EUROPHYSICS LETTERS 1 June 1999
Europhys. Lett., **46** (5), pp. 565-570 (1999)

Macroscopic model for collective behavior of chaotic coupled map lattices

A. Lemaître[1,2] and H. Chaté[2,1]

[1] *Laboratoire d'Hydrodynamique, Ecole Polytechnique - 91128 Palaiseau, France*
[2] *CEA - Service de Physique de l'Etat Condensé, Centre d'Etudes de Saclay 91191 Gif-sur-Yvette, France*

(received 30 September 1998; accepted in final form 17 March 1999)

PACS. 05.45−a – Nonlinear dynamics and nonlinear dynamical systems.
PACS. 05.70Ln – Nonequilibrium and irreversible thermodynamics.

Abstract. – We present a simple model for the collective behavior exhibited by chaotic coupled map lattices (CMLs) in any space dimension which takes into account local fluctuations and short-range correlations. This reduces the dynamics of CMLs to the evolution of a one-body distribution function coupled to a coherence length. Quantitatively good results are obtained, especially in the case of coupled tent maps, even though no free parameter is available.

The emergence of collective behavior in large dynamical systems made of coupled chaotic units is now well documented [1-5]. Whether the connections are local or global, form a lattice or a random network, one often observes that quantities averaged over the whole collection of individual units evolve in time, usually in some regular fashion, while strong chaos is present at the microscopic level, and no synchronisation occurs between sites. This constitutes some of the main features of non-trivial collective behavior (NTCB). To them one must add the sometimes implicit requirement that the collective behavior observed is a true macroscopic attractor, *i.e.* that it has a well-defined infinite-size/time limit, and that almost any initial condition leads to it. (The uniqueness of the collective motion is usually granted if the coupling between units is strong enough, with the weak-coupling regime being characterized by multistability [6].)

In lattice dynamical systems such as coupled map lattices (CMLs), NTCB can be seen as long-range order accompanied by the temporal evolution of spatially averaged quantities. Whereas globally or randomly coupled systems are somewhat easier to handle, being closer to some infinite-dimension limit, the case of lattice systems with local interactions is notably hard to approach analytically, especially if one is interested in the strong-coupling case where there is no hope of extending zero-coupling results (like in the so-called anti-integrable limit [7]). One of the outstanding problems concerning NTCB in lattice systems is that of the prediction of the collective motion from the local dynamical rules. In this letter, we present a model accounting for NTCB in lattices of coupled chaotic maps at the macroscopic level.

Specifically, we consider the discrete-time dynamics of a set of variables $\mathbf{X} = (\mathbf{X}_{\vec{r}})_{\vec{r}\in\mathcal{L}}$ lying at the nodes of a d-dimensional hypercubic lattice $\mathcal{L}$. The nodes are updated synchronously at discrete timesteps which involve two stages: firstly, the operator $\mathbf{S}$, applied to $\mathbf{X}^t$, transforms each local variable $\mathbf{X}^t_{\vec{r}}$ by a non-linear local map S, yielding an intermediate configuration

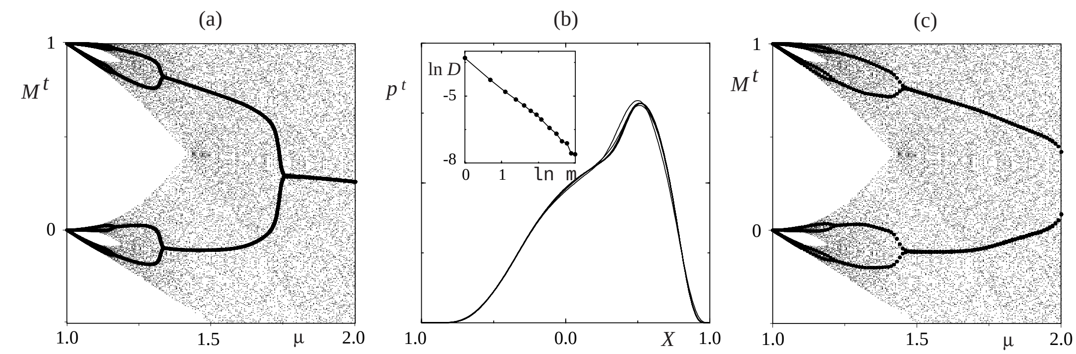

Fig. 1. – Non-trivial collective behaviour in lattices of coupled tent maps ($g = 1/(2d+1)$) with periodic boundary conditions. (a) $d = 2$ bifurcation diagram of the instantaneous spatial average M^t of a lattice of 2048^2 sites (thick curves) and bifurcation diagram of the single tent map (shaded area). (b) Asymptotic single-site pdf p in the stationary regime of $\mathbf{\Delta}^m \circ \mathbf{S}$ at $\mu = 2$, for m varying from 1 to 32; the distance $D = \int (p_m - p_{32})^2$ (insert) shows evidence of convergence with m. (c) Same as (a) for $d = 3$ (128^3 sites).

$\mathbf{X}^{t+\frac{1}{2}}$. Secondly, $\mathbf{X}^{t+\frac{1}{2}}$ is transformed by the diffusive coupling operator $\mathbf{\Delta}$,

$$[\mathbf{\Delta}(\mathbf{X})]_{\vec{r}} = (1 - 2dg)\mathbf{X}_{\vec{r}} + g \sum_{\vec{e}} \mathbf{X}_{\vec{r}+\vec{e}} \, , \tag{1}$$

where g is the coupling strength, and the sum runs over the $2d$ nearest neighbors of every site. The CML dynamics is thus simply written $\mathbf{X}^{t+1} = \mathbf{\Delta} \circ \mathbf{S}(\mathbf{X}^t)$.

In the following, we consider tent maps $S(X) = 1 - \mu|X|$ with $\mu \in [0, 2]$, which leave the interval $I = [-1, 1]$ invariant. For $\mu > \mu_\infty$ ($= 1$ for the tent map), and for strong enough coupling, CMLs $\mathbf{\Delta}\circ\mathbf{S}$ exhibit NTCB [3,8]. The bifurcation diagram of the instantaneous spatial average $M^t \equiv \langle \mathbf{X} \rangle^t$ is displayed in fig. 1 for $g = 0.2$ in dimensions 2 and 3. All sites evolve chaotically and are statistically equivalent. In fact, the instantaneous one-body distribution p^t is smooth, wide, and follows the same collective behavior as its mean M^t. Almost all initial conditions lead to the same macroscopic attractor, well-defined in the infinite-size limit. Only periodic regimes are observed in these cases, but quasiperiodic dynamics appears in higher space dimensions, indicating that the collective motion may have no direct equivalent at the microscopic level.

Standard mean-field approximations, relying only on the simple mean M^t, fail to account for NTCB [3, 9]. In this work, we focus on the instantaneous pdf p^t, which encodes more information about lattice variables than M^t. We now define an effective non-linear operator which governs the dynamics of p^t and accounts for the behavior observed in CMLs. A first attempt towards such an effective description was proposed in [9], via the definition of a "conditional mean-field approximation". We first recall this approach. Applying operator $\mathbf{S}$ to an infinite lattice, p^t is transformed by $\mathcal{P}_S$, the Perron-Frobenius operator associated to the local map:

$$p^{t+\frac{1}{2}}(X) = \mathcal{P}_S[p^t] \equiv \int p^t(Y) \delta\big(X - S(Y)\big) \, \mathrm{d}Y \, .$$

When $\mathbf{\Delta}$ is then applied to $\mathbf{X}^{t+\frac{1}{2}}$, the resulting pdf p^{t+1} depends on the joint pdf of $\mathbf{X}_{\vec{r}}^{t+\frac{1}{2}}$ and of the *local mean field* $\mathbf{m}_{\vec{r}}^{t+\frac{1}{2}} = \sum_{\vec{e}} \mathbf{X}_{\vec{r}+\vec{e}}^{t+\frac{1}{2}}$ at intermediate times. Neglecting short-range correlations leads to the usual mean-field approximations unable to account for NTCB [3, 9]. On the other hand, it is known that keeping only short-range correlations is often sufficient to account for the collective motion [10, 11]. In [9], we proposed that the local mean field $\mathbf{m}_{\vec{r}}$ around a "typical" site with value X should be represented by a fluctuating variable $m = m_{|X}$ correlated with X. This led us to write the local mean field at intermediate times as a sum involving X and N effective uncorrelated variables $\widetilde{X}_j$ distributed according to $p^{t+\frac{1}{2}}$. In this framework, physical space disappears and the coupling stage is replaced by a simple sum

$$X_i^{t+1} = (1-G)X_i^{t+\frac{1}{2}} + \frac{G}{N}\sum_{j=1}^{N} \widetilde{X}_j^{t+\frac{1}{2}} ,$$

where G is an effective coupling strength *a priori* different from $2dg$. From the point of view of the pdf $p^{t+\frac{1}{2}}$, this amounts to apply the convolution operator $\mathcal{D}_{G,N}$, defined by

$$\mathcal{D}_{G,N}\left[\, p\,\right](X) = \frac{1}{1-G}\int p\left(\frac{X-GY}{1-G}\right)\, p_N\left(Y\right)\, \mathrm{d}Y ,$$

where p_N is the distribution of the average of N independent variables distributed according to p. The computation of p_N is done *via* Fourier transform, $\hat{p}_N(\xi) = (\hat{p}(\xi/N))^N$, and can be performed for any $N \geq 1$, not necessarily integer. Parameters G and N can be shown [9] to be related to correlations and fluctuations in the original CML via the relations

$$\frac{\langle X^{t+1} X^{t+\frac{1}{2}}\rangle_c}{\langle X^2\rangle_c^{t+\frac{1}{2}}} = 1-G \quad \text{and} \quad \frac{\langle X^2\rangle_c^{t+1}}{\langle X^2\rangle_c^{t+\frac{1}{2}}} = (1-G)^2 + \frac{G^2}{N} , \tag{2}$$

with $\langle XY\rangle_c = \langle XY\rangle - \langle X\rangle\langle Y\rangle$ and where $\langle\,.\,\rangle$ denotes the p-average. Indeed, the corresponding quantities in the CML are correlations like $\langle[\mathbf{\Delta}(\,\mathbf{X}\,)]_{\vec{r}}\;\mathbf{X}_{\vec{r}}\rangle_c^{t+\frac{1}{2}}$,... where the brackets indicate the spatial average. Since $\mathbf{\Delta}$ is a linear operator, they are simple combinations of the correlations at time $t+\frac{1}{2}$ in a typical neighborhood.

At this stage, the approximation cannot provide any description of the dynamics of the correlations, and G and N were assumed in [9], to be stationary with numerical values chosen somewhat arbitrarily, but in a range suggested by numerical estimates of the above-mentioned correlations. Surprisingly, it accounted well for the shape and dynamics of p^t observed in CMLs, and this for a wide range of N and G, at least for not too high-period collective cycles.

We now go further, and present a new model without any free parameter which insures that the large-period collective cycles observed when $\mu \to \mu_\infty$ are better accounted for. To this aim, we need to incorporate the dynamics of short-range correlations in the model. Their dynamical evolution, for usual CMLs, leads to write hierarchical equations which cannot be easily handled [11]. However, this dynamics takes a particularly simple form in the continuous-space limit defined in [8]. We now present this limit in which most properties of NTCB are preserved, and then proceed to the definition of a closed model for the dynamics of the distribution p^t coupled to the short-range correlations.

The continuous-space limit of CMLs is reached by applying the coupling operator an increasing number of times per iteration, *i.e.* when considering CMLs of the form $\mathbf{\Delta}^m \circ \mathbf{S}$. When $m \to \infty$, $\mathbf{\Delta}^m$ converges to a diffusive operator with a Gaussian kernel, $\mathbf{\Delta}^\infty = \exp[\frac{1}{2}\lambda^2\nabla^2]$. Its coupling range $\lambda = \sqrt{2gm}\|\vec{e}\|$ diverges with m but $\|\vec{e}\|$, the lattice spacing, can be chosen to scale like $1/\sqrt{m}$ so as to keep λ constant. Operator $\mathbf{\Delta}^\infty$ is universal in the sense that all $\mathbf{\Delta}^m$ converge to it provided $\mathbf{\Delta}$ satisfies very general constraints [8].

In space dimensions $d = 2$ and 3, $\mathbf{\Delta}^\infty \circ \mathbf{S}$ presents the same periodic behavior as usual discrete-space CMLs. For example, the pdf p^t for a two-dimensional lattice of coupled tent maps in a fixed-point collective regime is displayed in fig. 1b for increasing values of m. Fast convergence towards a well-defined $m \to \infty$ limit is observed, corresponding to a fixed point of operator $\mathbf{\Delta}^\infty \circ \mathbf{S}$. For $d = 2$ and 3, the bifurcation diagrams obtained at different m are hardly distinguishable from those of fig. 1a, c, although the bifurcation points tend to be shifted to the right as m increases. Moreover, CMLs $\mathbf{\Delta}^m \circ \mathbf{S}$ obey a renormalisation group equation which shows that, in any dimension, the system near μ_∞ is conjugate to another with a larger m [8]. Thus, a faithful model of this continuous limit will also be valid for usual discrete-space CMLs.

Consider the correlation function $C(\vec{x}) = \langle \mathbf{X}_{\vec{r}} \mathbf{X}_{\vec{r}+\vec{x}} \rangle_c$ and its Fourier transform, the structure function $\hat{C}(\vec{k}\,) = |\hat{\mathbf{x}}_{\vec{k}}|^2$, where $\hat{\mathbf{x}}_{\vec{k}}$ is the spatial Fourier transform of the centered field $\mathbf{x}_{\vec{r}} = \mathbf{X}_{\vec{r}} - \langle \mathbf{X}_{\vec{r}} \rangle$. In the continuous limit, the diffusive coupling operator insures that the field $\mathbf{X}_{\vec{r}}$ is continuous and smooth in space at all times: therefore, the correlation function is Gaussian at short range (Porod's law). We can thus write, neglecting long-range correlations,

$$C\left(\vec{x}\right) = \left\langle \mathbf{X}^2 \right\rangle_c \exp\left[-\frac{x^2}{2\Lambda^2}\right] \quad \text{and} \quad \hat{C}(\vec{k}\,) = \left\langle \mathbf{X}^2 \right\rangle_c \left(\Lambda\sqrt{2\pi}\right)^d \exp\left[-\frac{\Lambda^2}{2}k^2\right], \tag{3}$$

with $\left\langle \mathbf{X}^2 \right\rangle_c$ the variance of p and Λ the *coherence length* of the field. The spatial order induced by the coupling is thus completely encoded by Λ, while the distribution p^t, an infinite-dimensional object, potentially bears all the complexity of the observed NTCB.

Next, we define an effective coupled dynamics for the system (p^t, Λ^t). When the local map is applied, p^t is transformed by the Perron-Frobenius operator $\mathcal{P}_S$. The coherence length decreases because of the divergence of nearby trajectories induced by the local chaos. The evolution of p^t during the coupling stage requires to evaluate the instantaneous parameters G and N, which in turn are related to short-range correlations at times $t + \frac{1}{2}$ and $t + 1$ via relation (2). The effect of the coupling on Λ is straightforward and permits to estimate these parameters: applying $\mathbf{\Delta}^\infty$ in Fourier space, the structure function verifies $\hat{C}^{t+1}(\vec{k}\,) = \exp\left[-\lambda^2 k^2\right] \hat{C}^{t+\frac{1}{2}}(\vec{k}\,)$. The coupling operator respects the Gaussian form (3) and simply increases Λ^2 by a constant term: $(\Lambda^{t+1})^2 = (\Lambda^{t+\frac{1}{2}})^2 + 2\lambda^2$. Moreover, identifying the prefactors yields a relation which fixes the dynamics of the variance $\left\langle \mathbf{X}^2 \right\rangle_c$ under the coupling operator. The direct calculation of the correlation $\langle \mathbf{X}_{\vec{r}}^{t+1} \mathbf{X}_{\vec{r}}^{t+\frac{1}{2}} \rangle_c$ provides a similar equation. Finally, we have

$$\frac{\langle \mathbf{X}_{\vec{r}}^{t+1} \mathbf{X}_{\vec{r}}^{t+\frac{1}{2}} \rangle_c}{\langle \mathbf{X}^2 \rangle_c^{t+\frac{1}{2}}} = \left(1 + \frac{\lambda^2}{(\Lambda^{t+\frac{1}{2}})^2}\right)^{-\frac{d}{2}} \quad \text{and} \quad \frac{\langle \mathbf{X}^2 \rangle_c^{t+1}}{\langle \mathbf{X}^2 \rangle_c^{t+\frac{1}{2}}} = \left(1 + \frac{2\lambda^2}{(\Lambda^{t+\frac{1}{2}})^2}\right)^{-\frac{d}{2}}. \tag{4}$$

These relations determine the effective parameters $G(\Lambda/\lambda)$ and $N(\Lambda/\lambda)$ of the convolution operator of the approximation: $\mathcal{D}_{G,N} \equiv \mathcal{D}_{\Lambda/\lambda}$.

We now have to estimate how the coherence length is changed when $\mathbf{S}$ is applied to the continuous field $\mathbf{X}_{\vec{r}}$. For this purpose, the correlation function is written

$$C(\vec{x}\,) = \langle \mathbf{X}^2 \rangle_c \left(1 - \frac{1}{2}\frac{\langle (\mathbf{X}_{\vec{r}+\vec{x}} - \mathbf{X}_{\vec{r}})^2 \rangle}{\langle \mathbf{X}^2 \rangle_c}\right)$$

to explicitly introduce the difference $\mathbf{X}_{\vec{r}+\vec{x}} - \mathbf{X}_{\vec{r}}$. Applying S to $\mathbf{X}_{\vec{r}}^t$, it comes, for small $\vec{x}$:

$$\left\langle (\mathbf{X}_{\vec{r}+\vec{x}} - \mathbf{X}_{\vec{r}})^2 \right\rangle^{t+\frac{1}{2}} \simeq \left\langle \left(S'(\mathbf{X}_{\vec{r}})\right)^2 (\mathbf{X}_{\vec{r}+\vec{x}} - \mathbf{X}_{\vec{r}})^2 \right\rangle^t = \mu^2 \left\langle (\mathbf{X}_{\vec{r}+\vec{x}} - \mathbf{X}_{\vec{r}})^2 \right\rangle^t,$$

since, for the tent map, $S'(X) = \pm\mu$ everywhere. Finally, expanding $C(\vec{x}\,)$ as $C(\vec{x}\,) \simeq \langle \mathbf{X}^2 \rangle_c (1 - \frac{1}{2}\frac{\vec{x}^{\,2}}{\Lambda^2})$ near $\vec{x} = \vec{0}$ yields, for the coherence length,

$$\frac{\Lambda^{t+\frac{1}{2}}}{\Lambda^t} \simeq \frac{1}{\mu}\sqrt{\frac{\langle \mathbf{X}^2 \rangle_c^{t+\frac{1}{2}}}{\langle \mathbf{X}^2 \rangle_c^t}} = \frac{1}{\mu}\sqrt{\frac{\langle \mathbf{S}(\mathbf{X})^2 \rangle_c^t}{\langle \mathbf{X}^2 \rangle_c^t}} \equiv D(p^t)\,. \tag{5}$$

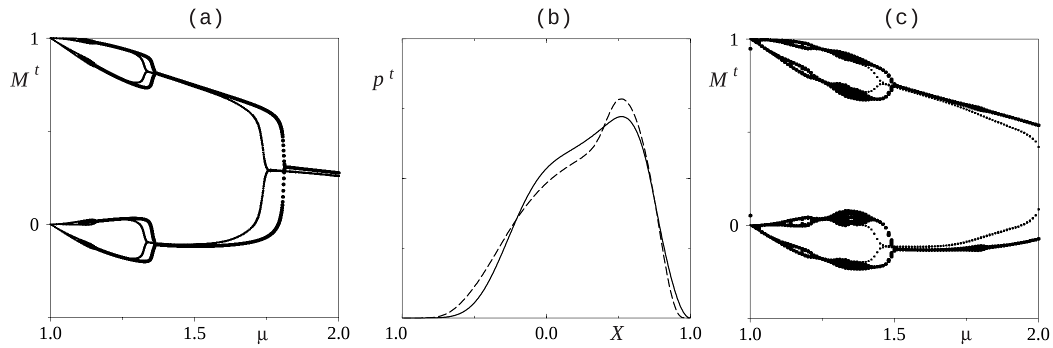

Fig. 2. – Comparison of the model with the non-trivial collective behaviour in democratically coupled tent maps of fig. 1. (a) Bifurcation diagram of the instantaneous spatial average M^t in two dimensions for the CML ($m = 1$, small circles) and for the model (large circles). (b) Asymptotic single-site pdf $p(X)$ in the stationary regime of $\mathbf{\Delta}^m \circ \mathbf{S}$ at $\mu = 2$ and $m = 32$ (dashed line) compared with the fixed-point regime obtained in the model (solid line). (c) Bifurcation diagrams for $d = 3$.

This relation couples the dynamics of Λ^t with p^t. Coefficient $D(p^t) \leq 1$ opposes the divergence of nearby trajectories, measured by μ, to the overall dilatation of the pdf p^t, measured by the ratio of standard deviations ([1]). The effective dynamics of p and Λ is now fully defined by

$$p^{t+1} = \mathcal{D}_{\Lambda^{t+\frac{1}{2}}/\lambda} \circ \mathcal{P}_S \left(p^t\right) , \quad \text{with} \quad \Lambda^{t+\frac{1}{2}} = D(p^t)\,\Lambda^t \quad \text{and} \quad \Lambda^{t+1} = \Lambda^{t+\frac{1}{2}} + 2\lambda^2 .$$

This model for the collective behavior of coupled tent maps in the continuous limit is self-consistent, without free parameters. Operator $\mathcal{D}_{\Lambda/\lambda}$ is one of the simplest which respects relations (4), and it is worth noting that the space dimension d comes in only through these expressions. Like in the original system, changing λ amounts to dilate Λ and has no influence on p. The dynamics of Λ^t is particularly simple, and it is immediate to check that it is stable for any periodic motion of p^t (the reverse is, of course, not true). Our model also satisfies a renormalisation group relation similar to that of the original system [8], which guarantees that it displays self-similarity and an infinite subharmonic cascade of global bifurcations.

We studied the dynamics of the model by direct numerical simulations. All initial pdfs tried led, at given parameter values, to the same asymptotic behavior. Figure 2 displays the bifurcation diagram of the average M^t in the model and the original, $m = 1$ CML in dimension $d = 2$ and 3. For $d = 2$, the whole bifurcation diagram is reproduced although the bifurcation points are slightly shifted to the right. The bifurcations seem to be discontinuous, although the hysteresis loops are very small (of the order of 10^{-2}). We are currently investigating whether the subcritical character of the bifurcations is a numerical effect of the method used for implementing pdf dynamics. The shapes of the pdfs obtained in the model compare fairly well to those measured in CMLs; they display the same asymmetry and inflexion points (fig. 2b). For $d = 3$, the model accounts for periodic behavior and also for the shift of the transition points to higher μ values. However, a window of quasi-periodic behavior of p^t is observed, which does not seem to be observed in the continuous limit (it is not observed for discrete CMLs with $m = 2, 3, 4$).

([1]) Note, in particular, that when the support of the pdf p^t lies on an interval which does not contain $X = 0$ (which is the case at every other timestep when the local map has two bands), the local map is linear, $D(p^t) = 1$, and Λ is unchanged; on the contrary, $D(p^t) < 1$ when the pdf p^t crosses the inversion/folding point $X = 0$, *i.e.*, when the chaotic map "destructurates" the spatial organization induced by the coupling.

Altogether, these results are of slightly lower quality than those obtained within the conditional mean-field approximation when the free parameters N and G were chosen optimally [9]. Nevertheless, the present approximation is far more powerful: in addition to being free of any arbitrary parameter, it accounts well for the high-period cycles observed near μ_∞ in any space dimension.

In higher space dimensions, the model often presents chaotic macroscopic behavior. Unfortunately, we are unable, at this stage, to compare with the NTCB exhibited by the original CMLs in their continuous limit because this requires very large calculations, possibly out of reach of today's most powerful computers.

The model has also been tested with other local maps than the tent map. Similar results are obtained for piecewise linear maps with constant $|S'|$ (which justifies (5)). But this relation can also be used for more general maps, such as the logistic map, assuming that the field difference $\mathbf{X}_{\vec{r}} - \mathbf{X}_{\vec{r}+\vec{x}}$ depends only on $\vec{x}$ and is therefore independent of $\mathbf{X}_{\vec{r}}$. This also yields qualitatively good agreement, reproduces the observed periodic behavior, but the bifurcation points given by the model are shifted to higher μ values. Better quantitative agreement can be expected from a more precise evaluation of coefficient $D(p^t)$.

Our work departs from standard mean-field approximations by the incorporation of fluctuations of the local mean field and by taking into account, in the space-continuous limit of CMLs, the dynamics of short-range correlations. For coupled tent maps, our approximation leads to a parameter-free model in which the non-linear Perron-Frobenius operator dynamics of the pdf p^t is coupled to a single scalar variable. In so far as the continuous-space limit is representative of the behavior of the original CMLs, the results obtained are qualitatively and even quantitatively good. For other local maps, such as the logistic map, we expect equally good results with a better treatment of the effect of the map on a pdf.

To our knowledge, this work constitutes the first macroscopic model for collective motion displayed by spatially extended, chaotic, dynamical systems in the strong-coupling limit. We hope it will trigger more rigorous, mathematical approaches, which have, so far, remained largely confined to the weak-coupling limit [12].

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