Critical Properties of Phase Transitions in Lattices of Coupled Logistic Maps

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We numerically demonstrate that collective bifurcations in two-dimensional lattices of locally coupled logistic maps share most of the defining features of equilibrium second-order phase transitions. Our simulations suggest that these transitions between distinct collective dynamical regimes belong to the universality class of Miller and Huse model with synchronous update [Marcq et al., Phys. Rev. Lett. **77** (1996), 4003].

§1. Introduction

For strong enough coupling, mutually interacting oscillators tend to synchronize. Professor Kuramoto first gave a solid mathematical grounding to this rather intuitive idea in 1975.[1)] More precisely, he showed analytically that an ensemble of nonidentical phase oscillators with distributed natural frequencies, and coupled through their mean-field, synchronize above a critical value of the coupling constant.[2)] This dynamical phase transition is a robust phenomenon, generically occurring in a wide class of similar coupled dynamical systems.[3)]

The system of interest in the present work is complementary to Kuramoto's model: N identical units with aperiodic individual (discrete time) dynamics sit on the nodes of a regular lattice, and are updated synchronously with local (diffusive) coupling.[4)] For a large enough coupling constant, the spatially-averaged activity of such coupled map lattices is time-periodic.[5)] However, synchronization of individual units is *not* involved, at least in the usual sense: macroscopic coherence coexists with microscopic disorder, as evidenced by broad distribution functions of local activity and rapidly decaying spatial correlation functions. Extensive numerical simulations have confirmed that these emergent macroscopic cycles are global attractors, well-defined in the infinite-time, infinite-size (thermodynamic) limit:[5)] fluctuations of the spatially-averaged activity about the collective cycle vanish when $N \to \infty$.

For a large enough control parameter, a single logistic map exhibits an inverse bifurcation cascade between regimes of banded chaos. In the same parameter region, and for strong enough coupling, locally coupled logistic maps exhibit an inverse bifurcation cascade between macroscopic cycles of period 2^n.[5)] Generalized mean-field arguments give a satisfactory understanding of the build-up of correlations at the origin of the dynamical long-range order involved here,[6] at least far from bifurcation points.

In this work, we wish to characterize, thanks to numerical simulations, the first bifurcation points in the cascade.[7] Upon defining adequate order parameters, we demonstrate that these macroscopic bifurcations harbor the characteristic features of *equilibrium* second-order phase transitions. The relevance of standard finite-size scaling theory indicates that they persist in the infinite-size limit. Many properties of an equilibrium system close to a second-order transition turn out to be largely independent of the microscopic details of the interactions between individual components: they fall instead into a small number of universality classes, each defined by global features such as the symmetries of the underlying Hamiltonian or the spatial dimensionality of the system. We numerically evaluate the critical exponents of period-doubling phase transitions, and discuss the relevance of the notion of universality to *temporal* spontaneous symmetry breaking.

§2. Critical properties of a period-doubling phase transition

We consider the dynamics of a set of $N = L^2$ variables $x^t_{i,j}$ sitting on the nodes of a two-dimensional square lattice. Time is discrete. The update rule consists of two stages: each site is first updated according to the logistic map with parameter $r \in [0, 4]$:

$$\begin{aligned} f: \quad [0,1] &\to [0,1], \\ x &\mapsto r\,x\,(1-x), \end{aligned} \tag{2·1}$$

then transformed according to a diffusive coupling operator. Interaction is restricted to nearest-neighbors. All sites are updated *synchronously* according to the rule:

$$x^{t+1}_{i,j} = (1-4g)\, f(x^t_{i,j}) + g\,\left(f(x^t_{i-1,j}) + f(x^t_{i,j-1}) + f(x^t_{i+1,j}) + f(x^t_{i,j+1})\right), \tag{2·2}$$

where g is the coupling constant, set to $g = 0.2$ in this section (democratic coupling). The *microscopic* control parameter of the lattice dynamical system is r.

The bifurcation diagram of the mean activity

$$x^t = \frac{1}{L^2} \sum_{i,j=1}^{L} x^t_{i,j} \tag{2·3}$$

is given in Fig. 1 for $r > r_\infty = 3.57\ldots$. The statistical behavior of the coupled map lattice may be interpreted as long-range order accompanied by the temporal evolution of spatially-averaged quantities: fixed point (or period 1) above $r_{1-2} \simeq 3.86$, period 2 for $r \in [r_{2-4}, r_{1-2}]$, with $r_{2-4} \simeq 3.63$, period 2^n, $n \geq 2$, for $r \in [r_\infty, r_{2-4}]$. An infinite cascade of period-doubling bifurcations is expected to occur in the limit $r \to r_\infty$.[8]

For simplicity, we focus on the period 1-period 2 bifurcation. The (time-asymptotic) distribution of site values in these two regimes is shown for typical parameters in Fig. 2. The Lyapunov spectrum scales linearly with the system size: chaos is *extensive*. Dynamical quantifiers such as the largest Lyapunov exponent and

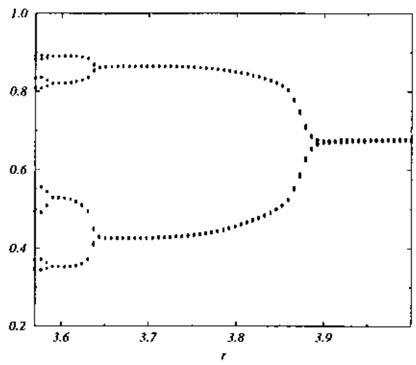

Fig. 1. Bifurcation diagram of democratically coupled ($g = 0.2$) logistic maps on a two-dimensional lattice of $L^2 = 1024^2$ sites: for each parameter value $r \geq r_\infty$, 20 consecutive values of the mean activity x^t are plotted vs r. Initial site values are randomly distributed over the interval $[0, 1]$, and a transient of duration $t_0 = 10^4$ is discarded. Period-doubling bifurcations are rounded by finite-time, finite-size effects.

the Kolmogorov entropy remain continuous close to the bifurcation point (see also Ref. 9)).

The local "magnetization" is defined by $m_{1-2}^t = x^{2t+1} - x^{2t}$. The order parameter of the transition reads:[7),10)]

$$M_{1-2} = \left\langle \left| m_{1-2}^t \right| \right\rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \left| x^{2t+1} - x^{2t} \right| . \tag{2·4}$$

The period 1 and period 2 phases are respectively "paramagnetic" ($M_{1-2} = 0$) and "ferromagnetic" ($M_{1-2} \neq 0$). Through Eq. (2·4), we wish to draw a strong analogy with the Ising model. Instead of spin-reversal invariance, time-translation invariance is broken in the ordered phase of the lattice dynamical system. Further, the correlation length ξ exhibits a sharp maximum close to the transition, where ξ characterizes the exponential decay of the equal-time, two-point correlation function of the field $x_{i,j}^t$. Large scale simulations suggest the presence of well-defined power laws controlling the behavior close to the transition of the correlation length ξ, the magnetization M_{1-2}, and the susceptibility χ_{1-2}, defined by $\chi_{1-2} = L^2 \left\langle \left(|m_{1-2}^t| - M_{1-2} \right)^2 \right\rangle$.

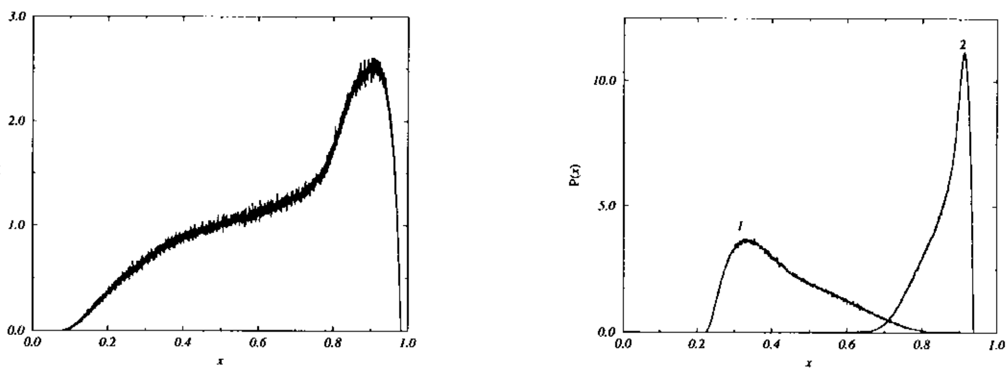

Fig. 2. Probability distribution function of the lattice's site values, measured for $g = 0.2$, $L = 1024$, in the period 1 (left-hand graph, $r = 3.93$) and period 2 (right-hand graph, $r = 3.75$) collective states. The pdfs measured over 4 consecutive time-steps are superposed. The breadth of the lines is related to finite-size statistical fluctuations.

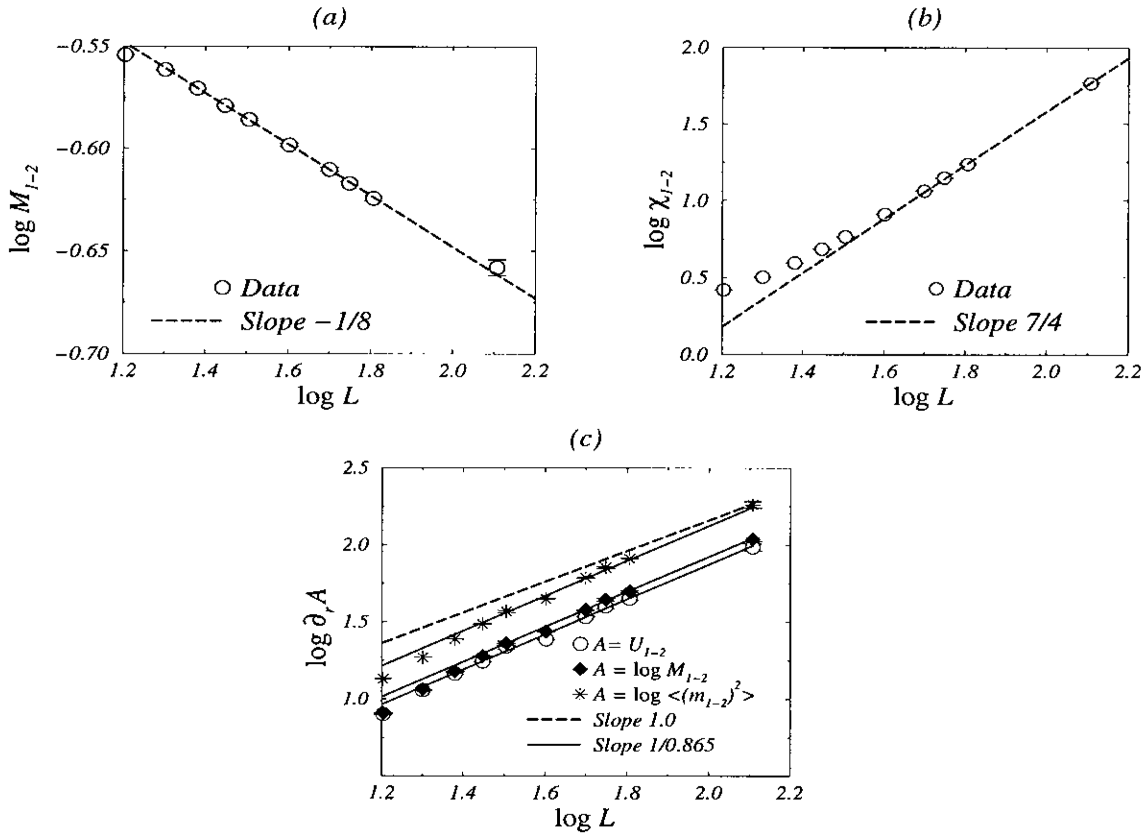

Fig. 3. Period 1-period 2 transition, $g = 0.2$, $r = r^{\infty}_{1-2} = 3.86212$. The finite-size scaling laws given in Eq. (2·5) allow to measure the exponent ratios β/ν (a), γ/ν (b), $1/\nu$ (c). Corrections to scaling are apparent in the three graphs. The dashed lines correspond to the exact exponent values of the two-dimensional Ising model.

The precise location of the transition point is determined using Binder's method:[11) the value of the finite-size cumulant $U^L_{1-2}(r) = -3 + \langle (m^t_{1-2})^4 \rangle / \langle (m^t_{1-2})^2 \rangle^2$ at size $N = L^2$ is independent of L at criticality: $\forall L$, $U^L_{1-2}(r^{\infty}_{1-2}) = U^{\infty}_{1-2}$. Using system sizes ranging from $L = 32$ to $L = 128$, we find $r^{\infty}_{1-2} = 3.86212(12)$, $-U^{\infty}_{1-2} = 1.819(12)$, where numbers within parentheses correspond to the uncertainty over the last digit(s). The exponent ratios β/ν, γ/ν, $1/\nu$ are then obtained from the following scaling laws:

$$
\begin{aligned}
M_{1-2}(L) &\propto L^{-\beta/\nu}, \\
\chi_{1-2}(L) &\propto L^{\gamma/\nu}, \\
\partial_r U^L_{1-2} &\propto L^{1/\nu}, \\
\partial_r \log M_{1-2}(L) &\propto L^{1/\nu}, \\
\partial_r \log \langle m^t_{1-2}(L)^2 \rangle &\propto L^{1/\nu},
\end{aligned}
\qquad (2{\cdot}5)
$$

where statistical averages are computed at the infinite-size critical parameter value r^{∞}_{1-2}. For all the above observables $\mathcal{O}$, our data is consistent with the simplest equilibrium correction to scaling: $\mathcal{O} = L^{\phi_{\mathcal{O}}} (a_0 + a_1 L^{-\omega_{\mathcal{O}}} + \ldots)$, where $\phi_{\mathcal{O}}$ and $\omega_{\mathcal{O}}$ are respectively the scaling exponent defined in Eq. (2·5) and the first subdominant exponent. Taking into account this correction, our best estimates are (see Ref. 12) for a detailed account of the procedure we use):

$$\begin{aligned} \beta/\nu &= 0.126(4), \\ \gamma/\nu &= 1.76(3), \\ \nu &= 0.865(25). \end{aligned} \tag{2·6}$$

The exponent ratios β/ν and γ/ν are in excellent agreement with the two-dimensional equilibrium Ising model: $(\beta/\nu)_{\text{Ising}} = 1/8$, $(\gamma/\nu)_{\text{Ising}} = 7/4$. However, the correlation length exponent ν is *not* consistent with $\nu_{\text{Ising}} = 1$ (see Fig. 3). As a consequence, our estimates for $\beta = 0.108(6)$ and $\gamma = 1.52(7)$ are also incompatible with the Ising values. Unless corrections to scaling of an unusual nature are present in this system, our data suggests that the period 1-period 2 *phase transition* does not belong to the universality class of the two-dimensional equilibrium Ising model: the analogy we have drawn breaks down at a quantitative level.

§3. Universality

The evolution rule (2·1)–(2·2) involves two parameters: the coupling constant g and the nonlinear parameter r: a line of period 1-period 2 transitions is observed in the parameter plane for large enough coupling ($g \gtrsim 0.10$). We studied two other transition points on this line: a r-driven transition at $g = 0.11$, and a coupling-driven transition at a fixed value of $r = 3.831$. The same measurement protocol is used, with similar system sizes and statistical accuracy (see Ref. 7) for further details). In the three cases, we find exponent values mutually consistent within error bars (see Table I for a summary of results). This suggests that the period 1-period 2 transition line defines a universality class distinct from that of the two-dimensional Ising model. However, the hyperscaling relation $2\beta + \gamma = d\nu$ ($d = 2$) remains valid, and the critical value U^∞ of Binder's cumulant is consistent with the Ising value.

We also studied transitions between cycles of higher period, which are characterized by the same phenomenology as described above. Physical observables are defined as before, upon replacing m^t_{1-2} by the appropriate local magnetization, for instance $m^t_{2-4} = x^{4t+2} - x^{4t}$ for period 2-period 4 transitions. We find that: (*i*) these transitions do not belong to the Ising universality class; (*ii*) the measured exponent values are consistent with those found for period 1-period 2 transitions – even though the system sizes we use do not allow a clear-cut answer to this last question.

In fact, the same features are characteristic of other ordering transitions between chaotic phases. Miller and Huse introduced a two-dimensional coupled map lattice with a microscopic "up-down" symmetry, using a continuous, piecewise-linear, odd-symmetric map.[13)] In a previous work,[12)] we showed that the Ising-like transition of this extensively chaotic lattice dynamical system does not belong to the Ising universality class, with in particular $\nu = 0.89 \pm 0.03$. However, Ising exponents are recovered for the same geometry, coupling, and local map once the update rule becomes asynchronous (lattice sites updated one at a time). The nature of update, a dynamical feature, is a relevant "parameter" that distinguishes between universality classes. The critical exponents of Miller and Huse's model with synchronous update are recalled in the Table: we showed[12)] that they are representative of a

Table I. Summary of critical exponents: Ising model; Miller and Huse model; r-driven period 1-period 2 transition at $g = 0.2$; r-driven period 1-period 2 transition at $g = 0.11$; g-driven period 1-period 2 transition at $r = 3.831$.

	2D Ising	MH	$g = 0.2$	$g = 0.11$	$r = 3.831$
Threshold		0.20534(2)	3.86212(12)	3.8310(1)	0.1100(1)
$-U^{\infty}$	1.83	1.832(4)	1.819(12)	1.828(10)	1.83(3)
β/ν	0.125	0.125(4)	0.126(4)	0.131(10)	0.129(10)
γ/ν	1.75	1.748(10)	1.76(3)	1.75(4)	1.73(3)
$(2\beta+\gamma)/\nu$	2	2.00(2)	2.02(4)	2.01(6)	1.99(5)
β	0.125	0.111(5)	0.108(6)	0.107(21)	0.107(10)
γ	1.75	1.55(4)	1.52(7)	1.50(12)	1.39(10)
ν	1	0.887(18)	0.865(25)	0.860(45)	0.80(4)

genuine universality class in the sense that changes of, e.g., the local map do not alter them provided that the up-down symmetry is preserved (see also Refs. 14) and 15)). Remarkably, these exponents are consistent (within error bars) with those obtained for the period 1-period 2 transitions of two-dimensional coupled logistic maps. The same universality class encompasses ordering transitions between chaotic phases of synchronously-updated lattice dynamical systems, whether due to the (discrete) breaking of an up-down symmetry in phase space or to that of time-translation invariance.

§4. Conclusion

Our numerical simulations demonstrate that *equilibrium* finite-size scaling laws allow to characterize the period-doubling macroscopic bifurcations of lattices of locally coupled chaotic logistic maps. This relevance provides in itself further evidence that such bifurcations are indeed continuous phase transitions, well-defined in the infinite-size limit, where the equal-time correlation length diverges as the system's linear size. We evaluate the static critical exponents β, γ and ν of fixed point-period 2 and period 2-period 4 transitions. The best interpretation of our (finite-size) data is the following: these transitions belong to the universality class of the Miller and Huse model with synchronous update, *not* to the equilibrium Ising universality class: $\nu \simeq 0.89 \neq \nu_{\rm Ising} = 1$.

Period-doubling phase transitions are by all means unusual: they separate microscopically chaotic states with different (regular) macroscopic *dynamics*. The existence of an ordered phase is related to the discrete breaking of time-translation invariance at the macro-scale, while microscopic dynamical quantifiers, such as the Lyapunov spectrum, remain continuous close to the transition. The lattice dynamical system is a priori far from equilibrium: microreversibility is not expected to hold. The non-Ising value of the correlation length exponent ν, at odds with standard coarse-graining arguments,[13),16)] remains a puzzle.

Two comments are in order. First, the control parameters we use are *microscopic*. This is also true of related models with similar critical properties.[14),15),17)] The various phases we observe arise due to a balance between nearest-neighbor interactions and fluctuations of deterministic origin. Since the critical exponents we measure are not those of mean-field theory,[18] we know that fluctuations cannot be neglected close to the transition points. One would certainly like to be able to satisfactorily define the temperature of a given extensively chaotic lattice dynamical system, i.e. a macroscopic, intensive parameter, well-defined in the infinite-size limit, that quantifies at a coarse-grained level the degree of microscopic disorder. This remains an open, challenging problem.

Second, the synchronous nature of the update rule of a coupled map lattice is never insignificant.[19),20] In the case of Miller and Huse model, Ising exponents are recovered with asynchronous update: the nature of update is a relevant "parameter" in the sense of critical phenomena. For coupled logistic maps, the long-range order leading to periodic collective behavior is destroyed by asynchronous update. The fully synchronized state, where all lattice site values are equal to that of the unstable fixed point of the local map, remains unstable under synchronous update. However, synchronization transitions occur as soon as the update rule includes some degree of asynchrony.[21] In all cases, the choice of a given update rule is a crucial modeling issue.

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