{nl} is the physical space counterpart of the local nonlinear dispersion relation $\omega = \Omega^{nl}(k, R^2, X)$ to be defined later.

A nonlinear global mode is then defined as a time-periodic solution of Eq. (4) of *real* frequency ω , satisfying vanishing boundary conditions at infinity. Thus it can be written as

$$\Psi(x, t) = \psi(x, \omega) e^{-i\omega t},$$

where the global frequency ω and the amplitude ψ are governed by the differential equation

$$\left[\omega - \Omega^{nl} \left(-i \frac{\partial}{\partial x}, |\psi|^2, X \right) \right] \psi = 0 \tag{5}$$

with the boundary conditions

$$\lim_{x \rightarrow \pm\infty} \psi(x, \omega) = 0. \tag{6}$$

In the particular case of the complex Ginzburg–Landau equation [11],

$$i \frac{\partial \Psi}{\partial t} = (\omega_0(X) + \frac{1}{2} \omega_{kk}(X) k_0^2(X)) \Psi + i \omega_{kk}(X) k_0(X) \frac{\partial \Psi}{\partial x} - \frac{1}{2} \omega_{kk}(X) \frac{\partial^2 \Psi}{\partial x^2} + \gamma(X) |\Psi|^2 \Psi, \tag{7}$$

the nonlinear dispersion operator Ω^{nl} may be separated into linear and nonlinear parts as

$$\Omega^{nl} \left(-i \frac{\partial}{\partial x}, |\Psi|^2, X \right) \equiv \Omega^{\ell} \left(-i \frac{\partial}{\partial x}, X \right) + \gamma(X) |\Psi|^2,$$

where the linear dispersion operator Ω^{ℓ} reads

$$\Omega^{\ell} \left(-i \frac{\partial}{\partial x}, X \right) \equiv \omega_0(X) + \frac{1}{2} \omega_{kk}(X) k_0^2(X) + i \omega_{kk}(X) k_0(X) \frac{\partial}{\partial x} - \frac{1}{2} \omega_{kk}(X) \frac{\partial^2}{\partial x^2}. \tag{8}$$

The complex functions $\omega_0(X)$, $\omega_{kk}(X)$, $k_0(X)$, and $\gamma(X)$ which solely depend on the slow space variable X account for the weak inhomogeneities of the medium and their meaning is discussed in Section 3.

The goal of the present study is to solve the nonlinear eigenvalue problem (5),(6) where the global mode amplitude ψ is $O(1)$ in an interval of finite extent. We construct such global mode solutions in the régime $\epsilon \ll 1$ and determine how the global frequencies ω_g depend on the characteristics of the “basic flow”, as specified by the nonlinear dispersion relation.

Since global modes must decay at infinity, they are governed by the linearized equation for sufficiently large X . The solutions of the associated linear equation are therefore required in outer linear regions (OL^\pm), as reviewed in Section 3.

3. Outer linear WKB regions OL^\pm

In the present section we examine the structure of the outer linear regions (OL^\pm) in Fig. 1 and recall the main features of the solutions of the linearized equation

$$i \frac{\partial \Psi}{\partial t} = \Omega^\ell \left(-i \frac{\partial}{\partial x}, X \right) \Psi \quad (9)$$

for an arbitrary frequency ω of order unity, as studied by Chomaz et al. [6] and Le Dizès et al. [16].

Upon substituting $\partial/\partial x \rightarrow ik$ and $\partial/\partial t \rightarrow -i\omega$ in (8) and (9) one obtains the associated local linear dispersion relation of the Ginzburg–Landau model in the form

$$\omega = \Omega^\ell(k, X) \equiv \omega_0(X) + \frac{1}{2} \omega_{kk}(X) (k - k_0(X))^2. \quad (10)$$

In order to enforce causality we assume throughout that sufficiently large wave numbers are damped, i.e., $\omega_{kk,i}(X) \equiv \text{Im}[\omega_{kk}(X)] < 0$ for all X . According to (10) the frequency ω is a simple quadratic function of the complex wave number k . Thus there exist for each value of the frequency two linear spatial branches given by

$$k^{\ell\pm}(X, \omega) \equiv k_0(X) \pm \sqrt{2 \frac{\omega - \omega_0(X)}{\omega_{kk}(X)}}. \quad (11)$$

Under the assumption $\omega_{kk,i}(X) < 0$ for all X , the square root branch in (11) can always be chosen such that $k_i^{\ell+}(X) > 0$ and $k_i^{\ell-}(X) < 0$ for large enough ω . The introduction of the functions $k_0(X)$ and $\omega_0(X)$ may now be justified: at each station X , k_0 and ω_0 denote the complex absolute wave number and frequency defined by $(\partial \Omega^\ell / \partial k)(k_0, X) = 0$, $\omega_0 = \Omega^\ell(k_0, X)$, as extensively discussed in [11]. In other words, $\omega_0(X)$ is a branch point of the function $k^\ell(X, \omega)$ in the complex ω -plane, as clearly displayed in (11).

Under the slowly varying medium hypothesis ($\epsilon \ll 1$) a solution of Eq. (9) with frequency ω can be obtained in terms of WKB approximations [1]. The spatial behavior of the solution is described by a rapidly varying phase, corresponding to the local wave number, and a slowly varying envelope which, for a given value of the frequency ω , read

$$\psi^\ell(X, \omega) = A^\ell(X, \omega) \exp \frac{i}{\epsilon} \int^X k^\ell(s, \omega) ds, \quad (12)$$

where $k^\ell(X, \omega)$ is one of the spatial branches (11).

In classical WKBJ fashion, the envelope¹ $A^\ell(X)$ is expanded in powers of ϵ

$$A^\ell(X) \sim A_0^\ell(X) + \epsilon A_1^\ell(X) + \epsilon^2 A_2^\ell(X) + \dots, \quad (13)$$

and the spatial derivative operator is decomposed into

$$\partial_x = ik^\ell(X) + \epsilon \partial_X. \quad (14)$$

Upon substituting (12)–(14) into the governing equation (9), one obtains

$$\omega(A_0^\ell + \epsilon A_1^\ell + \dots) = \Omega^\ell(k^\ell(X) - i\epsilon \partial_X, X)(A_0^\ell + \epsilon A_1^\ell + \dots). \quad (15)$$

Note that differentiation with respect to the fast variable, i.e., multiplication by $ik^\ell(X)$, does not commute with the slow derivative operator ∂_X [14]. Thus $\Omega^\ell(k^\ell(X) - i\epsilon \partial_X)$ admits the expansion

$$\begin{aligned} \Omega^\ell(k^\ell(X) - i\epsilon \partial_X) &= \Omega^\ell - i\epsilon \frac{1}{2}(\Omega_k^\ell \partial_X + \partial_X \Omega_k^\ell) + O(\epsilon^2) \\ &= \Omega^\ell - i\epsilon(\Omega_k^\ell \partial_X + \frac{1}{2}\Omega_{kk}^\ell k_X^\ell(X)) + O(\epsilon^2). \end{aligned} \quad (16)$$

On the r.h.s. of these equations, Ω^ℓ and its derivatives with respect to k are evaluated at $k^\ell(X)$. In Eq. (15) the slow derivative operator ∂_X only applies to the wave function, hence to $k^\ell(X)$, but *not* to the second argument X of $\Omega^\ell(k, X)$.

At lowest order, Eq. (15) reduces to the dispersion relation (10), which yields the local wave number $k^\ell(X)$ in terms of ω , as given by (11).

At order ϵ , we obtain

$$[\omega - \Omega^\ell(k^\ell(X), X)]A_1^\ell = -i\Omega_k^\ell(X) \frac{dA_0^\ell}{dX} - \frac{i}{2}\Omega_{kk}^\ell(X)k_X^\ell(X)A_0^\ell$$

with the notations

$$\Omega_k^\ell(X) = \frac{\partial \Omega^\ell}{\partial k}(k^\ell(X), X) \quad \text{and} \quad \Omega_{kk}^\ell(X) = \frac{\partial^2 \Omega^\ell}{\partial k^2}(k^\ell(X), X).$$

By invoking the local dispersion relation (10), one obtains the obvious solvability condition for A_0^ℓ :

$$\frac{dA_0^\ell}{dX} = -\frac{1}{2}k_X^\ell(X) \frac{\Omega_{kk}^\ell(X)}{\Omega_k^\ell(X)} A_0^\ell(X).$$

Thus, at first order, the solution of the linearized equation reads

$$\psi^\ell(X) \sim A_0^\ell(X_1) \exp\left(-\frac{1}{2} \int_{X_1}^X k_X^\ell(s) \frac{\Omega_{kk}^\ell(s)}{\Omega_k^\ell(s)} ds\right) \exp\left(\frac{i}{\epsilon} \int_{X_1}^X k^\ell(s) ds\right), \quad (17)$$

where X_1 is some arbitrarily specified reference point.

The linear Ginzburg–Landau equation admits two spatial branches $k^{\ell\pm}(X)$ given by (11); thus Eq. (17) yields two WKBJ approximations $\psi^{\ell\pm}(X)$ for two linearly independent solutions of Eq. (9). Let us assume that $k_1^{\ell+}(X) > 0$ and $k_1^{\ell-}(X) < 0$ for sufficiently large $|X|$ (for a discussion of the relationship between this assumption and causality see [16]). Under this assumption the WKBJ approximation $\psi^{\ell-}(X)$ is subdominant when $X \rightarrow -\infty$, i.e., exponentially

¹ For simplicity we no longer explicitly mention the frequency dependence.

small with respect to $\psi^{\ell^+}(X)$; similarly the solution $\psi^{\ell^+}(X)$ becomes subdominant for $X \rightarrow +\infty$. Boundary conditions (6) require the choice of $\psi^{\ell^\pm}(X)$ for $X \rightarrow \pm\infty$. A global mode solution is thus solely composed of the branch k^{ℓ^-} for $X \rightarrow -\infty$ and of the branch k^{ℓ^+} for $X \rightarrow +\infty$. In the following sections a nonlinear solution is derived which allows a smooth crossover from one linear branch to the other as X varies from $-\infty$ to $+\infty$.

4. Central nonlinear WKB regions CNL $^\pm$

In this section a *fully nonlinear* solution of Eq. (4) is constructed in the central regions (CNL $^\pm$) of Fig. 1 by means of the nonlinear dispersion relation, for an arbitrary frequency of order unity.

Let us first consider in the same notations as Eq. (4) the *homogenous* nonlinear equation with *constant* coefficients

$$i \frac{\partial \Psi}{\partial t} = \Omega^{n\ell} \left(-i \frac{\partial}{\partial x}, |\Psi|^2 \right) \Psi.$$

This equation admits nonlinear travelling wave Stokes solutions of the form

$$\Psi(x, t) = R e^{i(kx - \omega t)}, \quad (18)$$

where the amplitude R , wave number k , and frequency ω are *real* parameters bound by the *nonlinear* dispersion relation

$$\omega = \Omega^{n\ell}(k, R^2). \quad (19)$$

This complex equation completely defines the family of solutions (18). It is also convenient to introduce a *reduced nonlinear dispersion relation* as follows.

Taking the imaginary part of complex equation (19) shows that the amplitude R^2 is slaved to the wave number k and is implicitly given by

$$0 = \Omega_i^{n\ell}(k, R^2).$$

Replacing the amplitude R^2 as a function of k in the real part of (19),

$$\omega = \Omega_r^{n\ell}(k, R^2),$$

yields the reduced nonlinear dispersion relation in the form

$$\omega = \Omega^{NL}(k) \equiv \Omega_r^{n\ell}(k, R^2(k)).$$

This reduced form clearly demonstrates that Stokes travelling waves (18) indeed constitute a one-parameter family of finite amplitude solutions, of real parameter k .

In the context of the complex Ginzburg–Landau model (7), the *complete* nonlinear dispersion relation reads

$$\omega = \Omega^{n\ell}(k, R^2) \equiv \omega_0 + \frac{1}{2} \omega_{kk} (k - k_0)^2 + \gamma R^2 = \Omega^\ell(k) + \gamma R^2, \quad (20)$$

and the reduced dispersion relation may explicitly be determined by following the procedure outlined above. Extracting R^2 from the imaginary part of (20), one finds

$$R^2 = \frac{\text{Im}[\Omega^\ell(k)]}{\text{Im}[\gamma^*]}, \quad (21)$$

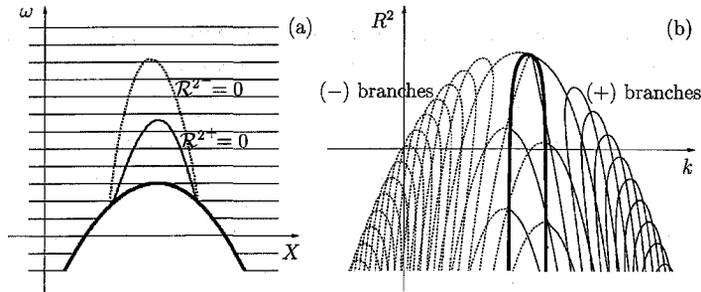


Fig. 2. Nonlinear spatial branches in the $X-\omega$ and $k-R^2$ planes. In (a) the horizontal lines $\omega = \text{const.}$ indicate in the $X-\omega$ plane the allowable domain, delineated by the bold critical curve, where the nonlinear dispersion relation (23) admits real solutions in the $k-R^2$ plane sketched in (b). Each of the lines $\omega = \text{const.}$ is mapped into two spatial branches ($k^{\text{nl}\pm}(X), \mathcal{R}^{2\pm}(X)$) in the $k-R^2$ plane: the (+) branches are represented by solid curves, the (-) branches by dotted curves. For frequencies such that the line $\omega = \text{const.}$ intersects the boundary of the allowable domain (bold curve in (a)), the two branches in the $k-R^2$ plane meet at the corresponding image critical curve (bold curve in (b)). The subregions of the allowable domain corresponding to $R^2 > 0$ are delimited in (a) by the curves $\mathcal{R}^{2+}(X, \omega) = 0$ (solid) and $\mathcal{R}^{2-}(X, \omega) = 0$ (dotted).

where a * superscript denotes the complex conjugate, and by substitution into the real part of (20), one readily obtains the reduced nonlinear dispersion relation

$$\omega = \Omega^{\text{NL}}(k) = \frac{\text{Im}[\gamma^* \Omega^\ell(k)]}{\text{Im}[\gamma^*]} \tag{22}$$

Since (22) is quadratic in k , to each frequency ω are associated, under suitable conditions, two nonlinear spatial branches $k^{\text{nl}\pm}(\omega)$, and, via (21), two amplitudes $\mathcal{R}^{2\pm}(\omega)$. Thus the complete nonlinear dispersion relation (20) is seen to possess two pairs of solutions

$$(k, R^2) = (k^{\text{nl}\pm}(\omega), \mathcal{R}^{2\pm}(\omega)).$$

Such pairs of solutions only exist if (20) can be solved for *real* k and *real non-negative* R^2 . For given coefficients in (20) these conditions correspond to a range of allowable frequencies.

In a *weakly inhomogeneous* medium where the coefficients of the nonlinear equation are now functions of the slow space variable X (Eqs. (4) and (7)), a *local* nonlinear dispersion relation may still be derived of the form

$$\omega = \Omega^{\text{nl}}(k, R^2, X) \equiv \omega_0(X) + \frac{1}{2} \omega_{kk}(X)(k - k_0(X))^2 + \gamma(X)R^2 = \Omega^\ell(k, X) + \gamma(X)R^2, \tag{23}$$

and the corresponding local reduced nonlinear dispersion relation may be formally expressed as

$$\omega = \Omega^{\text{NL}}(k, X) \equiv \frac{\text{Im}[\gamma^*(X) \Omega^\ell(k, X)]}{\text{Im}[\gamma^*(X)]}. \tag{24}$$

Local wave number and amplitude are obtained in the same way as in the homogenous case; they are real functions of the frequency and the slow space variable:

$$(k, R^2) = (k^{\text{nl}\pm}(X, \omega), \mathcal{R}^{2\pm}(X, \omega)), \tag{25}$$

solutions of (23) and (24).

Let us now examine the properties of the double-valued mapping between the $X-\omega$ and $k-R^2$ planes defined by solution pairs (25). The $X-\omega$ plane separates into an allowable region where two *real* solution pairs (25) satisfy the local nonlinear dispersion relation (23) and a region where no such real solutions are obtained (Fig. 2(a)). For given X , real wave number solutions of the quadratic equation (24) are found to exist when $\omega \rightarrow +\infty(-\infty)$ if $\omega_{kk,r} - \omega_{kk,i} \gamma_r / \gamma_i > 0 (< 0)$. Let us assume that $\omega_{kk,r}(X) - \omega_{kk,i}(X) \gamma_r(X) / \gamma_i(X) > 0$ for all X . As ω decreases

for a given X , the two pairs merge at a critical value of ω when the discriminant of the quadratic equation (24) vanishes. The set of such points in the $X-\omega$ plane defines the critical line (bold curve of Fig. 2(a)), i.e., the boundary of the allowable domain indicated by the hatched region of Fig. 2(a). Below this critical line the nonlinear dispersion relations (23) or (24) admit no real solutions. Each point (X, ω) of the allowable domain then maps into two points in the $k-R^2$ plane (Fig. 2(b)) through the mapping functions (25). A solution of real uniform frequency is in fact represented in the $X-\omega$ plane by a straight horizontal line $\omega = \text{const.}$ As X varies for a given ω , solutions of eq.(25) move along two distinct branches $(k^{n\ell^+}(X), \mathcal{R}^{2^+}(X))$ and $(k^{n\ell^-}(X), \mathcal{R}^{2^-}(X))$ in the $k-R^2$ plane, as indicated by the continuous and dashed curves in Fig. 2(b). If the line $\omega = \text{const.}$ is high enough and does not meet the critical line delimiting the allowable domain in the $X-\omega$ plane, the two corresponding branches in the $k-R^2$ plane do not intersect; if by contrast the horizontal line $\omega = \text{const.}$ is not entirely contained in the allowable hatched domain, spatial branches in the $k-R^2$ plane are not defined for the portion of the line situated outside the allowable domain. In fact, upon hitting the critical line in the $X-\omega$ plane, (+) and (-) branches in the $k-R^2$ plane meet along the image of the critical line (bold curve of Fig. 2(b)) and cease to exist.

Note also that each branch of (25) has physical meaning only if $R^2 \geq 0$. Thus the subregions of the allowable domain, where meaningful nonlinear solutions are obtained, are further restricted to $\mathcal{R}^{2^\pm}(X, \omega) \geq 0$. These regions are located within the curves labelled $\mathcal{R}^{2^+} = 0$ and $\mathcal{R}^{2^-} = 0$ in Fig. 2(a). If one assumes $\gamma_1(X) < 0$ uniformly in X (locally supercritical behavior), the condition $\mathcal{R}^{2^\pm}(X, \omega) \geq 0$ corresponds to linearly unstable wave numbers $k^{n\ell^\pm}(X, \omega)$, as seen by inspection of (21). Elsewhere in the $X-\omega$ plane, $\mathcal{R}^{2^\pm}(X, \omega) < 0$, nonlinear solutions pertaining to (+) or (-) branches do not exist and only linear solutions are allowed.

As in the outer linear regions, the spatial distribution is taken to be a rapidly oscillating wave modulated by a slowly varying envelope

$$\psi^{n\ell}(X) = A^{n\ell}(X) \exp \frac{i}{\epsilon} \int^X k^{n\ell}(s) ds. \tag{26}$$

Upon expanding the envelope in powers of ϵ

$$A^{n\ell}(X) \sim A_0^{n\ell}(X) + \epsilon A_1^{n\ell}(X) + \epsilon^2 A_2^{n\ell}(X) + \dots,$$

interpreting the spatial derivatives as

$$\partial_x = ik^{n\ell}(X) + \epsilon \partial_X,$$

and substituting these expansions into Eq. (4), one readily obtains the nonlinear counterpart of (15), i.e.,

$$\omega(A_0^{n\ell} + \epsilon A_1^{n\ell} + \dots) = \Omega^{n\ell}(k^{n\ell}(X) - i\epsilon \partial_X, |A_0^{n\ell} + \epsilon A_1^{n\ell} + \dots|^2, X)(A_0^{n\ell} + \epsilon A_1^{n\ell} + \dots). \tag{27}$$

At lowest order, we recover the local nonlinear dispersion relation (23), i.e.,

$$\omega = \Omega^{n\ell}(k^{n\ell}(X), |A_0^{n\ell}(X)|^2, X), \tag{28}$$

which determines the local wave number $k^{n\ell}(X)$ and the leading order amplitude $|A_0^{n\ell}(X)|^2 = \mathcal{R}^2(X)$ as functions of the slow space variable X for a given frequency ω . Note that the phase $\theta_0(X)$ of $A_0^{n\ell}(X)$ is as yet undetermined.

After expanding the operator $\Omega^{n\ell}$ in powers of ϵ in the same fashion as in (16), the ϵ terms of (27) yield

$$(\omega - \Omega^{n\ell})A_1^{n\ell} = -i\Omega_k^{n\ell} \frac{dA_0^{n\ell}}{dX} - \frac{i}{2} \Omega_{kk}^{n\ell} k_X^{n\ell} A_0^{n\ell} + \Omega_{R^2}^{n\ell} (A_0^{n\ell*} A_1^{n\ell} + A_1^{n\ell*} A_0^{n\ell}) A_0^{n\ell},$$

where $\Omega^{n\ell}$ and its partial derivatives are evaluated at $(k^{n\ell}(X), \mathcal{R}^2(X), X)$. In the following, $\Omega^{n\ell}(X)$ stands for $\Omega^{n\ell}(k^{n\ell}(X), \mathcal{R}^2(X), X)$.

Upon exploiting the identity $\omega = \Omega^{n\ell}(X)$, the fact that $dA_0^{n\ell}/dX = (i d\theta_0/dX + \mathcal{R}_X^2/2\mathcal{R}^2)A_0^{n\ell}$, and the real nature of $A_0^{n\ell*}A_1^{n\ell} + A_1^{n\ell*}A_0^{n\ell}$, we readily obtain a solvability condition for $\theta_0(X)$ of the form

$$\frac{d\theta_0}{dX} = \frac{1 \operatorname{Re}[\Omega_{R^2}^{n\ell}(X)^* \Omega_k^{n\ell}(X)] \mathcal{R}_X^2(X)}{2 \operatorname{Im}[\Omega_{R^2}^{n\ell}(X)^* \Omega_k^{n\ell}(X)] \mathcal{R}^2(X)} - \frac{1 \operatorname{Re}[\Omega_{R^2}^{n\ell}(X)^* \Omega_{kk}^{n\ell}(X)]}{2 \operatorname{Im}[\Omega_{R^2}^{n\ell}(X)^* \Omega_k^{n\ell}(X)]} k_X^{n\ell}(X). \quad (29)$$

Eqs. (26), (28) and (29) entirely specify the nonlinear solution at leading order.

5. Saddle point region SP: Leading-order nonlinear frequency selection criterion

In this section the global frequency ω_g is obtained at lowest order by matching the nonlinear spatial branches $k^{n\ell+}(X)$ and $k^{n\ell-}(X)$.

Solving the linear dispersion relation (10) yields the local complex wave numbers $k^{\ell\pm}(X, \omega)$ as functions of the frequency at any location. By contrast the nonlinear dispersion relation (23) admits solution pairs $(k^{n\ell\pm}(X, \omega), \mathcal{R}^{2\pm}(X, \omega))$ only in the allowable region of the X - ω plane. Moreover only nonlinear solutions with $\mathcal{R}^{2\pm}(X, \omega) > 0$ have physical meaning.

The relationship between linear and nonlinear spatial branches as a function of frequency is illustrated in Fig. 3. In the X - ω plane a solution of uniform frequency is represented by the line $\omega = \text{const}$. This line is mapped, in the complex k -plane, into two linear spatial branches $k^{\ell\pm}(X)$ defined in (11).

Let us recall that, for large enough frequencies $\omega = \omega_a$ (see X - ω plane of Fig. 3), the linear branches do not cross the k_r -axis in the complex k -plane as long as $\omega_{kk,i}(X) < 0$ for all X (Fig. 3(a1)), and that $k^{\ell\pm}(X)$ denotes the linear branch located in the $k_i > 0$ and $k_i < 0$ half-planes, respectively. Corresponding variations of $k_r^{\ell\pm}$ with X are displayed on Fig. 3(a2).

When the frequency is decreased to $\omega = \omega_b$ (see X - ω plane of Fig. 3), the linear branches move about in the complex k -plane. The branch $k^{\ell\pm}(X)$ crosses the k_r -axis (Fig. 3(b1)) whenever the frequency is such that the line $\omega = \omega_b$ intersects, in the X - ω plane, the region $\mathcal{R}^{2\pm}(X, \omega) > 0$. Indeed a local solution of the nonlinear dispersion relation with $R^2 = 0$ is precisely a solution of the linear dispersion relation, and this linear solution happens to have a *real* wave number.

Thus at locations where $R^2 = 0$, linear and nonlinear branches meet. Let us denote by $k^{n\ell\pm}(X, \omega)$ the nonlinear branches that match to the corresponding linear branches $k^{\ell\pm}(X, \omega)$. In the specific case of Fig. 3(b), the line $\omega = \omega_b$ is partly contained within the region $\mathcal{R}^{2-} > 0$ of the X - ω plane and, as a result, the nonlinear branch $k^{n\ell-}$ is seen to take over from $k_r^{\ell-}$ in a finite X -subinterval.

For frequencies such that the line $\omega = \omega_c$ in the X - ω plane intersects both regions $\mathcal{R}^{2\pm}(X, \omega) > 0$, both linear branches cross the k_r -axis (Fig. 3(c1)) and are connected to the corresponding nonlinear branches (Fig. 3(c2)). Let $X_1^\pm(\omega)$ and $X_2^\pm(\omega)$ be the locations where $\mathcal{R}^{2\pm}(X, \omega) = 0$, whereby $k^{\ell\pm}(X_{1,2}^\pm) = k^{n\ell\pm}(X_{1,2}^\pm)$. A nonlinear solution with local wave number $k^{n\ell\pm}(X, \omega)$ then exists in the range $X_1^\pm < X < X_2^\pm$. The locations $X_{1,2}^\pm$ depend on the frequency, and in Section 6 we show that, for any frequency, the linear and nonlinear solutions can be matched at all orders in ϵ through transition layers centered around $X_{1,2}^\pm(\omega)$. Recall that boundary conditions (6) require keeping only the linear branch $k^{\ell\pm}(X)$ in the regions $X \rightarrow \pm\infty$, respectively. At the locations X_1^- and X_2^+ where the linear branches $k^{\ell-}$ and $k^{\ell+}$ cross the k_r -axis they change over to the corresponding nonlinear branches $k^{n\ell-}$ and $k^{n\ell+}$. Thus the nonlinear region is delimited by X_1^- (where $k^{\ell-} = k^{n\ell-}$) and X_2^+ (where $k^{n\ell+} = k^{\ell+}$), and a continuous global mode solution must necessarily connect both these nonlinear branches.

A smooth crossover between the nonlinear branches $k^{n\ell-}$ and $k^{n\ell+}$ (Fig. 3(s2)) can only be achieved for frequencies such that the line $\omega = \text{const}$. meets the critical curve limiting the allowable domain in the X - ω plane.

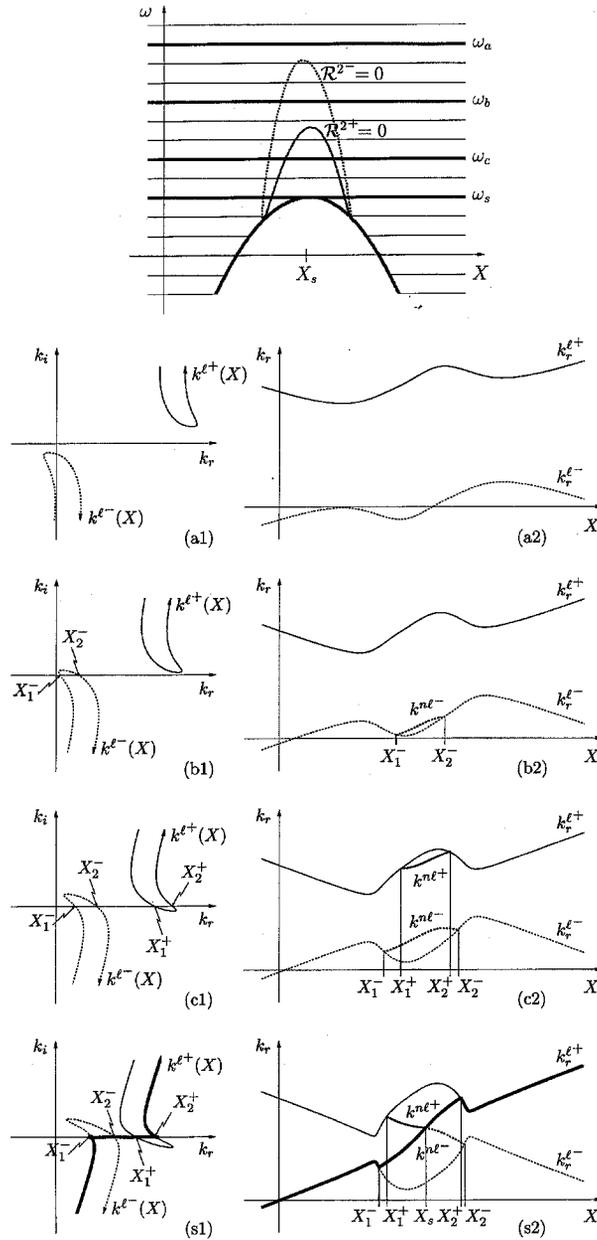


Fig. 3. Relationship between linear and nonlinear spatial branches for different values of the frequency. For a large frequency ω_a , the line $\omega = \omega_a$ does not intersect the regions $\mathcal{R}^{2\pm}(X, \omega) \geq 0$ in the X - ω plane, the linear branches $k^{\ell\pm}(X)$ do not cross the k_r -axis in the complex k -plane (a1), and nonlinear branches do not exist; the real parts $k_r^{\ell\pm}(X)$ are represented in (a2). For ω_b such that the line $\omega = \omega_b$ intersects the region $\mathcal{R}^{2-}(X, \omega) \geq 0$ in the X - ω plane, the branch $k^{\ell-}(X)$ crosses the k_r -axis for $X = X_1^-$ and $X = X_2^-$ in (b1); as a result, the nonlinear $k^{nl-}(X)$ exists for $X_1^- < X < X_2^-$ and is connected to $k_r^{\ell-}(X)$ in (b2). The line $\omega = \omega_c$ intersects both regions $\mathcal{R}^{2\pm}(X, \omega)$, thus both linear branches cross the k_r -axis in (c1); both nonlinear branches exist in the ranges $X_1^+ < X < X_2^+$, respectively (c2). When the frequency equals ω_s such that the line $\omega = \omega_s$ is tangent to the boundary of the allowable domain at X_s in the X - ω plane, the nonlinear branches meet at X_s (s2), and the crossover between branches is possible; the global spatial branch selected by the global mode is indicated in bold in (s1,2).

Furthermore, the line $\omega = \text{const.}$ has to be entirely contained within the allowable domain. Thus, *the leading order global frequency ω_s is obtained by requiring that the line $\omega = \omega_s$ be tangent to the allowable domain in the $X-\omega$ plane at a point X_s .* Since, to each point in the allowable domain of the $X-\omega$ plane, correspond two pairs $(k^{n\ell^+}, \mathcal{R}^{2^+})$ and $(k^{n\ell^-}, \mathcal{R}^{2^-})$, the allowable domain may be considered as the superposition of two sheets, respectively, labelled (+) and (–) which necessarily join along the critical curve where $(k^{n\ell^+}, \mathcal{R}^{2^+}) = (k^{n\ell^-}, \mathcal{R}^{2^-})$. A plane $\omega = \text{const.}$ cuts the surface made up of the two sheets along two lines, one on each sheet, which correspond to the (+) and (–) spatial branches, respectively. Passing from one sheet to the other on such a line is possible only for $\omega = \omega_s$. The additional condition that the line $\omega = \omega_s$ be entirely contained in the allowable domain guarantees that the surface made up of the two sheets exhibits a saddle point when cut by the plane $\omega = \omega_s$. Thus, *at leading order, the global frequency ω_g is given by a saddle point of the nonlinear dispersion relation.* Moreover nonlinear global modes only exist if the saddle point is located in a region where the wave number $k_s \equiv k^{n\ell^+}(X_s, \omega_s) = k^{n\ell^-}(X_s, \omega_s)$ is linearly unstable, i.e., $R_s^2 \equiv \mathcal{R}^{2^+}(X_s, \omega_s) = \mathcal{R}^{2^-}(X_s, \omega_s) > 0$.

An explicit condition for the existence of a saddle point may be derived from the local reduced nonlinear dispersion relation (24)

$$\omega = \Omega^{\text{NL}}(k, X).$$

This relation defines a surface in (ω, k, X) space which bears the same features as the complete nonlinear dispersion relation (23). For a given frequency ω , the spatial branches $k^{n\ell^\pm}(X, \omega)$ are obtained as the intersection of this surface and the plane $\omega = \text{const.}$ The condition that the branches $k^{n\ell^\pm}(X)$ be connected, requires that the global frequency be given by a saddle point of the surface $\omega = \Omega^{\text{NL}}(k, X)$. Thus, the global frequency ω_g is obtained, at leading order in the WKBJ approximation, for a saddle point of the reduced nonlinear dispersion relation, i.e.,

$$\omega_g \sim \omega_s = \Omega^{\text{NL}}(k_s, X_s), \tag{30}$$

where the pair (k_s, X_s) satisfies

$$\frac{\partial \Omega^{\text{NL}}}{\partial k}(k_s, X_s) = \frac{\partial \Omega^{\text{NL}}}{\partial X}(k_s, X_s) = 0. \tag{31}$$

The additional condition ²

$$(\Omega_{kX,s}^{\text{NL}})^2 - \Omega_{kk,s}^{\text{NL}} \Omega_{XX,s}^{\text{NL}} > 0 \tag{32}$$

guarantees that (ω_s, k_s, X_s) is actually a saddle point and not a local extremum.

Criterion (30), (31) may also be expressed in terms of the complete dispersion relation (23)

$$\omega = \Omega^{n\ell}(k, R^2, X).$$

Bearing in mind that

$$\Omega^{\text{NL}}(k, X) = \Omega_i^{n\ell}(k, R^2(k, X), X),$$

where $R^2(k, X)$ is implicitly defined by

$$\Omega_i^{n\ell}(k, R^2(k, X), X) = 0,$$

one may relate the derivatives of Ω^{NL} to those of $\Omega^{n\ell}$. After elementary calculations, one finds that criterion (31) takes the alternate form

$$\text{Im}[(\Omega_{R^2,s}^{n\ell})^* \Omega_{k,s}^{n\ell}] = \text{Im}[(\Omega_{R^2,s}^{n\ell})^* \Omega_{X,s}^{n\ell}] = 0,$$

² Throughout, subscript (s) denotes evaluation at the saddle point (k_s, R_s^2, X_s) .

i.e., the derivatives of the complex function $\Omega^{n\ell}$ with respect to each of its arguments have the same phase. Thus the leading order global frequency ω_s and the associated quantities (k_s, R_s^2, X_s) are equivalently determined by

$$\omega_s = \Omega^{n\ell}(k_s, R_s^2, X_s) \quad \text{and} \quad \text{Arg } \Omega_{k,s}^{n\ell} = \text{Arg } \Omega_{R^2,s}^{n\ell} = \text{Arg } \Omega_{X,s}^{n\ell}.$$

Additional condition (32) may also be expressed in terms of $\Omega^{n\ell}$ after cumbersome but straightforward calculations.

The essential result of this section is summarized by the saddle point conditions (30) and (31) applicable to the reduced nonlinear dispersion relation $\omega = \Omega^{\text{NL}}(k, X)$: to leading order, the *nonlinear* global frequency ω_s satisfies a criterion that is formally identical to its *linear* counterpart (1) and (2). However it should be emphasized that the linear criterion involves *complex* values of the wave number, frequency and spatial coordinate, whereas the nonlinear criterion only involves purely real values for these quantities.

It is possible to compute higher order corrections to the global frequency ω_g by a detailed analysis of the saddle point region (SP). As in the case of linear global modes with a double turning point [6,11,16,20], it is found that its characteristic thickness is $O(\epsilon^{1/2})$. Matching between the SP layer and the surrounding CNL $^\pm$ regions leads to an expansion of the form

$$\omega_g \sim \omega_s + \epsilon \omega_{1,n},$$

where the $O(\epsilon)$ frequency correction term $\omega_{1,n}$ is parametrized by a nonnegative integer n . Detailed calculations of this quantified set of fully nonlinear global modes are postponed to a future publication.

6. Spatial structure of transition layers TL^{1,2}

At the locations X_1^- and X_2^+ where the amplitude of nonlinear WKB approximations vanishes, nonlinearities become weak, and, in the semi-infinite regions (OL $^\pm$) extending towards $X = \pm\infty$, the solution is described by linear WKB approximations. The structure of the transition layer connecting the finite amplitude nonlinear solution in (CNL $^\pm$) to exponentially decaying linear solutions in (OL $^\pm$) is analyzed in this section.

It is essential to bear in mind that this transition layer plays no role in the global frequency selection mechanism which solely depends on the local nonlinear dispersion relation at the saddle point X_s , in the core of the nonlinear region. The locations of the transition layers indeed depend on the frequency ω_g but the linear and nonlinear solutions can be matched for any frequency. For this reason matching is carried out for an arbitrary fixed real frequency ω .

With the definitions of Section 5, the linear branch $k^{\ell-}(X)$ meets the nonlinear branch $k^{n\ell-}(X)$ at $X = X_1^-$. Let $k_1 \equiv k^{\ell-}(X_1^-) = k^{n\ell-}(X_1^-)$ be the local real wave number at ³ X_1 . For $X < X_1$ the global mode is approximated by the linear WKB ψ^ℓ approximation (17) and for $X > X_1$ by the nonlinear WKB $\psi^{n\ell}$ approximation (26). None of these expressions are uniformly valid in a full neighborhood of X_1 . An inner solution must then be constructed in an inner layer TL¹ around X_1 to allow a smooth transition between both outer WKB approximations.

The form of the inner solution is obtained from the behavior of the outer solutions as $X \rightarrow X_1$. Near X_1 the nonlinear solution (26) expands into

$$\psi^{n\ell}(X) \sim (X - X_1)^{1/2} \sqrt{\mathcal{R}_{X,1}^2} e^{i\theta_0(X_1)} e^{ik_1 x} \exp\left(\frac{i}{2\epsilon} k_{X,1}^{n\ell} (X - X_1)^2\right) \quad (33)$$

³ For simplicity we drop from now on the superscript (-).

with⁴

$$\mathcal{R}_{X,1}^2 = -\frac{\text{Im}[(\Omega_{k,1}^{n\ell})^* \Omega_{X,1}^{n\ell}]}{\text{Im}[(\Omega_{k,1}^{n\ell})^* \Omega_{R^2,1}^{n\ell}]} \quad \text{and} \quad k_{X,1}^{n\ell} = -\frac{\text{Im}[(\Omega_{R^2,1}^{n\ell})^* \Omega_{X,1}^{n\ell}]}{\text{Im}[(\Omega_{R^2,1}^{n\ell})^* \Omega_{k,1}^{n\ell}]}, \quad (34)$$

while the linear solution displays the following asymptotic behavior:

$$\psi^\ell(X) \sim A_0^\ell(X_1) e^{ik_1 X} \exp\left(\frac{i}{2\epsilon} k_{X,1}^\ell (X - X_1)^2\right) \quad (35)$$

with

$$k_{X,1}^\ell = -\Omega_{X,1}^\ell / \Omega_{k,1}^\ell.$$

The expansion of the outer solutions (33) and (35) shows that the inner solution proceeds in the form

$$\tilde{\psi}(\tilde{X}) \sim \epsilon^{1/4} (\tilde{A}_0(\tilde{X}) + \epsilon^{1/2} \tilde{A}_{1/2}(\tilde{X}) + \dots) e^{ik_1 X}, \quad (36)$$

where $\tilde{X} = (X - X_1)/\epsilon^{1/2}$ is the inner local space variable in the TL¹ layer.

From the governing equation for the inner solution $\tilde{\psi}$ in the transition layer

$$\omega(\tilde{A}_0 + \epsilon^{1/2} \tilde{A}_{1/2} + \dots) = \Omega^{n\ell}(k_1 - i\epsilon^{1/2} \partial_{\tilde{X}}, \epsilon^{1/2} |\tilde{A}_0 + \epsilon^{1/2} \tilde{A}_{1/2} + \dots|^2, X_1 + \epsilon^{1/2} \tilde{X}) \times (\tilde{A}_0 + \epsilon^{1/2} \tilde{A}_{1/2} + \dots), \quad (37)$$

we recover at lowest order

$$\omega = \Omega^{n\ell}(k_1, 0, X_1) = \Omega^\ell(k_1, X_1). \quad (38)$$

At order $\epsilon^{1/2}$ Eq. (37) reduces to

$$(\omega - \Omega_1^\ell) \tilde{A}_{1/2} = -i\Omega_{k,1}^{n\ell} \frac{d\tilde{A}_0}{d\tilde{X}} + \Omega_{R^2,1}^{n\ell} |\tilde{A}_0|^2 \tilde{A}_0 + \Omega_{X,1}^{n\ell} \tilde{X} \tilde{A}_0.$$

Bearing in mind (38), this leads to a solvability condition for \tilde{A}_0

$$i\Omega_{k,1}^{n\ell} \frac{d\tilde{A}_0}{d\tilde{X}} = (\Omega_{X,1}^{n\ell} \tilde{X} + \Omega_{R^2,1}^{n\ell} |\tilde{A}_0|^2) \tilde{A}_0. \quad (39)$$

This equation is of the form

$$\frac{d\tilde{A}_0}{d\tilde{X}} = ((a_r + ia_i)\tilde{X} + (b_r + ib_i)|\tilde{A}_0|^2) \tilde{A}_0, \quad (40)$$

and admits the solution

$$\tilde{A}_0(\tilde{X}) = \frac{\tilde{A}_0(0) \exp[\frac{1}{2}(a_r + ia_i)\tilde{X}^2 - \frac{1}{2}i(b_i/b_r) \ln(1 - 2b_r|\tilde{A}_0(0)|^2 \int_0^{\tilde{X}} e^{a_r u^2} du)]}{\sqrt{1 - 2b_r|\tilde{A}_0(0)|^2 \int_0^{\tilde{X}} e^{a_r u^2} du}}. \quad (41)$$

⁴ Subscript (1) always denotes evaluation at $k = k_1$, $R_1^2 = 0$, $X = X_1$. Note that $\Omega_1^{n\ell} = \Omega_1^\ell$ because $\mathcal{R}^2(X_1) = 0$.

The inner solution $\tilde{\psi}$ in the TL¹ layer has to be matched for $\tilde{X} \rightarrow \pm\infty$ with the outer solutions ψ^{nl} and ψ^ℓ in the CNL⁻ and OL⁻ regions, respectively. The square root behavior of ψ^{nl} in (33) requires that $\tilde{A}_0(\tilde{X}) = O(\tilde{X}^{1/2})$ as $\tilde{X} \rightarrow +\infty$. This implies that

$$2b_r |\tilde{A}_0(0)|^2 \int_0^\infty e^{a_r u^2} du = 1 \quad \text{or} \quad |\tilde{A}_0(0)|^2 = \frac{1}{4b_r} \sqrt{\frac{\pi}{-a_r}}.$$

With this value for $|\tilde{A}_0(0)|$ the inner solution (41) admits for $\tilde{X} \rightarrow +\infty$ the asymptotic expansion

$$\tilde{A}_0(\tilde{X}) \sim \sqrt{-\frac{a_r}{b_r}} \tilde{X} \exp \frac{i}{2} \left[\left(a_i - \frac{b_i}{b_r} a_r \right) \tilde{X}^2 + \frac{b_i}{b_r} (\ln \tilde{X} + \text{const.}) \right]. \quad (42)$$

Comparison of the coefficients in (39) and (40), and use of (34) leads to

$$-\frac{a_r}{b_r} = \mathcal{R}_{X,1}^2 \quad \text{and} \quad a_i - \frac{b_i}{b_r} a_r = k_{X,1}^{\text{nl}}.$$

Thus the choice of

$$\tilde{A}_0(0) = \left(\frac{\pi}{-a_r} \right)^{1/4} \frac{e^{i\theta_0(X_1)}}{(4b_r)^{1/2}}$$

completely matches, at leading order, the inner solution $\tilde{\psi}$ given by (36) and (42) with the outer nonlinear solution ψ^{nl} in the limit (33).

For this value of $\tilde{A}_0(0)$ the asymptotic behavior of the inner solution (41) as $\tilde{X} \rightarrow -\infty$ is

$$\tilde{A}_0(\tilde{X}) \sim \frac{\tilde{A}_0(0)}{\sqrt{2}} 2^{-ib_i/2b_r} \exp\left(\frac{1}{2}(a_r + ia_i)\tilde{X}^2\right).$$

As $a_r + ia_i = ik_{X,1}^\ell$, the inner solution $\tilde{\psi}$ matches the outer linear WKB approximation ψ^ℓ given by (35) as long as

$$A_0^\ell(X_1) = \epsilon^{1/4} \frac{\tilde{A}_0(0)}{\sqrt{2}} 2^{-ib_i/2b_r}.$$

This completes the proof that, for a given frequency, the linear and nonlinear WKB approximations can be matched at leading order through a transition layer of width $\epsilon^{1/2}$ (in terms of X). A similar analysis holds for the transition layer at X_2^+ .

7. Conclusions and discussion

A fully nonlinear frequency selection criterion has been derived for global modes on a doubly infinite interval, governed by the complex Ginzburg–Landau equation with spatially varying coefficients.

Let $\omega = \Omega^{\text{NL}}(k, X)$ denote the local reduced nonlinear dispersion relation obtained by eliminating the amplitude R^2 from the complete local nonlinear dispersion relation $\omega = \Omega^{\text{nl}}(k, R^2, X)$ pertaining to finite amplitude travelling waves. The global frequency ω_g on the doubly infinite interval is then, at leading order, a *real saddle point* ω_s of Ω^{NL} associated with the *real* wave number k_s and station X_s such that

$$\omega_g \sim \omega_s = \Omega^{\text{NL}}(k_s, X_s) \quad \text{and} \quad \frac{\partial \Omega^{\text{NL}}}{\partial k}(k_s, X_s) = \frac{\partial \Omega^{\text{NL}}}{\partial X}(k_s, X_s) = 0. \quad (43)$$

This saddle point condition is selected so as to ensure a smooth crossover between the two nonlinear spatial branches $k^{\text{nl}-}$ and $k^{\text{nl}+}$ present on either side of X_s .

The nonlinear global mode spatial structure is illustrated in Fig. 1. A spatially evolving complex $k^{\ell-}$ branch issuing from $X = -\infty$ and governed by the local linear dispersion relation $\omega = \Omega^\ell(k, X)$ in the OL^- region experiences a transition to the fully nonlinear régime within a layer TL^1 of thickness $\text{O}(\epsilon^{1/2})$ centered around a point X_1^- of local neutral stability. In a “downstream” CNL^- region of finite extent, the wave is fully nonlinear and its real wave number $k^{\text{nl}-}$ and amplitude \mathcal{R}^{2-} are governed by the local nonlinear dispersion relation $\omega = \Omega^{\text{nl}}(k, R^2, X)$. As the wave enters the SP layer centered around X_s , it is smoothly converted into a $k^{\text{nl}+}$ spatial branch since condition (43) is satisfied. The CNL^+ region mirrors, for the $k^{\text{nl}+}$ wave, the CNL^- region for the $k^{\text{nl}-}$ wave. A second transition layer TL^2 centered around the locally neutral station X_2^+ allows a return to a linear $k^{\ell+}$ spatial wave in the OL^+ region, and a final exponential decay at $X = +\infty$.

These results should be compared with the corresponding double turning point *linear* frequency selection criterion derived in [6,16,20]. For infinitesimal waves governed by a local linear dispersion relation $\omega = \Omega^\ell(k, X)$ between *complex* wave number k and frequency ω , the complex global frequency ω_g is given by the saddle point condition (1), (2), where the frequency ω_s , wave number k_s , and spatial coordinate X_s are all *complex*. The linear global mode spatial structure is thus made up of complex $k^{\ell-}$ and $k^{\ell+}$ spatially evolving waves governed by the local linear dispersion relation $\omega = \Omega^\ell(k, X)$ in corresponding OL^- and OL^+ regions near $X = \mp\infty$. The saddle point condition (2) ensures a smooth crossover from the $k^{\ell-}$ to the $k^{\ell+}$ branch in an inner region SP of thickness $\text{O}(\epsilon^{1/2})$ centered around the *complex* saddle point X_s . In the linear case, the $\text{TL}^{1,2}$ and CNL^\pm regions are naturally absent, but an inner SP layer structure is necessary in both linear and nonlinear studies. Indeed, both frequency selection criteria are formally identical. Paradoxically the nonlinear criterion involves only real variables and therefore appears simpler to implement, without any of the complexities associated with analytic continuation in the complex X -plane.

According to [6,16], a necessary condition for the existence of a growing *linear* global mode is that the basic state be linearly absolutely unstable in a region of finite extent. According to the present study, a necessary condition for the existence of a self-sustained *nonlinear* global mode appears to be that the basic state only contain a linearly unstable region of finite extent corresponding to $\mathcal{R}^{2\pm} > 0$. Fully nonlinear global modes of the type shown here may in principle occur even if the medium is only convectively unstable.

The latter result is entirely consistent with the studies of Chomaz [4] and Couairon and Chomaz [7]. In these recent investigations, nonlinear global modes on the semi-infinite interval $0 < X < \infty$ and governed by the real Ginzburg–Landau equation with constant coefficients and both cubic and quintic nonlinearities have been examined from the dynamical systems point of view. In the supercritical case (stabilizing nonlinear terms), it is found that nonlinear global modes may exist even if the basic state is only linearly convectively unstable. Thus, there is clearly more and more evidence that the relation between local *linear* stability properties and global modes becomes somewhat tenuous as the fully nonlinear régime is approached.

The recent numerical simulations of wakes in shallow layers performed by Schär and Smith [22] underscore the importance of nonlinearities in the frequency selection process. When nonlinearities were turned off, the wake was found to oscillate at a Strouhal frequency $\omega \sim 0.17$, to be compared with $\omega_s = 0.19$ resulting from the linear frequency selection criterion (1), (2). Upon retaining nonlinear terms, the “observed” wake frequency increased substantially to $\omega \sim 0.27$. The criterion proposed in the present study could possibly account for this discrepancy.

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