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A model for transitional plane Couette flow

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Abstract. A simplified model of plane Couette flow is derived by means of a cross-stream (y) Galerkin expansion in terms of trigonometric functions appropriate for idealized stress-free boundary conditions at the plates. A set of partial differential equations is obtained, governing the in-plane (x–z) space-dependence of a velocity field taken in the form: $u = U_0(x,z,t) + [1 + U_1(x,z,t)]\sin(\pi y/2)$, $v = V_1(x,z,t)\cos(\pi y/2)$, $w = W_0(x,z,t) + W_1(x,z,t)\sin(\pi y/2)$. Beyond Lorenz-like Waleffe's modeling (Waleffe 1997), this Swift–Hohenberg type of approach is expected to give an access to the microscopic mechanism of spatiotemporal intermittency typical of the transition to turbulence in plane Couette flow (Pomeau 1986, Bergé et al. 1998). © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

plane Couette flow / transition to turbulence / Galerkin expansion

Un modèle pour l'écoulement de Couette transitionnel

Résumé. *Un modèle simplifié d'écoulement de Couette plan est obtenu au moyen d'un développement de Galerkin de la dépendance spatiale transverse* (y) *sur une base de fonctions trigonométriques appropriée à des conditions aux limites libres aux plaques. Il se présente comme un système d'équations aux dérivées partielles gouvernant la dépendance spatiale dans le plan* (x–z) *de l'écoulement pris sous la forme :* $u = U_0(x,z,t) + [1 + U_1(x,z,t)]\sin(\pi y/2)$, $v = V_1(x,z,t)\cos(\pi y/2)$, $w = W_0(x,z,t) + W_1(x,z,t)\sin(\pi y/2)$. *Au-delà de la modélisation à la Lorenz réalisée par Waleffe (Waleffe 1997), ce type d'approche à la Swift–Hohenberg devrait donner accès au mécanisme microscopique de l'intermittence spatio-temporelle typique de la transition vers la turbulence de l'écoulement de Couette plan (Pomeau 1986, Bergé et al. 1998). © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

écoulement de Couette plan / transition vers la turbulence / développement de Galerkin

Version française abrégée

Les premières étapes de la transition vers la turbulence sont plus faciles à comprendre pour les écoulements qui sont instables vis-à-vis de perturbations infinitésimales saturant au-dessus du seuil et pour lesquels le concept de chaos spatio-temporel a pris, en premier, son sens. Au contraire, les systèmes qui ne se déstabilisent que sous l'effet de perturbations d'amplitude finie sont beaucoup plus difficile à traiter. Le cas de l'écoulement de Couette plan, l'écoulement de cisaillement qui se réalise entre deux plaques en translation relative, est l'un des plus dramatiques puisque l'on peut montrer qu'il reste linéairement stable à tout nombre de Reynolds alors qu'une transition directe (sous-critique) vers la turbulence est observée par nuclation de poches (« spots ») turbulentes à des nombres de Reynolds intermédiaires dont la valeur précise dépend des conditions expérimentales [1].

Note présentée par Paul CLAVIN.

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D'un côté, la coexistence de la turbulence à l'intérieur des poches et de l'écoulement resté laminaire à l'extérieur a été analysée par Pomeau ([2,3]) qui a introduit pour cela le concept d'*intermittence spatio-temporelle*. D'un autre côté, la nature de l'écoulement à l'intérieur des poches et le mécanisme maintenant la turbulence a été l'objet de conjectures [4] mettant en jeu un processus d'effondrement et de régénération de trainées turbulentes longitudinales («streaks») observées dans certaines expériences [5] et peut-être reliées à certaines solutions non-linéaires non-triviales des équations de Navier–Stokes.

Un modèle concret d'écoulement de Couette plan a été obtenu par Waleffe [6] pour rendre compte des couplages à l'origine du processus. Ce modèle suppose une structure spatiale gelée périodique dans le plan de l'écoulement et s'exprime sous la forme d'un système d'équations différentielles couplées pour les amplitudes des modes décrivant la structure postulée. En un sens, il est donc l'équivalent du modèle de Lorenz pour la convection de Rayleigh–Bénard. Or, dans ce dernier cas, un progrès important a consisté à considérer des modèles de type Swift–Hohenberg (SH) [7] qui «dégèlent» la structure spatiale. Dans cette note, suivant la stratégie développée dans [8] où le modèle SH était étendu pour incorporer l'effet des écoulements de dérive, nous déterminons la généralisation spatio-temporelle du système de Waleffe.

Les équations de Navier–Stokes (2) sont ici adimensionnées par la demi-épaisseur h de la couche fluide et la vitesse U_b des plaques. Le nombre de Reynolds est alors défini par $R = U_b h/\nu$, où ν est la viscosité cinématique du fluide. Suivant Waleffe, l'écoulement de Couette plan original avec conditions de non-glissement aux plaques est remplacé par un écoulement fictif avec conditions aux limites libres, engendré par une force en volume $\mathbf{F}$. La dépendance spatiale transverse (en y) de l'écoulement peut dès lors être développée sur une base de simples lignes trigonométriques. L'écoulement de base $u = u_b = \sin(\beta y)$ avec $\beta = \pi/2$ est obtenu en prenant $\mathbf{F}(y) = F\sin(\beta y)\hat{\mathbf{x}}$ et $F = \beta^2/R$. Le développement de Galerkin de la dépendance transverse est tronqué pour ne retenir que $u = U_0(x,z,t) + [1 + U_1(x,z,t)]\sin(\beta y)$, $v = V_1(x,z,t)\cos(\beta y)$, $w = W_0(x,z,t) + W_1(x,z,t)\sin(\beta y)$, où les trainées turbulentes sont associées aux composantes (U_0, W_0) du champ de vitesse, la pression étant prise sous la forme $p = P_0(x,z,t) + P_1(x,z,t)\sin(\beta y)$. La projection de l'équation de continuité sur les fonctions indépendantes de y et $\sin(\beta y)$ conduit respectivement aux équations (3) et (4). De même la projection des équations de Navier–Stokes (2) sur les fonctions constantes, $\sin(\beta y)$ et $\cos(\beta y)$ donne le système (5)–(9). On vérifie facilement que les termes d'advection du modèle conservent l'énergie cinétique (en moyenne sur y) contenue dans la perturbation, soit $E = (U_0^2 + W_0^2) + \frac{1}{2}(U_1^2 + V_1^2 + W_1^2)$. L'élimination de la pression conduit à un système de trois équations aux dérivées partielles (11)–(13) gouvernant trois champs Ψ_0, Ψ_1 et Φ_1 associés aux parties rotationnelles et irrotationelle des différentes contributions au champ de vitesse.

Une analyse de stabilité classique de l'écoulement de base permet de vérifier que celui-ci est bien linéairement stable pour tout nombre de Reynolds. D'un autre côté, une hypothèse appropriée pour Ψ_0, Ψ_1 et Φ_1 conduit directement aux modèles de Waleffe [6].

Conformément à ce qu'on était en droit d'attendre, des simulations numériques préliminaires du modèle ont montré l'existence de transitoires turbulents assez comparables à ceux qui sont observés dans les expériences [5]. La *figure 1* illustre l'évolution de l'amplitude des champs Ψ_0, Ψ_1 et Φ_1 et des énergies associées E_0, E_1 et $E = E_0 + E_1$ au cours d'un tel transitoire. Un risque de développement de singularité en temps fini apparaît néanmoins pour certaines conditions initiales de trop grande amplitude lorsque R augmente. Dans la gamme de nombres de Reynolds concernée par le processus de transition, ce défaut pourrait être corrigé en résolvant plus finement la dépendance en y du champ de vitesse.

Plus riche que le modèle de Waleffe puisque s'affranchissant de l'hypothèse de structure spatiale périodique gelée, notre modèle devrait conduire à une description semi-réaliste de la transition vers la turbulence par intermittence spatio-temporelle dans l'écoulement de Couette plan, et peut-être donner des indications intéressantes sur des situations moins académiques où les poches turbulentes sont présentes, tels les écoulements de Poiseuille plan ou de couche limite.

The first steps of the transition to turbulence are easier to understand for flows that experience instabilities against infinitesimal perturbations saturating above threshold, and for which the classical tools of weakly nonlinear analysis, e.g., the envelope formalism, are appropriate. Rayleigh–Bénard convection cells and Taylor–Couette vortices are classical examples. In these cases, keywords are mainly 'patterns' and 'spatio-temporal chaos' and the successive bifurcated states involved in the transition remain in some sense close to each other. 'Turbulence' sets in at later stages of what can be called a globally supercritical scenario. By contrast, systems that become unstable against finite amplitude perturbations but remain stable against smaller ones, at least in some range of control parameters, are much more difficult to deal with. The case of plane Couette flow, i.e., the shear flow achieved between two plates in relative motions, is perhaps the most dramatic case since linear stability holds for all Reynolds numbers whereas a direct (subcritical) transition to turbulence via the nucleation of turbulent spots is observed at intermediate Reynolds numbers, the precise value of which depends on the experimental conditions [1].

On the one hand, the coexistence of turbulence within the spots and laminar flow outside them has been theoretically analyzed by Pomeau [2,3] who introduced the concept of *spatiotemporal intermittency*. On the other hand, the nature of the flow inside the spots and the mechanism sustaining turbulence has been the subject of conjectures [4] involving the breakdown and regeneration of streamwise structures called *streaks* observed in some experiments [5] and possibly connected to the existence of nonlinear nontrivial solutions to the Navier–Stokes equations.

A concrete model of plane Couette flow has been derived by Waleffe [6] in order to account for the couplings leading to the expected behavior. This model rests on the assumption of a frozen spatial structure supposed to be periodic in the plane of the flow, which leads to a low-order system of ordinary differential equations involving the amplitudes of the modes accounting for the postulated structure. In a sense, it is thus the strict analog to Lorenz' for Rayleigh–Bénard convection. In the latter case, much progress has been done in the understanding of the transition to turbulence by turning to the Swift–Hohenberg (SH) model [7] that relaxes the assumption about the spatial structure. In this note, following the strategy developed in [8] where the SH model was extended to include drift flow effects, we derive the equivalent space-time generalization of Waleffe's system.

Using the half-gap h as length unit, the plate speed U_b as velocity unit and defining the Reynolds number as $R = U_b h/\nu$, where ν is the kinematic viscosity of the fluid, the continuity and Navier–Stokes equations read:

$$\nabla \cdot \mathbf{V} = 0 \tag{1}$$

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + R^{-1} \nabla^2 \mathbf{V} + \mathbf{F} \tag{2}$$

Following Waleffe [6] we replace the original no-slip boundary conditions at the moving plates by stress-free conditions. While this change allows us to perform a Galerkin expansion of the cross-stream dependence of the hydrodynamic fields using trigonometric lines, the plate motion is no longer able to drive the flow, so that a fictitious bulk force $\mathbf{F}$ has to be added to equation (2). Taking $\mathbf{F}(y) = F \sin(\beta y)\hat{\mathbf{x}}$ with $\beta = \pi/2$ and $F = \beta^2/R$ leads to an exact (strictly streamwise) solution of (2) in the form $u = u_b = \sin(\beta y)$ (the scaling adopted here is different from that in [6] by a factor $\sqrt{2}$).

In Waleffe's picture, the streaks are associated with a streamwise velocity component uniform in the y-direction, $\tilde{u}_{\mathrm{str}} = U_0(x, z, t)$ and a related span-wise component $\tilde{w}_{\mathrm{str}} = W_0(x, z, t)$ linked together by the projection of the continuity condition onto functions independent of y:

$$\partial_x U_0 + \partial_z W_0 = 0 \tag{3}$$

Other contributions to the flow are assumed to depend on y through single trigonometric lines. The streamwise part is taken as $u = u_b + \tilde{u} = [1 + U_1(x, z, t)] \sin(\beta y)$ and equation (1) shows that the associated span-wise and cross-stream components necessarily read $w = W_1(x, z, t) \sin(\beta y)$ and $v =$

$V_1(x,z,t)\cos(\beta y)$, with U_1, V_1, and W_1 linked by the projection of the continuity condition onto $\sin(\beta y)$:

$$\partial_x U_1 + \partial_z W_1 = \beta V_1 \tag{4}$$

Accordingly, the pressure field must be taken as $p = P_0(x,z,t) + P_1(x,z,t)\sin(\beta y)$. Inserting these expressions in the Navier–Stokes equations and truncating the Galerkin expansion at lowest significant order (first-harmonic approximation), denoting $\Delta_2 = \partial_{xx} + \partial_{zz}$ the in-plane Laplacian, we obtain:

(i) projection of the Navier–Stokes equations onto constant y-functions:

$$\partial_t U_0 + N_{U_0} = -\partial_x P_0 - \tfrac{1}{2}\partial_x U_1 - \tfrac{1}{2}\beta V_1 + R^{-1}\Delta_2 U_0 \tag{5}$$
$$N_{U_0} = U_0\partial_x U_0 + W_0\partial_z U_0 + \tfrac{1}{2}U_1\partial_x U_1 + \tfrac{1}{2}\beta V_1 U_1 + \tfrac{1}{2}W_1\partial_z U_1$$
$$\partial_t W_0 + N_{W_0} = -\partial_z P_0 - \tfrac{1}{2}\partial_x W_1 + R^{-1}\Delta_2 W_0 \tag{6}$$
$$N_{W_0} = U_0\partial_x W_0 + W_0\partial_z W_0 + \tfrac{1}{2}U_1\partial_x W_1 + \tfrac{1}{2}\beta V_1 W_1 + \tfrac{1}{2}W_1\partial_z W_1$$

(ii) projection onto $\sin(\beta y)$ (streamwise and span-wise directions):

$$\partial_t U_1 + N_{U_1} = -\partial_x P_1 - \partial_x U_0 + R^{-1}(\Delta_2 - \beta^2)U_1 \tag{7}$$
$$N_{U_1} = U_1\partial_x U_0 + U_0\partial_x U_1 + W_1\partial_z U_0 + W_0\partial_z U_1$$
$$\partial_t W_1 + N_{W_1} = -\partial_z P_1 - \partial_x W_0 + R^{-1}(\Delta_2 - \beta^2)W_1 \tag{8}$$
$$N_{W_1} = U_1\partial_x W_0 + U_0\partial_x W_1 + W_1\partial_z W_0 + W_0\partial_z W_1$$

(iii) projection onto $\cos(\beta y)$ (cross-stream direction):

$$\partial_t V_1 + U_0\partial_x V_1 + W_0\partial_z V_1 = -\beta P_1 + R^{-1}(\Delta_2 - \beta^2)V_1 \tag{9}$$

The origin of each term in the model is easily identifiable. All inertial terms (time dependent or nonlinear) are on the left hand side of (5)–(9) and it is easily checked that the advection terms conserve the kinetic energy contained in the perturbation, namely $E = E_0 + E_1 = (U_0^2 + W_0^2) + \frac{1}{2}(U_1^2 + V_1^2 + W_1^2)$. On the right hand side of (5)–(9) two kinds of linear terms are to be found, those coming from the linearization of $\mathbf{V}\cdot\nabla\mathbf{V}$ around the basic solution (which give its nonnormal character to the linear stability operator) and those expressing the effect of the viscous dissipation.

The pressure field can be eliminated by introducing appropriate stream functions and potentials: a stream function Ψ_0 for the flow component independent of y given by $U_0 = -\partial_z\Psi_0$ and $W_0 = \partial_x\Psi_0$, and two fields Ψ_1 and Φ_1 such that $U_1 = \partial_x\Phi_1 - \partial_z\Psi_1$ and $W_1 = \partial_z\Phi_1 + \partial_x\Psi_1$ for the y-dependent part that contains both irrotational and rotational contributions. The velocity component V_1 is then derived from equation (4) that reads:

$$\beta V_1 = \Delta_2\Phi_1 \tag{10}$$

The proposed model for the three-dimensional modified plane Couette flow is therefore a system of three partial differential equations for the three two-dimensional fields, Ψ_0, Ψ_1, Φ_1. The equations for Ψ_0 and Ψ_1 are obtained by cross-differentiation and subtraction of (5), (6) and (7), (8), which yields:

$$\partial_t\Delta_2\Psi_0 + \partial_x N_{W_0} - \partial_z N_{U_0} = \tfrac{1}{2}\Delta_2[\partial_z\Phi_1 - \partial_x\Psi_1] + R^{-1}(\Delta_2)^2\Psi_0 \tag{11}$$
$$\partial_t\Delta_2\Psi_1 + \partial_x N_{W_1} - \partial_z N_{U_1} = -\partial_x\Delta_2\Psi_0 + R^{-1}(\Delta_2 - \beta^2)\Delta_2\Psi_1 \tag{12}$$

and the equation for Φ_1 by taking the divergence of equations (7)–(9) using (10):

$$\partial_t(\Delta_2 - \beta^2)\Delta_2\Phi_1 + \Delta_2(U_0\partial_x + W_0\partial_z)\Delta_2\Phi_1 - \beta^2[\partial_x N_{U_1} + \partial_z N_{W_1}] = R^{-1}(\Delta_2 - \beta^2)^2\Delta_2\Phi_1 \tag{13}$$

The characteristic linear stability of the basic flow is easily seen to be preserved by the model, the three eigenvalues corresponding to infinitesimal perturbations in the form $\exp[st + \mathrm{i}(\alpha x + \gamma z)]$ being always negative or having always negative real part. $\widehat{\Phi}_1$ turns out to be an eigen-mode with growth rate $s = -(\alpha^2 + \beta^2 + \gamma^2)/R \leqslant -\beta^2/R$ for all (α, γ). $\widehat{\Psi}_0$ and $\widehat{\Psi}_1$ are coupled to each other. Their growth rates are solutions of $s^2 + s(2\alpha^2 + \beta^2 + 2\gamma^2)/R + (\alpha^2 + \gamma^2)(\alpha^2 + \beta^2 + \gamma^2)/R^2 + \frac{1}{2}\alpha^2 = 0$ and a straightforward calculation shows that they remains always bounded from above by $s_{\max} = -\beta^2/2R < 0$.

At the nonlinear stage, steady nontrivial solutions corresponding to those obtained by Waleffe can be found by inserting his ansatz for the flow field, i.e., in his notations [6]:

$$\Psi_0 = -(1/\gamma)U\sin(\gamma z) + (1/\alpha)A\sin(\alpha x) - B\cos(\alpha x)\cos(\gamma z)$$

$$\Psi_1 = -(1/\alpha)C\cos(\alpha x) + D\sin(\alpha x)\cos(\gamma z)$$

$$\Phi_1 = (\beta/\gamma)V\cos(\alpha x) - \beta E\cos(\alpha x)\sin(\gamma z)$$

Preliminary numerical simulations (pseudo-spectral scheme with $N_x = 128$ streamwise modes for $L_x = 64$ and $N_z = 64$ span-wise modes for $L_z = 32$; second-order in time with $\delta t = 0.01$) fulfill our expectations satisfactorily. They have shown the existence of turbulent transients similar to those observed experimentally [5]. *Figure 1* displays the evolution of the amplitude of the fields Ψ_0, Ψ_1, and Φ_1, and the associated energies E_0, E_1, and $E = E_0 + E_1$ during such a transient. Some care is however required in order to prepare initial conditions that do not lead to a run-away of the solution. This phenomenon seems to result from an accumulation in the large scales of the energy extracted from the mean flow. This phenomenon is possibly linked to the low order of the y-truncation, itself implying an inappropriate management of the energy transfer towards small scales in that direction. As a matter of fact, in view of an accurate simulation of the stress-free model, the Galerkin expansion should be continued by adding all terms generated by $\sum_{k\geqslant 1} U_{2k}(x,z,t)\cos(2k\beta y) + U_{2k+1}\sin((2k+1)\beta y)$ and projecting the Navier–Stokes equations onto the relevant series of basis functions. Increasing by one the order of the truncation is a straightforward task. In the range of Reynolds numbers considered, these higher order modes are presumably enslaved to the modes kept up to now. Their adiabatic elimination would thus lead to a set of equations similar to (11)–(13) but with more complicated, expectedly saturating, formally cubic non-linearities.

Focusing on the space-time behavior, our approach appears to be complementary to that followed by Eckhardt et al. [9,10] who were able to point out the existence of transient and sustained chaos by staying

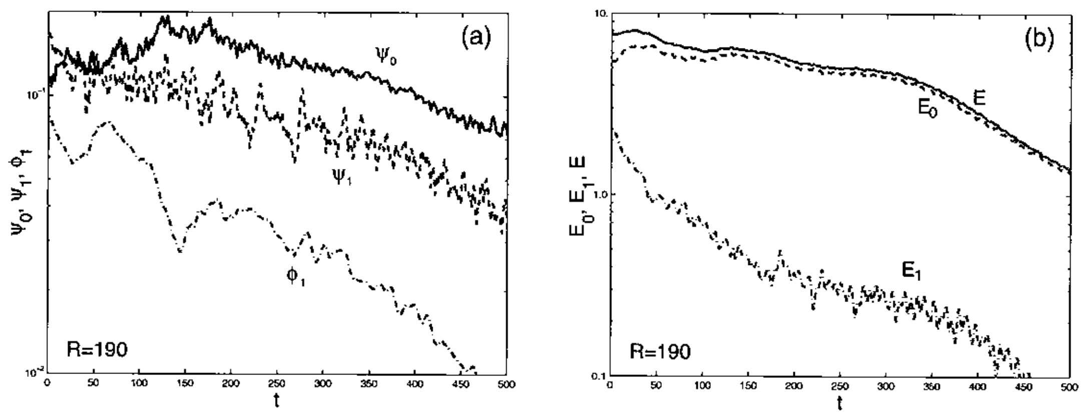

Figure 1. Evolution of the L_∞ norms of Ψ_0, Ψ_1, and Φ_1 (a) and the energies $E_0 = (U_0^2 + W_0^2)$, $E_1 = \frac{1}{2}(U_1^2 + V_1^2 + W_1^2)$, and $E = E_0 + E_1$ contained in the perturbation (b) for $R = 190$.

Figure 1. *Évolution de l'amplitude des champs* Ψ_0, Ψ_1 *et* Φ_1 *et des énergies associées* E_0, E_1 *et* E.

in the context of low-dimensional dynamical systems and taking in-plane periodic conditions at some small size while increasing the cross-stream resolution. We expect that the numerical simulations presently under way will rather allow us to illustrate directly the spatio-temporally intermittent nature of the transition to turbulence in the plane Couette flow and perhaps give interesting hints about less academic situations where turbulent spots are involved, notably Poiseuille and boundary layer flows.

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