Universal Critical Behavior in Two-Dimensional Coupled Map Lattices

Philippe Marcq,[1] Hugues Chaté,[1,2] and Paul Manneville[1,2]
[1]*Service de Physique de l'Etat Condensé, Commissariat à l'Energie Atomique–Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France*
[2]*LadHyX–Laboratoire d'Hydrodynamique, Ecole Polytechnique, 91128 Palaiseau, France*
(Received 20 October 1995)

We numerically investigate the critical properties of nonequilibrium continuous phase transitions in two-dimensional, synchronously updated lattices of coupled chaotic maps. A finite-size scaling analysis provides evidence for the existence of a new universality class, characterized by a correlation-length exponent $\nu = 0.89 \pm 0.03 < \nu_{\rm Ising} = 1.0$, while the exponent ratios β/ν, γ/ν, and the amplitude ratio U^* are consistent with the 2D Ising universality class. The standard value of ν is recovered for *asynchronous* updating rules. [S0031-9007(96)01566-9]

PACS numbers: 05.45.+b, 05.70.Jk, 47.27.Cn, 64.60.Cn

Nonequilibrium phase transitions in driven diffusive systems have attracted substantial interest recently [1]. It is natural to ask which notions and results derived from equilibrium statistical mechanics remain relevant when detailed balance is broken. For stochastic systems at least, the interplay of numerical experiments and field-theoretic techniques has led to rapid progress, indicating that nonequilibrium transitions generally display more variety than their equilibrium counterparts.

Phase transitions in deterministic, chaotic, spatially extended dynamical systems [2] remain at present far less well understood, partly because experimental [3] and numerical [4] studies of systems large enough to meaningfully approximate the infinite-size, "thermodynamic" limit have only recently become possible. That averaged quantities are well defined in this limit is ensured by an apparently generic feature of homogeneous, extended dynamical systems: Through its quantifiers, such as Lyapunov spectra and entropies, deterministic chaos becomes *extensive* above a given (and small) system size [5]. This property validates statistical analyses, since such extended systems can be viewed as ensembles of smaller, coupled, identical subsystems. Examples include elements of all classes of extended dynamical systems, from lattice models such as cellular automata and coupled map lattices (CMLs) to partial differential equations (PDEs).

CMLs, i.e., lattices of interacting dynamical systems with continuous phase space and discrete time, were specifically designed to overcome numerical limitations associated with simulations of PDEs (see [6] for recent reviews). In their usual form, they are the simplest examples of a wider class of extensively chaotic reaction-diffusion systems. Numerical simulations have shown that a variety of transitions between chaotic phases occur in generic CMLs when a continuous control parameter is varied [7]. These transitions are characterized by qualitative changes of statistical quantifiers, e.g., invariant densities and space-averaged activities. Although analytical results remain scarce and, so far, limited to behavior far from transition points [8], these transitions are generally believed to persist in the thermodynamic limit [7].

At a theoretical level, attempts to understand phase transitions in extended dynamical systems have been made within the framework of field-theoretic techniques originally developed for stochastic systems. For a nonconserved, scalar order parameter, it was argued in [9] that phase transitions in generic nonequilibrium systems made up of locally interacting subunits should fall into the equilibrium Ising universality class, provided that an Ising-type symmetry be respected microscopically. This result, later generalized to encompass cases where no microscopic symmetry is present [10], depends on two assumptions, so far unchecked for extended dynamical systems: (i) The system can be accurately described by a stochastic differential equation arbitrarily close to the transition (and thus for large space-time scales); and (ii) an expansion in powers of $\epsilon = d_c - d$ is valid, where the upper critical dimension d_c is assumed to be equal to 4. Recently the relevance of these ideas to deterministic systems has been tested numerically [11]. A two-dimensional CML with a built-in microscopic up-down symmetry was introduced and shown to exhibit an Ising-like continuous ordering transition. Its critical behavior was found to be "consistent" with the Ising universality class. However, the system sizes considered (up to 16^2 for a finite-size scaling analysis) were too small to allow a direct, reliable measurement of critical exponents [12]. In this Letter, we report extensive numerical simulations of the CML introduced in [11], as well as of other two-dimensional CMLs with the same symmetry. The system sizes we consider and the statistical accuracy we reach permit a direct measurement of critical indices along the lines of *equilibrium* finite-size scaling theory [13,14]. The fact that finite-size scaling theory applies in this context provides additional evidence that the transitions persist in the infinite-size limit [15]. Our results temper the conclusions of [11], in that the CML introduced there is found *not* to belong to the 2D Ising universality class. While the exponent ratios β/ν

and γ/ν agree (within error) with their respective Ising values of 1/8 and 7/4, the value of the correlation-length exponent ν is significantly lower than $\nu_{\rm Ising} = 1$. Mutually consistent results are obtained for all the nonequilibrium continuous phase transitions we study, provided that Ising-like up-down symmetry is respected. Our global estimate for ν is 0.89 ± 0.03 [16].

Microscopic details such as lattice geometry or the choice of a local chaotic map are thus irrelevant. Furthermore, this result is resistant to the addition of a small level of Gaussian noise in the evolution rule, indicating that non-Ising critical properties are not related to the boundedness of the (nonthermal) microscopic noise produced by chaotic maps. We find that this new universality class derives from the particular nature of updating in extended dynamical systems. Indeed, CMLs, as well as numerical implementations of PDEs, are updated *synchronously*, as opposed to the site-by-site updating scheme associated with, say, standard Monte Carlo simulations of equilibrium systems. In all the cases considered, the Ising exponents predicted in [9] are recovered as soon as the CML's updating rule is made asynchronous. In analogy with far-from-equilibrium phase transitions of stochastic systems [1], nontrivial *static* critical behavior is determined by a *dynamical* property.

Our results are illustrated by the following example (a more extensive discussion is deferred to an upcoming article [16]). The choice of model is motivated by the undesirable presence of strong corrections to scaling in the CML introduced in [11], where crossover to asymptotic scaling occurs for a size as large as 32^2 [16]. Replacing the original nearest neighbor coupling by a locally anisotropic, three-neighbor rule allows most corrections to scaling to be eliminated. A possible explanation for this observation is the absence, in this case, of the small, antiferromagnetically ordered domains observed in [11], which may obscure the dominant ferromagnetic order for small sizes. We retain the original local map:

$$f(x) = \begin{cases} -3x - 2 & \text{for } -1 \le x \le -\frac{1}{3}, \\ 3x & \text{for } -\frac{1}{3} \le x \le \frac{1}{3}, \\ -3x + 2 & \text{for } \frac{1}{3} \le x \le 1. \end{cases}$$

This piecewise linear, odd function of $[-1, 1]$ onto itself is everywhere expanding, and thus chaotic, with an invariant density uniform on $[-1, 1]$. We consider two-dimensional, square lattices of L^2 sites. The CML's evolution rule reads

$$\begin{aligned} x_{2i,j}^{t+1} &= (1 - 3g)\, f(x_{2i,j}^t) + g[f(x_{2i,j-1}^t) \\ &\quad + f(x_{2i-1,j}^t) + f(x_{2i+1,j}^t)], \\ x_{2i-1,j}^{t+1} &= (1 - 3g)\, f(x_{2i-1,j}^t) + g[f(x_{2i-1,j+1}^t) \\ &\quad + f(x_{2i,j}^t) + f(x_{2i-2,j}^t)], \end{aligned} \tag{1}$$

where superscripts and subscripts, respectively, denote temporal and Cartesian spatial coordinates, and g is the coupling strength. Although different rules are used for sites with odd and even i index, we checked that the CML does not decouple into two independent sublattices. The analogy with equilibrium Ising models is built upon the definition of an order parameter $M_L = \langle |m_L^t| \rangle$, where m_L^t is the fluctuating "magnetization"

$$m_L^t = \frac{1}{L^2} \sum_{i,j} \mathrm{sgn}(x_{i,j}^t).$$

Dynamical invariance under "spin" reversal results from the combination of an odd local map f with a linear coupling rule. For both synchronous and asynchronous updating, numerical experiments performed on lattices of sizes up to $L^2 = 1024^2$ indicate the existence of two distinct nonequilibrium steady states: a "paramagnetic" phase at low g, with vanishing magnetization M_L, and a "ferromagnetic" phase at larger g, characterized by a nonzero magnetization and long-range spatial order.

We now focus on the case of synchronous update: All sites are updated simultaneously according to Eq. (1). We demonstrate that both phases are extensively chaotic [5] by studying Lyapunov spectra of the lattice dynamical system for different sizes. The spectra superpose for $L^2 \ge 8^2$ when plotted as a function of l/L^2, where l indexes exponents of decreasing magnitude. Chaotic activity persists for nonzero values of g, and is distributed over the whole lattice. A coupling-driven phase transition occurs at $g = g_c \sim 0.251$, bearing all the hallmarks of equilibrium continuous transitions. No hysteresis is observed. The magnetization M_L goes smoothly to zero at g_c. The "susceptibility" χ_L, defined as

$$\chi_L = \left\langle \frac{1}{L^2} \sum_{i,j} [\mathrm{sgn}(x_{i,j}^t) - M_L]^2 \right\rangle,$$

and the correlation length ξ exhibit sharp maxima close to g_c, where ξ characterizes the exponential decay of equal-time, two-point correlation functions of the field x_i^t. Simulations of large systems (up to 1024^2) suggest the presence of well-defined, equilibriumlike power laws in the infinite-size limit:

$$M \propto (g - g_c)^\beta, \quad \text{for } g \ge g_c, \tag{2}$$

$$\chi \propto |g - g_c|^{-\gamma}, \tag{3}$$

$$\xi \propto |g - g_c|^{-\nu}. \tag{4}$$

In this context, *equilibrium* finite-size scaling theory is the only available framework in which critical exponents can be measured reliably (see, e.g., [14] for a review). For isotropic systems, its underlying assumption is the existence of a unique relevant length scale close to the transition: the correlation length ξ. This assumption has recently been substantiated by a numerical study: Other "natural" length scales derived from dynamical properties of CMLs, such as the dimension correlation length, are indeed insensitive to the onset of long-range spatial order

[17]. Dimensional analysis then implies that $\xi \propto L$ at criticality. Combined with Eqs. (2) and (4), this leads to the following scaling form of the probability distribution function P of the fluctuating magnetization m:

$$P(L, m, g) \simeq L^{\beta/\nu}\, \tilde{P}(mL^{\beta/\nu}, (g - g_c)L^{1/\nu}). \quad (5)$$

Our analysis of finite-size effects is performed on lattices of size up to 128^2 with periodic boundary conditions. Random initial conditions are uniformly distributed on $[-1, 1]$. The typical duration of discarded transients is $\mathcal{O}(10^5)$ time steps. For all sizes considered, satisfactory statistical accuracy is reached when total sampling times are $\mathcal{O}(10^7)$ time steps, accumulated over several independent realizations when necessary. In the least favorable case, this amounts to $\mathcal{O}(10^3)$ coherence times. Following a standard procedure, the infinite-size transition point g_c^∞ is first determined independently of other quantities, using Binder's method [13] based on the cumulant ratio $U_L(g) = 3 - \langle (m_L^t)^4 \rangle / \langle (m_L^t)^2 \rangle^2$. When Eq. (5) holds, U_L adopts the scaling form

$$U_L(g) \simeq \tilde{U}((g - g_c^\infty)L^{1/\nu}).$$

Curves of $U_L(g)$ vs g for different L are expected to intersect at a unique, L-independent point of coordinates $(g_c^\infty, U^* = \tilde{U}(0))$ in the scaling regime. For $L < 48$, the abscissas of intersection points of two such curves for consecutive L are found to drift systematically when L increases. Our estimate for g_c^∞ is based on the spatial extension of the (stabilized) intersection region for sizes between 48^2 and 128^2 (see inset in Fig. 1). We find $g_c^\infty = 0.25118(4)$ and $U^* = -1.823(3)$, an amplitude ratio U^* in rough agreement with the 2D Ising universality class. Once g_c^∞ is known, ratios of exponents are derived from the scaling laws [14]

$$M_L|_{g_c^\infty} \propto L^{-\beta/\nu}, \quad (6)$$

$$L^2\, \chi_L|_{g_c^\infty} \propto L^{\gamma/\nu}, \quad (7)$$

$$\partial_g U_L|_{g_c^\infty} \propto L^{1/\nu}, \quad (8)$$

$$\partial_g \ln M_L|_{g_c^\infty} \propto L^{1/\nu}, \quad (9)$$

$$\partial_g \ln\langle (m_L^t)^2 \rangle|_{g_c^\infty} \propto L^{1/\nu}. \quad (10)$$

The values of β/ν, γ/ν, and ν listed in Table I are obtained from linear fits of plots of the relevant observables vs L in log-log scale. The sizes considered are large enough for corrections to scaling to be negligible: $L > 12$ for $1/\nu$, $L > 24$ for other exponent ratios. The values of β/ν and γ/ν are compatible with the 2D Ising universality class. However, Eqs. (8) to (10) consistently yield a value of ν incompatible with $\nu_{\rm Ising} = 1.0$ (see Fig. 1). For all observables A defined on the left hand side of Eqs. (6) to (10), our data are qualitatively consistent with the corrections to scaling expected at equilibrium [14]:

$$A = L^{\varphi}[a_0 + a_1 L^{-\omega} + \cdots + b_1(g - g_c^\infty)L^{1/\nu} + \cdots], \quad (11)$$

where φ is the appropriate critical exponent. In renormalization group theory, the subdominant exponent ω and the last term in Eq. (11) express corrections caused, respectively, by an irrelevant scaling field and by nonlinearities of the relevant scaling fields. Analyzing Eqs. (8)–(10) in light of Eq. (11) shows that our data are incompatible with a crossover to $\nu = \nu_{\rm Ising}$ at larger sizes [16]. In the case of Eq. (6), fitting our data according to Eq. (11) allows data from smaller lattices to be incorporated, down to 12^2. This fit suggests that $\omega = 6(2)$. Finally, the relevance of Eq. (5) is illustrated in Fig. 2 by a collapse of magnetizations measured at numerous sizes and coupling strengths close to criticality.

A continuous phase transition occurs at a somewhat lower critical coupling when the update is asynchronous, i.e., when Eq. (1) is applied site by site, for instance, in fixed, sequential order. A finite-size scaling analysis performed close to $g_c^\infty = 0.15847(2)$, with the same level of accuracy as before, yields critical indices fully consistent

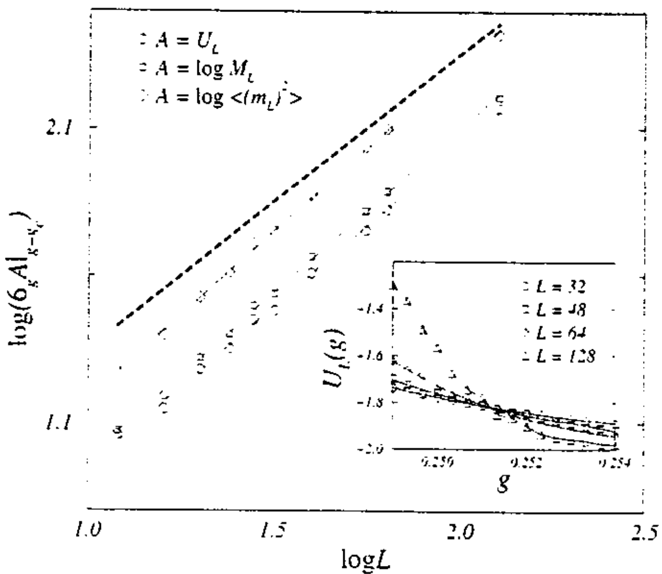

FIG. 1. Direct measure of ν. Error bars are smaller than symbols. The solid lines are fitted to our data, with slopes $1/\nu = 1.110$, 1.116, and 1.119 (from bottom to top). The slope of the dashed line is set to 1. Inset: measure of g_c^∞.

TABLE I. Comparison of 2D Ising exponents with results for CMLs with synchronous (S) and asynchronous (A) update. Numbers in brackets correspond to one standard deviation. The main contribution to error bars stems from the uncertainty on g_c^∞.

Model	ν	β/ν	γ/ν	β	γ
2D Ising	1.0	0.125	1.75	0.125	1.75
S	0.895(12)	0.131(6)	1.74(3)	0.117(7)	1.55(5)
A	1.04(5)	0.123(8)	1.72(2)	0.124(20)	1.79(11)

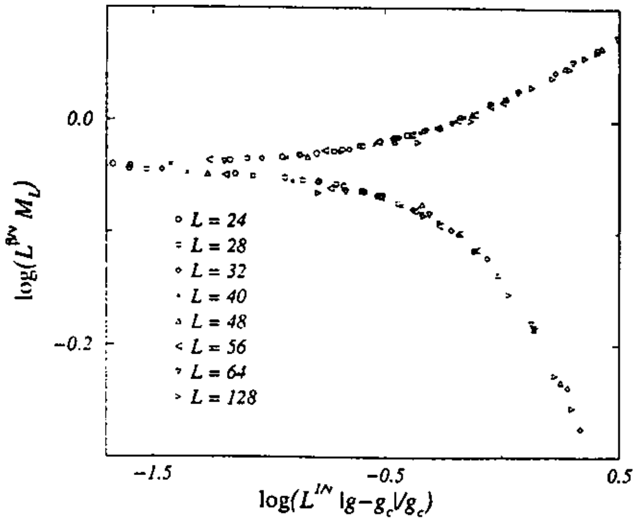

FIG. 2. Collapse of the magnetization M_L (synchronous update), for $0.24 \leq g \leq 0.26$ and $24 \leq L \leq 128$. The parameters are $g_c = g_c^* = 0.25118$, $\nu = 0.895$, $\beta/\nu = (\beta/\nu)_{\rm Ising} = 0.125$.

with 2D Ising values (see Table I). This result holds for other choices of the lattice geometry and/or the local map, even though accurate measurements may be made more difficult by critical coherence times much larger than for synchronous update [16]. Note, moreover, that the continuous Ising-like transition of a CML respecting detailed balance, but updated asynchronously on two sublattices, does belong to the 2D Ising universality class [18].

A few comments are in order. First, measured critical exponents are consistent with the hyperscaling relation $2\beta + \gamma = d\nu$ typical of fluctuation-dominated transitions [19], where $d = 2$ is the space dimensionality [15]. Second, the quantities U^*, β/ν, and γ/ν may be considered as "superuniversal," i.e., independent of the type of update. In the framework of the renormalization-group analysis of the ϕ^4 model, this remark translates into a distinction between the two relevant renormalization exponents y_h and y_t [19]: $y_h = (\beta + \gamma)/\nu$ is superuniversal, while $y_t = 1/\nu$ depends on the updating rule. We would like to emphasize once more that analogies with equilibrium results must be treated with great care, in particular since (i) the explicit derivation of an effective, coarse-grained description starting from the local evolution rule remains unfeasible; and (ii) the control parameter g is defined at microscopic scales only, and cannot be easily related to a macroscopic "temperature" which has still to be properly defined in such a context [20]. Finally, an interesting open question is whether this new universality extends to *dynamic* critical properties, which are notoriously dependent on the type of update when detailed balance is respected [21].

We wish to acknowledge useful discussions with Henry Greenside, Geoff Grinstein, and David Huse.

[1] B. Schmittmann and R. K. P. Zia, in *Statistical Mechanics of Driven Diffusive Systems*, Phase Transitions and Critical Phenomena Vol. XVII, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1995).
[2] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[3] See, e.g., N. B. Tufillaro, R. Ramshankar, and J. P. Gollub, Phys. Rev. Lett. **62**, 422 (1989); Y. Hu, R. Ecke, and G. Ahlers, Phys. Rev. E **51**, 3263 (1995).
[4] B. Shraiman *et al.*, Physica (Amsterdam) **57D**, 241 (1992); D. A. Egolf and H. S. Greenside, Phys. Rev. Lett. **74**, 1751 (1995); M. C. Cross and D. I. Meiron, Phys. Rev. Lett. **75**, 2152 (1995).
[5] D. Ruelle, Commun. Math. Phys. **87**, 287 (1982); P. Manneville, in *Macroscopic Modeling of Turbulent Flows*, Lecture Notes in Physics Vol. 230, edited by O. Pironneau (Springer-Verlag, New York, 1985).
[6] *A Focus Issue on Coupled Map Lattices*, edited by K. Kaneko [Chaos **2**, 3 (1992)]; *Coupled Map Lattices: Theory and Experiments*, edited by K. Kaneko (World Scientific, Singapore, 1993).
[7] H. Chaté and P. Manneville, Europhys. Lett. **17**, 291 (1992); Prog. Theor. Phys. **87**, 1 (1992).
[8] J. Losson and M. C. Mackey, Phys. Rev. E **50**, 843 (1994); J. Losson, J. Milton, and M. C. Mackey, Physica (Amsterdam) **81D**, 77 (1995).
[9] Y. He, G. Grinstein, and C. Jayaprakash, Phys. Rev. Lett. **55**, 2527 (1985); Y. He, T. Bohr, G. Grinstein, and C. Jayaprakash, Phys. Rev. Lett. **58**, 2155 (1987).
[10] K. Bassler and B. Schmittmann, Phys. Rev. Lett. **73**, 3343 (1994).
[11] J. Miller and D. Huse, Phys. Rev. E **48**, 2528 (1993).
[12] D. Huse (private communication).
[13] K. Binder, Z. Phys. B **43**, 119 (1980).
[14] *Finite-Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).
[15] J. M. Houlrik, I. Webman, and M. H. Jensen, Phys. Rev. A **41**, 4210 (1990); J. M. Houlrik and M. H. Jensen, Phys. Lett. A **16**, 3275 (1992).
[16] P. Marcq, H. Chaté, and P. Manneville (to be published).
[17] C. S. O'Hern, D. A. Egolf, and H. S. Greenside, Phys. Rev. E **53**, 3374 (1996).
[18] H. Sakaguchi, Prog. Theor. Phys. **80**, 7 (1992).
[19] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Oxford University Press, Oxford, 1989).
[20] B. I. Shraiman and P. C. Hohenberg, Physica (Amsterdam) **37D**, 109 (1989); M. S. Bourzutschky and M. C. Cross, Chaos **21**, 73 (1992).
[21] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **58**, 86 (1987).