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Conditional mean field for chaotic coupled map lattices

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Abstract. – A conditional mean-field approach to strongly chaotic coupled map lattices is presented. It focusses on the time evolution of the one-body probability distribution function $p(X)$ of instantaneous site values. The local environment of a site is modelled in terms of an effective number of independent neighbours, while keeping the Perron-Frobenius operator to account for the action of the local map. This approximation is shown to produce distributions $p(X)$ in agreement with empirical observations of non-trivial collective behaviour, and captures the essence of their dynamics.

Coupled map lattices (CMLs) constitute perhaps the most popular type of spatially extended dynamical systems for studying the generic properties and elementary mechanisms of spatiotemporal chaos [1]. These discrete-time, discrete-space models exhibit a rich phenomenology, in some instances very reminiscent of far-from-equilibrium physical situations. Yet, even for these "simple" systems, analytical methods [2], [3] for tackling their chaotic regimes are scarce and generally restricted to cases of limited physical relevance, such as very weak coupling coefficients and/or expanding piecewise linear maps. Unsurprisingly, simple mean-field approximations are also of limited use, since they neglect one of the essential features of spatiotemporal chaos, spatial correlations.

The situation is probably at its worst when spatiotemporal chaos takes the form of *non-trivial collective behaviour* (NTCB). This generic phenomenon of high-dimensional systems is characterized by the emergence of a temporal evolution of spatially averaged quantities out of local chaotic fluctuations [4]. Since natural extensions of the mean field can only be satisfactorily defined in one space dimension, where block probabilities factorize unambiguously [5], extensions of the simple mean-field approximation are notably difficult to derive for NTCB, which appears in space dimensions $d \geq 2$. Cluster-type expansions can be defined, but their practical implementation rapidly becomes intractable [6].

In this letter, a new type of approximation is introduced and shown to account remarkably well for the simple NTCB regimes of usual CMLs. More specifically, we consider real variables

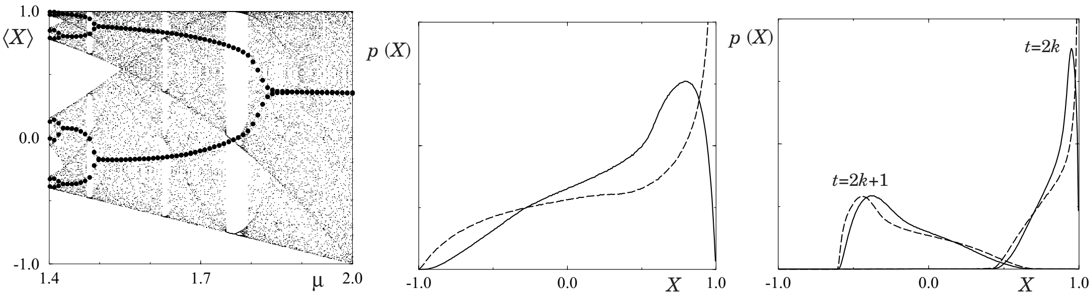

Fig. 1. – Non-trivial collective behaviour in democratically coupled logistic maps on a two-dimensional lattice of size $L = 1024$ with periodic boundary conditions ($g = 0.2$). Left: bifurcation diagram of the instantaneous spatial average M^t (filled circles) and bifurcation diagram of the single logistic map (small dots). Middle: asymptotic single-site pdf $p(X)$ in the stationary regime at $\mu = 2$ (solid line) and its iterate by the PFO $\mathcal{P}_S$ (dashed line). Right: same as middle panel, but in the collective period-2 regime at $\mu = 1.6$; up to statistical fluctuations, the two solid-line pdfs are iterates of each other during the CML evolution. The left dashed-line pdf is the iterate of the right solid-line pdf under the action of $\mathcal{P}_S$ (and vice-versa).

X_i at the nodes of d-dimensional hypercubic lattices updated synchronously *via*

$$X_i^{t+\frac{1}{2}} = S\left(X_i^t\right) \tag{1}$$

$$X_i^{t+1} = (1-2dg)X_i^{t+\frac{1}{2}} + g\sum_{j\in\mathcal{V}_i} X_j^{t+\frac{1}{2}}\,, \tag{2}$$

where S is a chaotic map, g is the (diffusive) coupling strength, and $\mathcal{V}_i$ denotes the set of the $2d$ nearest neighbours of site i.

Although our method is very general, for clarity's sake, it is applied only to lattices of logistic maps $S(X) = 1 - \mu X^2$, with $X \in [-1, 1]$ and $\mu \in [0, 2]$ in the following. In the band regime of the logistic map [7], $\mu > \mu_\infty \simeq 1.41...$, and for strong enough coupling (*e.g.* "democratic" coupling $g = 1/(2d + 1)$), the above CML exhibits NTCB [4]. Figure 1 (left) illustrates the $d = 2$, $g = 0.2$ case by showing the bifurcation diagram of the instantaneous spatial average $M^t \equiv \langle X\rangle^t$ superimposed on that of the (uncoupled) logistic map. In this case, only periodic collective motion is observed. These regimes are reached from almost every initial condition. All sites evolve chaotically, and their instantaneous distributions $p(X)$ are wide, smooth, and follow the same collective behaviour (fig. 1, middle and right).

Under the simplest mean-field approximation, the spatial average M^t is governed just by the local map S: $M^{t+1} = S(M^t)$. While this obviously fails to account for the collective motion, a similarity can nevertheless be seen, in this simple example, between the NTCB of the CML and the band structure of the uncoupled map: varying μ from 2 to μ_∞, and neglecting the periodicity windows, the logistic map exhibits banded chaos, with an increasing number of bands (1, 2, 4, 8, ...). In this work, we start from this naive remark and present an approximation of the CML (1-2) in terms of the evolution of a *ensemble* of variables with an appropriate modelling of the coupling.

Let us first consider the case of uncoupled variables evolving under the action of the map S. When the number of such variables tends to infinity, this problem reduces to the evolution of a probability distribution function (pdf):

$$p^{t+1}(Y) = \int p^t(X)\,\delta(Y - S(X))\,\mathrm{d}X \equiv \mathcal{P}_S\left[p^t\right]\,, \tag{3}$$

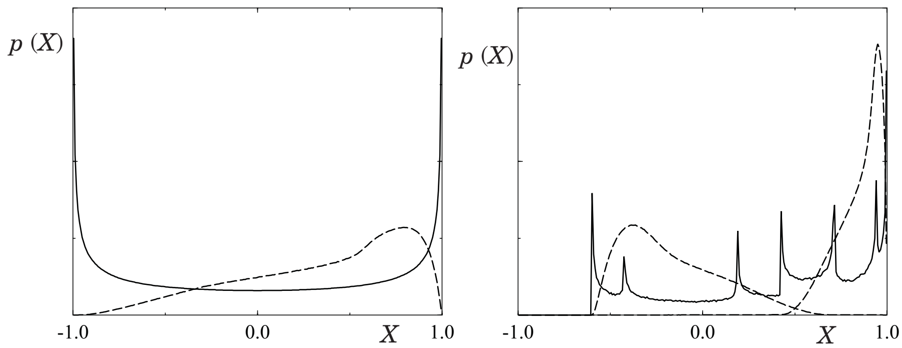

Fig. 2. – Asymptotic pdfs for the single logistic map (solid line) superimposed on the instantaneous distributions $p(X)$ produced by the CML of fig. 1 (dashed line). Left: $\mu = 2$. Right: $\mu = 1.6$.

where $\mathcal{P}_S$ is the Perron-Frobenius operator (PFO) associated to the map S.

In a banded chaos regime with q bands $\{B_i\}_{i=1}^q$, there exists a $\mathcal{P}_S$-invariant family $\mathcal{F} = \{\rho^i\}_{i=1}^q$ of disjoint distributions ρ^i of support B_i. At long times, p^t is a convex linear combination $\sum \lambda_i \rho_i$ of these *pure states*, and the action of the PFO on this decomposition is a permutation of the weights λ_i: $\mathcal{P}_S$ is said to be *asymptotically periodic* [8]. In the case of the logistic map, these pure states are not known exactly, except for $\mu = 2$. Numerical simulations reveal a very intricate structure with sharp peaks, in contrast to the smooth distributions of the CML (fig. 2).

In spite of this striking —and general— discrepancy, the PFO of the uncoupled map is essential to understanding the dynamics of the CML. After all, eq. (1) does correspond to the action of $\mathcal{P}_S$ on $p(X)$. And indeed, in the collective period-2 regime of our example, the PFO roughly transforms the two asymptotic distributions into one another (fig. 1, right), while the spatial correlations bring the corrections necessary to produce the regular cycle, and do not let the system evolve toward the uncoupled asymptotic distribution of fig. 2 (note that, in this case, the uncoupled map is still in a one-band regime). In other words, the periodicity of the CML is already contained in the local map, at least in its global features.

In the following, we numerically implement —without attempting to model— the action of the PFO on $p(X)$, and we focus on the spatial structure of the "typical" neighbourhood of a site. The object that evolves under our approximation is the pdf $p(X)$ of an infinite CML. In order to close equation (2) at first order, we must estimate the distribution of the *local* average at the site i, $M_i \equiv (1/2d)\sum_{j\in\mathcal{V}_i} X_j$, which depends on X_i.

Consider a "central" site with value X. Adopting a conditional approach, we want to estimate $m = m|_X$, the typical value of the local average around this site, and $y = y|_X$, the typical value taken by any of its neighbours. Due to invariance under lattice symmetries, the stochastic variables verify $\langle m|X\rangle = \langle y|X\rangle$, where the notation $\langle\,.\,|X\rangle$ indicates the conditional average taken for a fixed value of X.

Two limit cases can be considered: if all sites are synchronized, $y|_X = X$; if they are uncorrelated, $y|_X = x$, where x denotes a random variable independent of X and distributed according to p. In intermediate cases, the typical value $y|_X$ depends on p and X. This is the situation of interest in chaotic CMLs, since two-point correlations usually decay fast but are not zero. In particular, correlations are strong at very short distances, indicating the relative smoothness of the system's spatial profile. The corresponding "shoulder" of the two-point correlation function at small distances can be interpreted as the existence of a certain degree of synchronization between nearby sites. Our central assumption is to approximate this situation by linearly interpolating between complete synchronization and complete independence of neighbouring sites. We thus write

$$y|_X = \alpha\, \widetilde{X} + (1-\alpha)\, X\,, \tag{4}$$

where $\alpha \in [0,1]$ and $\widetilde{X}$ and X are independent random variables distributed according to p. The *local* mean field then reads: $\langle m|X\rangle = \langle y|X\rangle = \alpha M + (1-\alpha)X$, a relation well verified by numerical simulations of CMLs such as those described above. This shows that $(1-\alpha)$ expresses the "screening" of the mean field M due to local correlations. Indeed, $(1-\alpha)$ is nothing but the correlation coefficient between neighbours since, letting X vary, we have $\langle mX\rangle_{\rm c} = \langle yX\rangle_{\rm c} = (1-\alpha)\langle X^2\rangle_{\rm c}$, where the subscript "c" denotes cumulants. Consequently, $(1-\alpha)$ is a macroscopic quantity which accounts for the mutual dependence of sites at short range and is invariant under linear transformations of the lattice variables.

Next, we want to express $m|_X$ in terms of $y|_X$. However, the fluctuations of the local average $\langle m^2\rangle_{\rm c}$ do not depend solely on the correlations between the neighbours and the central site, as the neighbours are themselves correlated with one another. If two neighbours (of the same central site) are represented by variables $y|_X = \alpha\, \widetilde{X} + (1-\alpha)\, X$ and $y'|_X = \alpha\, \widetilde{X}' + (1-\alpha)\, X$, with $\widetilde{X}$ and $\widetilde{X}'$ independent, their conditional correlation $\langle yy'|X\rangle_{\rm c}$ vanishes, even though $\langle yy'\rangle_{\rm c} = (1-\alpha)^2\langle X^2\rangle_{\rm c} \neq 0$: y and y' are correlated; but their dependence comes out through X only: they are independent in a Bayesian sense. To account for the correlations between neighbours of a common central site, we write m as a sum of N *effective* Bayes-independent variables $y_j|_X$:

$$m|_X \equiv \frac{1}{N}\sum_{j=1}^{N} y_j|_X\,,$$

where the number N is not necessarily equal to the actual number of neighbours in the lattice. As a matter of fact, extending the notion of a sum of independent variables, it is even possible to deal with non-integer N's. This allows the modelling of the coupling in terms of synchronized and independent neighbours in arbitrary proportions. This amounts to considering the system as living on a tree structure of connectivity N. Equation (2) is then "replaced" by

$$\widetilde{X}_i^{t+1} = (1-N\tilde{g})\widetilde{X}_i^{t+\frac{1}{2}} + \tilde{g}\sum_{j=1}^{N}\widetilde{X}_j^{t+\frac{1}{2}}\,, \tag{5}$$

where $\tilde{g} = 2d\alpha g/N$ and the N effective independent neighbours $\widetilde{X}_j$ are distributed according to p.

At this stage, our approximation involves two free parameters, α and N. They express the local structure of correlations in the CML and are related to the cumulants $\langle Xm\rangle_{\rm c}$, $\langle X^2\rangle_{\rm c}$, and $\langle m^2\rangle_{\rm c}$:

$$\langle Xm\rangle_{\rm c} = (1-\alpha)\langle X^2\rangle_{\rm c}\,, \tag{6}$$

$$\langle m^2\rangle_{\rm c} = [(1-\alpha)^2 + \alpha^2/N]\langle X^2\rangle_{\rm c}\,. \tag{7}$$

A priori, in NTCB regimes, these quantities evolve in time. In fact, α and N being ratios of second-order cumulants, they are relatively independent of the underlying collective motion (this is also true for the mutual information between neighbouring sites [9]), at least in simple cases such as the lattices of coupled logistic maps described above (see fig. 3 (left), which shows the numerical values of α and N as defined by eqs. (6) and (7)). These observations lead us to our final assumption of *geometrical stationarity*: α and N are taken as fixed parameters.

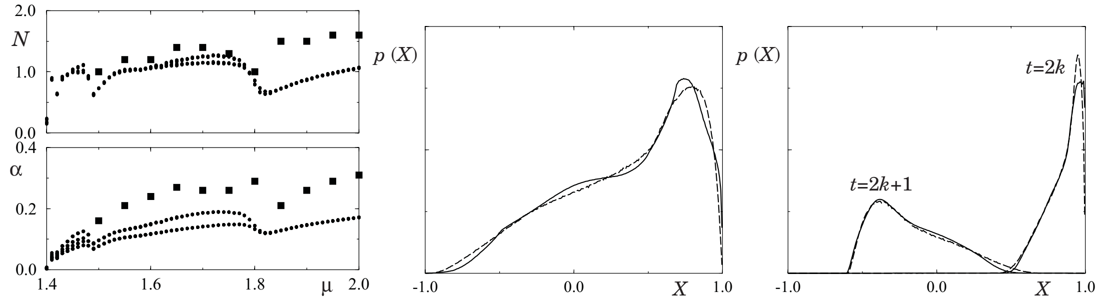

Fig. 3. – Left: α and N measured on the CML of fig. 1 at all timesteps in the asymptotic regime for different values of the parameter μ (small symbols), and best-fit values for the approximation, *i.e.* values of α and N for which the distance between the empirical pdfs and those produced by the approximation scheme is minimal (squares). Middle: asymptotic pdf $p(X)$ produced by the approximation for $\mu = 2$ and best-fit values of α and N (solid line); same for the original CML (dashed line). Right: same but for the collective period-2 regime at $\mu = 1.6$.

The dynamical evolution of $p(X)$ within our approximation is now complete: eq. (1) is replaced by the action of the PFO of the local map (eq. (3)), and eq. (2) by the action of a complex convolution operator $\Delta_{\tilde{g},N}$ (see eq. (5)):

$$\cdots p^t \xrightarrow{\mathcal{P}_S} p^{t+\frac{1}{2}} \xrightarrow{\Delta_{\tilde{g},N}} p^{t+1} \cdots .$$

The convolution operator $\Delta_{\tilde{g},N}$ is defined by

$$\Delta_{\tilde{g},N}\,[p]\,(X') = \int p(X)\, p_N\left(\frac{X' - (1 - N\tilde{g})X}{N\tilde{g}}\right) \mathrm{d}X\,, \tag{8}$$

where p_N is the distribution of the average $(1/N)\sum_{j=1}^{N}\widetilde{X}_j$ over the N effective independent neighbours $\widetilde{X}_j$ distributed according to p:

$$p_N(Y) \equiv \int \left(\prod_{j=1}^{N} p(\widetilde{X}_j)\mathrm{d}\widetilde{X}_j\right) \delta\left(Y - \frac{1}{N}\sum_{j=1}^{N}\widetilde{X}_j\right) . \tag{9}$$

The computation of this convolution integral can be performed for non-integer N *via* Fourier transform.

We will now describe the results of our approximation as applied to the two-dimensional lattice of democratically coupled logistic maps used throughout the text. First, we note that, for realistic values of α and N, eqs. (3), (8) and (9) usually converge rapidly to the same attractor for various initial distributions.

For $\mu = 2$, the CML is in a stationary state (fig. 1). For a large range of parameters including the numerically observed values of α and N (approximately $N \in [1, 4]$ and $\alpha \in [0.15, 0.32]$), $p^t(X)$ converges, under the approximation, to a stationary pdf strikingly similar to the original one. For $\mu = 1.6$, the approximation easily captures the collective motion, again for a rather large range of parameter values ($N \in [1, 3]$ and $\alpha \in [0.21, 0.26]$). In both cases, the agreement between the empirical and approximated pdfs is excellent and the results are remarkably robust, see fig. 3, middle and right, where the parameter values yielding the best agreement were used.

Similarly good results were obtained for the same CML and $d = 3$, as well as for coupled tent maps. However, it is more difficult, and sometimes impossible, to capture the larger-period cycles of the CML and the more complex collective evolutions observed for $d > 3$. We believe that this is due to the non-stationarity of α and N in these cases. This indicates the direction towards which the present work should be extended: the *dynamics* of the effective parameters α and N should be self-consistently incorporated. We are currently working on various schemes toward this end, as well as on an extension to cellular automata.

At the present stage, the two parameters α and N have to be chosen somewhat arbitrarily, but their physical significance is clear: α measures correlations between neighbouring sites, while N can be seen as the mean effective number of Bayes-independent neighbours of a given site. This approximation yields robust and reliable results, reproducing the dynamics of non-trivial distributions. The basic ideas underlying our approach shed some light on "fully developed" spatiotemporal chaos and, in particular, on the macroscopic motion displayed by synchronously updated systems in the strong-coupling limit. They confirm [10], [6] the local origin of the non-trivial collective behaviour observed in CMLs, and detail the "regularizing" role of the coupling. In some sense, the geometry can be seen to correct the action of the Perron-Frobenius operator on one-site pdfs so as to make them drift towards a new operating point associated with the collective regime.

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