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Non-trivial collective behavior in extensively-chaotic dynamical systems: an update

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Abstract

Extensively-chaotic dynamical systems often exhibit non-trivial collective behavior: spatially-averaged quantities evolve in time, even in the infinite-size, infinite-time limit, in spite of local chaos in space and time. After a brief introduction, we give *our* current thoughts about the important problems related to this phenomenon. In particular, we discuss the nature of non-trivial collective behavior and the properties of the dynamical phase transitions observed at global bifurcation points between two types of collective motion.

1. Introduction

Chaos in spatially-extended dynamical systems with local interactions is often *extensive*: the total amount of chaos in the system, as measured, e.g., by the number of positive Lyapunov exponents, is proportional to the system size [1]. In this case, spatiotemporal chaos takes on a rather better-defined character than usual. In particular, one can safely deal with the infinite-size, infinite-time limit, where some statistical analysis might be possible. Such a statistical approach is desirable indeed, as the traditional tools developed for chaos in "small" systems, e.g. the reconstruction of attractors from time-series, cannot be used or do not provide useful information.

The extent to which the methods and ideas of statistical mechanics are relevant to spatiotemporal chaos is a question currently attracting a lot of attention [2], partly because the physics involved is ideally suited, being more complex than chaos but considerably simpler than, say, fully-developed turbulence. In particular, the relevant physical situations present "basic units" upon which the dynamics is built (for example, convection rolls in a Rayleigh–Bénard experiment) [3], which justifies the study of "toy models" with discrete space/time/local variables. A priori simpler than partial

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differential equations, they are also closer to the systems studied in statistical mechanics and thus better candidates for statistical approaches.

"Equilibrium-minded" intuition leads to the belief that, in extensively-chaotic systems, as in their stochastic counterparts, spatially-averaged quantities do not evolve in time, apart from statistical fluctuations which vanish in the thermodynamic limit. This belief has been contradicted recently by the discovery of non-trivial collective behavior (NTCB) [4–6], where macroscopic quantities show a well-defined, usually regular, temporal evolution in spite of the presence of local disorder in space and time. In this paper we give an account of current knowledge about this phenomenon and, in particular, discuss various links that can be drawn with (hopefully better known) statistical mechanics problems in order to understand and describe this type of extensively chaotic regimes.

2. Numerical facts

Since NTCB was first reported [4], many studies were devoted to the subject, mostly involving numerical experiments [5–8]. In [5], a detailed account of extensive simulations of large classes of cellular automata and coupled map lattices (CML) was given and the basic features of NTCB were outlined. Here we focus on a typical example and take this opportunity to recall the general properties of NTCB.

Our example is a d-dimensional hypercubic lattice of logistic maps ($f(X) = 1 - \mu X^2$, $X \in [-1,1]$) coupled "democratically" to their nearest-neighbors,

$$X_i^{t+1} = \frac{1}{2d+1} \sum_{j \sim i} f(X_j^t) = 1 - \frac{\mu}{2d+1} \sum_{j \sim i} X_j^{t\,2}\,, \tag{1}$$

where subscripts denote the position of lattice sites, superscripts the discrete time, and the sum includes site i. In a typical numerical experiment, the simplest macroscopic quantity, i.e. the spatial average $c^t = (\sum_{i=1}^{N} X_i^t)/N$, where N is the number of sites in the lattice, is monitored along time. Figs. 1a–c show that c^t evolves regularly in time: this is a *collective* effect. For $\mu > \mu_\infty \simeq 1.41$, the sites of the lattice are not synchronized with each other, the system is chaotic, and the instantaneous distributions of site values are broad (Fig. 1d). Each site follows a chaotic evolution which has no apparent relation to the global one: this is a *non-trivial* collective effect. The global variables do show fluctuations (around the collective motion), but these fluctuations decrease like $1/\sqrt{N}$, leading to a well-defined thermodynamic limit. Local disorder is also evidenced in the absence of large-scale structures in space (Fig. 2).

More generally, numerical experiments have shown that:

- When observed, the same NTCB is reached from almost all initial conditions: NTCB are attracting regimes, with large, finite-measure basins of attraction.
- The collective motion is robust to small modifications of the local rules, and to the addition of some small amount of noise.

- Synchronous updating (intrinsic to dynamical systems) is essential for the emergence of NTCB.
- NTCB are all the more frequent and temporally complex as the space dimension d is large.
- The collective motion is sensitive to the precise geometry of the lattice and the coupling.

These numerical facts raise three important questions:

(i) How can the collective motion be predicted from the local rule?
(ii) What insures the existence of NTCB in the infinite-size, infinite-time limit?
(iii) What is the nature of the bifurcations observed at the global level (Fig. 1a)?

The answers are only partially known today, as we will show in the following.

3. Mean-field and cluster approximations

Numerical evidence for various types of NTCB has triggered a number of works discussing their nature, their existence, or even non-existence in the thermodynamic limit [7]. The controversy was fueled by the counterintuitive character of NTCB in an equilibrium statistical mechanics context, and by the inability of simple mean-field approximations to account for them (in particular, they produce an effective map for the evolution c^t which cannot have quasiperiodic states as in Fig. 1b). Progress was made recently when it was shown that cluster expansions incorporating an exact treatment of correlations up to a certain cut-off distance can reproduce the collective dynamics, albeit in an approximate way [9]. We now sketch how this is done on the CML introduced above.

Taking spatial averages, evolution rule (1) yields an infinite hierarchy of equations,

$$\begin{aligned}
\langle X_i\rangle^{t+1} &= 1-\mu\langle X_i^2\rangle^t\,,\\
\langle X_i^2\rangle^{t+1} &= 1-2\mu\langle X_i^2\rangle^t+\left(\frac{\mu}{2d+1}\right)^2\sum_{j,k\in\mathcal{V}_i}\langle X_j^2X_k^2\rangle^t\,,\\
\langle X_i^2X_j^2\rangle^{t+1} &= \cdots\,.
\end{aligned}\qquad (2)$$

This hierarchy can be reduced when taking symmetry properties into account. Moments $\langle X_{i_1}^{\alpha_1}\cdots X_{i_n}^{\alpha_n}\rangle^t$ can be classified according to their geometrical support and their set of weights α_i. A two-step truncation and closure is necessary to transform (2) into a self-consistent, finite-dimensional map: a geometrical truncation is performed on the cumulants, based on the fast decay of correlations in space. At the same time, the maximum total order of moments/cumulants is limited, which completes the closure scheme. This "analytical" truncation is founded on the observed decay of cumulants with their total order. Other problems, which are detailed in [9], arise along the way but we do not mention them here for the sake of simplicity. Similarly, we do not describe here the numerical implementation of the scheme, which involves rather complex programming.

Results on the simple case of the $d = 2$ lattice of logistic maps are in excellent agreement with the numerics (Fig. 1a). It is not clear at this point whether this scheme will be equally successful for more complex cases. At any rate, it possesses several advantages:

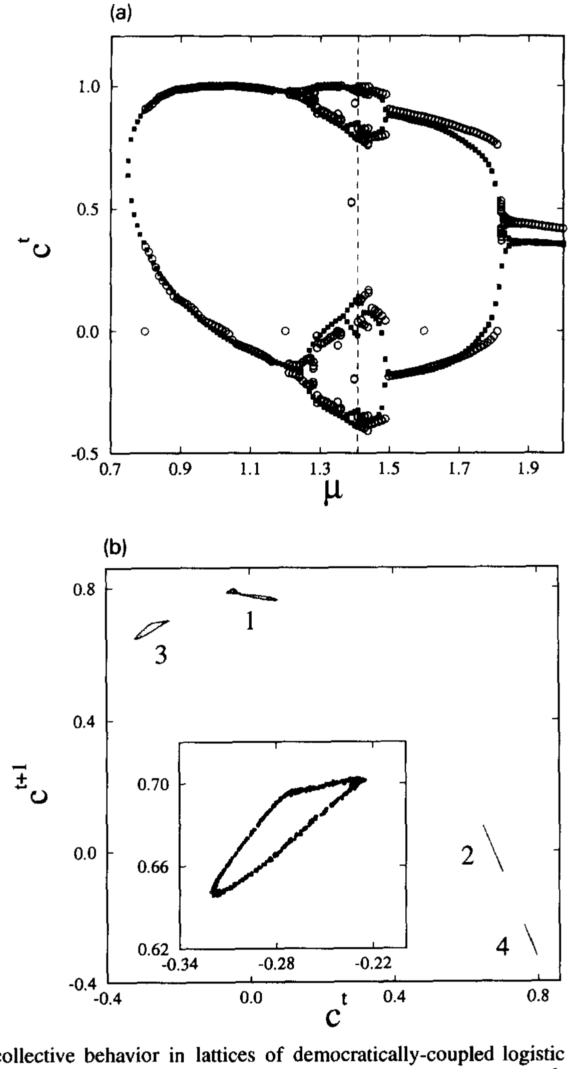

Fig. 1. Non-trivial collective behavior in lattices of democratically-coupled logistic maps. (a) Bifurcation diagram of the global variable c^t for $d = 2$, obtained from simulations of a $N = 1024^2$ site lattice with periodic boundary conditions (filled squares). Superimposed (open circles) is the result of the cluster expansion described in Section 3 for a maximum order 2 and a cut-off distance of 3 sites. Agreement is very good except near the transition points where the expansion is not expected to hold due to long-range correlations. The dashed line is drawn at $\mu = \mu_\infty$. (b) Return plot for c^t in a $d = 5$ lattice of $N = 35^5$ sites for $\mu = 1.71$. The collective motion consists of a period-4 main cycle composed with a quasiperiodic cycle. The numbers near the 4 tori indicate the order of the main periodic cycle. The insert is a close-up of one of these tori. (c) Time-series of c^t shown every 20 timesteps for the same system as in (b). (d) Instantaneous distributions of site values for the $d = 2$ lattice in a period-four collective cycle ($\mu = 1.48$).

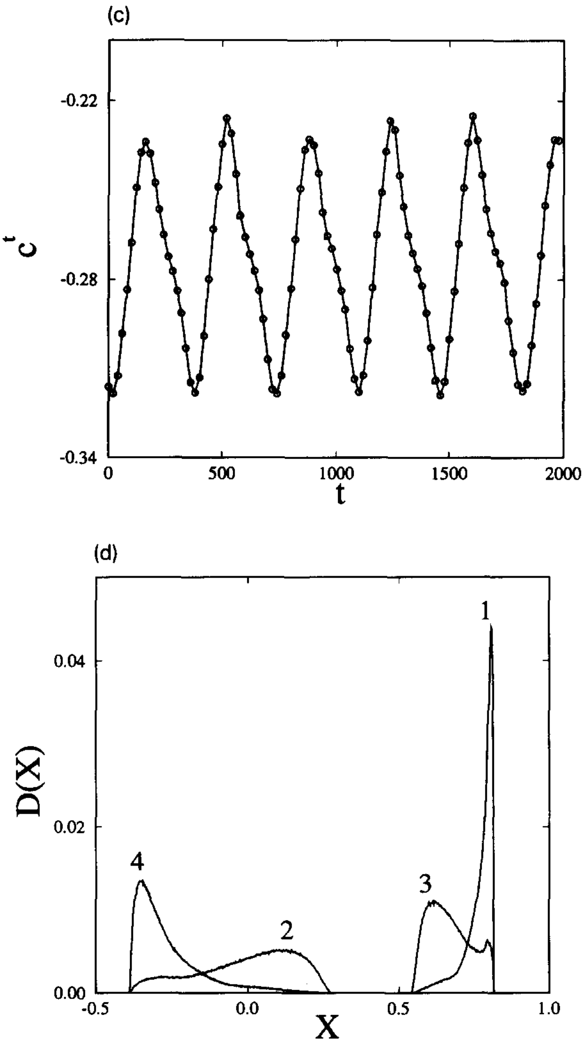

Fig. 1 — continued.

it is systematic, has no adjustable parameters, and depends only on the choice of the decorrelation distance and the total order retained. It can be applied easily to the case of cellular automata, where it was actually first tested [10].

It is thus possible to predict the collective motion from the local rules, at least in the simple cases treated so far. This, in turn, sheds light on the nature of NTCB.

4. Nature of non-trivial collective behavior

Apart from predicting the collective dynamics from the rules, the cluster expansion sketched above proves that the collective motion exists already at a mesoscopic scale

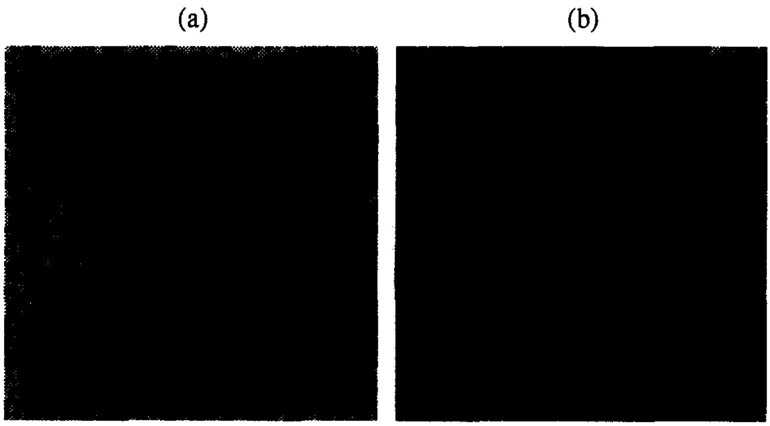

Fig. 2. Snapshots of a $d = 2$ lattice of 256^2 democratically-coupled logistic maps in a period-2 collective cycle ($\mu = 1.7$). Grey scale for X between -1 (black) and 1 (white). Two consecutive timesteps are shown.

(of the order of the cut-off distance), but in a noisy fashion. Consequently, the problem of the existence of NTCB in the thermodynamic limit can be seen as the problem of the "dynamical synchronization" of these noisy sub-units. Obviously, these sub-units are coupled together. The dominant mode of coupling is probably diffusive (the democratic coupling in (1) is a limit case of diffusive coupling). We have thus three main ingredients in this new picture of systems exhibiting NTCB: individual mesoscopic units with approximately the same dynamics as the collective motion are coupled diffusively to each other and subjected to some "noise".

In this perspective of Langevin-like equations, the problem of the thermodynamic limit is akin to that of the existence of long-range order in stochastic systems. One key specificity of the dynamical systems studied here, though, is that the "noise" is not external; it is intrinsic and arises from a large-scale, "renormalized" view of the local deterministic chaos. Two remarks have to be made at this point: first, it is not clear how well the emergent mesoscopic dynamics can be separated from this effective noise. Furthermore, it is by no means obvious that this effective noise should be delta-correlated white noise, since the underlying determinism could generate long-range correlations. In fact, this turns out to be a crucial question, as most of our knowledge on stochastic systems is confined to the case of Gaussian noise.

In spite of these difficulties, this approach has already borne fruit and established more firmly the existence of NTCB in the thermodynamic limit in the case where the collective motion can be described via a continuous phase variable (as in Fig. 1c,d) [10]. The mesoscopic units can also be described via a local phase variable $\phi(x,t)$. The problem of the thermodynamic limit is then equivalent to the existence of long-range rotating order in a noisy medium. If the phase variables now evolve on the real axis and not on the unit circle, this oscillating medium becomes a stochastic interface advancing in time with a velocity equal to the phase velocity. Long-range rotating order in the oscillating medium is realized when the interface is smooth, i.e. when its mean width does not diverge with system size.

Recently, it has been shown [10] that a $d = 3$ cellular automaton exhibiting quasiperiodic collective behavior is well described, at large scales, by the Kardar–Parisi–Zhang (KPZ) [11] model of stochastic interfaces in its smooth phase, insuring the existence of long-range rotating order, and thus of the NTCB in the thermodynamic limit.

This approach, if valid for all NTCB with a continuous phase variable, has general implications: the KPZ equation has no smooth phase for $d \leq 2$, and thus no such NTCB should be observed for $d = 1$ and $d = 2$. Similarly, knowledge of the KPZ equation provides predictions about the correlations on the original extensively-chaotic dynamical systems. Finally, the validity of this KPZ "Ansatz" is intimately related to the Gaussian character of the "effective noise" mentioned above.

Somewhat surprisingly, the status of *periodic* collective behavior is less satisfactory: no definitive argument or theory has as yet been given to establish their existence in the thermodynamic limit. As discussed by Pomeau [8], the problem probably lies in the subtle differences between the "effective noise" and externally-imposed Gaussian noise. A better understanding and direct measurements of the effective noise are needed in order to clarify this point.

5. Global bifurcations and phase transitions

When a control parameter can be continuously varied, transitions from one type of NTCB to another can be observed. Whereas, at the level of global, collective variables, these transitions are akin to bifurcations in small dynamical systems (see for example Fig. 1a and [5]), they can be looked at as *phase transitions*, in spite of an a priori broken detailed balance, of the absence of a free energy, etc. In fact, the transitions observed can be very complex, with no analog in equilibrium situations, but in the $d = 2$ CML of Fig. 1a and many other cases, all the features of traditional second-order phase transitions are present. Fig. 3a illustrates one of the simplest cases, the transition from a stationary state to a period-2 collective behavior. In the transition region, the system is not homogeneous in space and shows domains on all scales. The question of possible universality classes naturally arises. In particular, one can wonder whether these non-equilibrium dynamical phase transitions, occurring in extensively chaotic dynamical systems, belong to any of the well-known *equilibrium* universality classes. In this respect, the period-doubling transitions in Fig. 1a are natural candidates for the Ising universality class. Indeed a magnetization $M_{T:2T} = \langle \sum_i (X_i^{t+T} - X_i^t) \rangle / N$ and a susceptibility $\chi_{T:2T} = \langle \sum_i (X_i^{t+T} - X_i^t - M_{T:2T})^2 \rangle / N$ can be defined, and the machinery of finite-size scaling used to estimate critical exponents (Fig. 3). In such a nonequilibrium situation, one has – as of today – no other solution than to perform numerical simulations. Results obtained so far, if they confirm that the finite-size scaling hypothesis is indeed justified, indicate on the other hand that the critical exponents cannot be in the Ising universality class, nor do they have the mean-field values (Table 1) [12].

Ongoing work aims at determining whether some kind of universality can be detected

Table 1
Critical exponents for the period-doubling transitions of the $d = 2$ CML described in Fig. 1a

Transition	β	γ	ν
Ising	0.125	1.75	1.0
P1:P2	[0.096,0.128]	[1.36,1.69]	[0.81,0.94]
P2:P4	[0.109,0.161]	[1.31,1.65]	[0.76,0.89]
P4:P8	[0.139,0.185]	[1.33,1.49]	[0.82,0.88]
Mean-field	0.5	1.0	0.5

The brackets indicate the limit values allowed by our numerical data.

in various transitions of a similar type. It is too early to conclude on this point, but it should already be stressed that, coming back to the discussion above on the nature of NTCB, these "unexpected" critical properties also point out the role of chaotic fluctuations in building an "effective noise" which is probably at the origin of the departure from equilibrium universality classes.

More generally, the respective roles of synchronous updating, broken detailed balance, and non-Gaussian effective noise have to be carefully asserted before a satisfactory understanding can be reached.

6. Conclusion

If a coherent and hopefully convincing picture of NTCB has started to emerge, much remains to be done to complete it:

- The cluster expansion described in Section 3 has to be tested further in various CML and CA models. It could also provide a better understanding of the important correlations at the origin of the collective motion, a knowledge which could lead to

(a) (b)

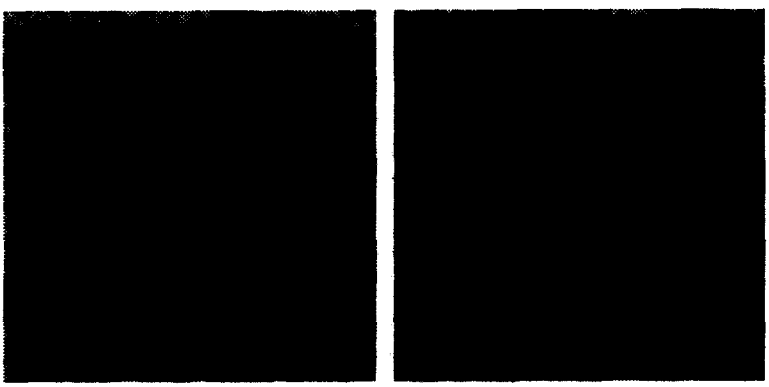

Fig. 3. Two-dimensional lattice of coupled logistic maps at the transition point between period-1 and period-2 collective cycle. (a),(b) Two consecutive snapshots showing the clustering typical of the critical region. (c),(d) Finite-size effects on the "magnetization" M and the "susceptibility" χ.

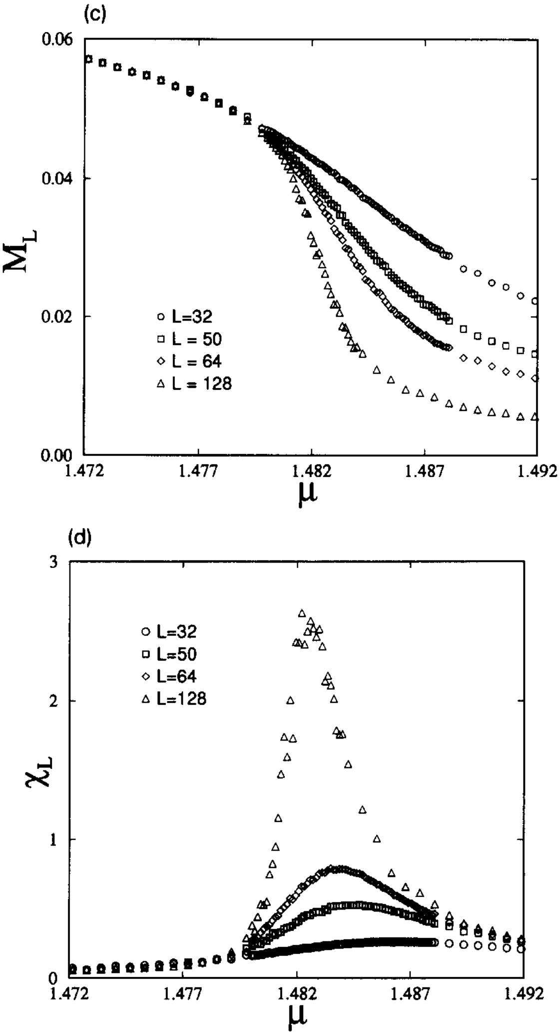

Fig. 3 — continued.

simple "phenomenological theories" of NTCB.

- A direct appreciation of the "effective noise" in a large-scale description of NTCB is a crucial step to further justify its existence in the thermodynamic limit. This should help reach a better understanding of periodic collective behavior, as well as provide clues to the problem of universality of the continuous phase transitions described in Section 5.

We would like now to briefly comment on the mathematical setting in which, we believe, NTCB is best approached. Losson and Mackey [13] have recently investigated the properties of the Perron–Frobenius operator (PFO) for coupled map lattices and, in particular, the notion of *asymptotic periodicity*, already introduced to describe ensemble properties of single maps. The PFO governs the evolution of probability densities in phase space. At this level of *ensemble dynamics*, NTCB reflects the spectral properties of the PFO. For example, periodic NTCB probably implies that the PFO is *asymptotically periodic* with a period at least equal to that observed numerically. To account for all the known types of NTCB, the notion of asymptotic periodicity has to be both extended and made more precise [14]. Unfortunately, the spectral properties of the PFO are very difficult, if not impossible, to determine exactly, even in the simplest cases such as lattices of piecewise-linear maps. It is likely that one has to turn to approximations to be able to deal with realistic cases. As a matter of fact, the cluster expansion described above provides, to some extent, a finite-dimensional functional approximation of the PFO, since it governs a finite set of moments of the probability densities which normally evolve under the action of the PFO [9]. The global bifurcations between two types of NTCB described in section 5 are probably related to a change in the spectral properties of the PFO [14]. This remains to be investigated in detail, in particular the question of the extent to which spectral transitions in the PFO imply phase transitions in the original dynamical system.

Finally, the case of *globally-coupled* systems should be studied in the light of the ideas developed for locally-coupled systems. Although reported under various other denominations ("breakdown of the law of large numbers", "hidden coherence", etc.) [15–18], NTCB do also appear in these systems, and they display an even richer spectrum of possibilities. In particular, it was noted by Nakagawa and Kuramoto [19] and by Rappel and Hakim [20] for oscillators and by Kaneko [21] for maps, that *chaotic* collective behavior is possible in globally-coupled systems, contrary to the locally-coupled case (where such behavior has not yet been reported). In this fascinating behavior, chaotic degrees of freedom produce a collective motion consisting of an altogether different type of chaos. The questions raised for NTCB in locally coupled systems hold in this case as well, perhaps even more acutely since no direct analogies with traditional problems of statistical mechanics are available. These questions...

- How can the collective motion be predicted?
- Can chaotic NTCB exist in the thermodynamic limit?
- What is a "phase transition" to collective chaos?

...are left for future work.

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