Cluster Expansion for Collective Behavior in Discrete-Space Dynamical Systems

Anaël Lemaître, Hugues Chaté, and Paul Manneville
LadHyX, Laboratoire d'Hydrodynamique, Ecole Polytechnique, 91128 Palaiseau, France
Commissariat à l'Energie Atomique, Service de Physique de l'Etat Condensé, Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France
(Received 17 October 1995)

We introduce an approximation scheme for determining the evolution of spatially averaged quantities in large classes of extensively chaotic dynamical systems. The case of lattices of diffusively coupled logistic maps is presented. Results in two space dimensions show that the scheme succeeds in reproducing the nontrivial collective behavior observed in this system. [S0031-9007(96)00709-0]

PACS numbers: 05.45.+b, 02.60.–x, 05.70.Ln, 82.20.–w

When studying spatiotemporal chaos in out-of-equilibrium systems, it is generally difficult to estimate even the simplest statistical quantifiers, such as spatial averages, from the (local) evolution rule at the origin of the observed phenomena. This task is made harder for systems exhibiting nontrivial collective behavior, i.e., extensively chaotic regimes in which spatially averaged quantities evolve in time—most often regularly—in sharp contrast with equilibriumlike situations [1]. In this Letter, we introduce a general scheme to attack this problem for discrete-space dynamical systems and detail its implementation and significance in the case of coupled map lattices (CML).

Spatiotemporal chaos refers to physical situations more complex than (temporal) chaos but simpler than, say, fully developed turbulence. This intermediate situation is mainly due to the existence of basic scales arising, for example, from a symmetry-breaking instability, as in Rayleigh-Bénard convection or Taylor-Couette flow [2]. Discrete-space systems are of particular interest in this context because they can be thought of as *a priori* incorporating these basic scales. Even when chaos is extensive, i.e., when quantities measuring the degree of chaos in the system are proportional to the system size [3], there is no general method [4] to estimate statistical properties despite the existence of a well-defined, infinite-size, infinite-time, "thermodynamic" limit. In any case, conventional mean-field theory is unable to account for nontrivial collective behavior [1]. Our approach relies on an exact treatment of local correlations, applies *a priori* to many infinite-size, discrete-space dynamical systems in any space dimension d, and provides the dynamical evolution of spatially averaged quantities. Related to BBGKY-type cluster expansions [5], it allows for a self-consistent calculation of the statistical properties of many extensively chaotic systems including those showing nontrivial collective behavior. Here, we treat only the case of CML whose evolution can be put into a polynomial form. For simplicity, we present preliminary results on hypercubic lattices of logistic maps $S(X) = 1 - \mu X^2$ with "democratic" (equal-weight), nearest-neighbor coupling. Their evolution rule reads

$$X_i^{t+1} = 1 - \frac{\mu}{2d+1} \sum_{j \in \mathcal{V}_i} X_j^{t2}, \tag{1}$$

where $X_i \in [-1, 1]$, $\mu \in [0, 2]$, subscripts denote space, and $\mathcal{V}_i$ is the neighborhood of site i (including itself). These CML are known to exhibit strict synchronization for $\mu < \mu_\infty \simeq 1.401$: all sites in the lattice eventually take on the same value and follow the period-2^k cycle of the (uncoupled) logistic map. This is a "trivial" collective behavior. For $\mu > \mu_\infty$ and space dimension $d \geq 2$, CML (1) displays nontrivial collective behavior, in which the sites are not synchronized: in spite of strong, local, chaotic fluctuations, spatially averaged quantities are not statistically stationary in time, even in the "thermodynamic" limit. In general, the collective motion is *not* directly related to the behavior of the local map. A striking example for CML (1) is the collective quasiperiodic cycle reported for $d = 5$ and $\mu \simeq 1.71$ [1], which cannot be accounted for by a single-variable iteration such as the local map. For $d = 2$ and $d = 3$, CML (1) exhibits periodic collective behavior, but the period of these cycles do not correspond to that of the banded-chaos regimes exhibited by the logistic map for $\mu > \mu_\infty$ (see below). Predicting the collective motion from the local dynamical rule is thus a major challenge which is addressed here.

Taking spatial averages, the evolution rule (1) yields an infinite hierarchy of equations,

$$\begin{aligned}
\langle X_i \rangle^{t+1} &= 1 - \mu \langle X_i^2 \rangle^t, \\
\langle X_i^2 \rangle^{t+1} &= 1 - 2\mu \langle X_i^2 \rangle^t + \left(\frac{\mu}{2d+1} \right)^2 \sum_{j,k \in \mathcal{V}_i} \langle X_j^2 X_k^2 \rangle^t, \\
\langle X_i X_j \rangle^{t+1} &= \cdots, \\
\langle X_i X_j^2 \rangle^{t+1} &= \cdots,
\end{aligned} \tag{2}$$

where the indice i is retained only for clarity. Even though this hierarchy can always be written, here we treat only the (common) case where the solutions of the CML

 0031-9007/96/77(3)/486(4)$10.00

do not break the symmetries of the lattice and where, consequently, all points share the same statistical properties. Leaving this experimental point of view for a more theoretical one, the variables of (2) can be taken as the *moments* of a probability distribution function governed by the Perron-Frobenius operator. Hierarchy (2) then represents the action of this operator on the subset of probability densities which describe the ensemble of statistically equivalent systems displaying the same collective behavior [6]. In the following, this point of view is adopted, and we discuss our approximation in terms of moments rather than spatially averaged quantities.

A moment $\langle X_{i_1}^{\alpha_1} \cdots X_{i_n}^{\alpha_n} \rangle^t$ is characterized by its geometrical support $(i_1, \ldots, i_n)$ and the set of weights $(\alpha_1, \ldots, \alpha_n)$, $\alpha_j \geq 1$, associated to the points of the support. Due to the (assumed) statistical equivalence of all points, the symmetry properties of the lattice $\mathcal{L}$ (translations *and* rotations) can be used to classify moments. Hierarchy (2) is then rewritten in terms of classes of moments with integer multiplicative factors coming out of the reduction process. Hierarchy (2) is an infinite set of linear equations derived from the nonlinear evolution rule (1). Each class of moments on the left-hand side is expressed in terms of classes of wider geometrical support and/or higher total order $\alpha = \sum_{j=1}^{n} \alpha_j$. Keeping only a given set C of moments (defined by a set of supports and weights), the truncated hierarchy can be symbolically written $\vec{M}^{t+1} = \mathcal{F}(\vec{M}^t + \vec{N}^t)$, where $\vec{M}^t$ is the set of moment values corresponding to C, and $\vec{N}^t$ is the set of moment values on the left-hand side corresponding to moments which are not in C. A closure relation is written symbolically $\vec{N}^{t+1} = \mathcal{G}(\vec{M}^{t+1})$. We now describe a physically motivated truncation and closure scheme. It is twofold: a "geometrical" truncation—similar to BBGKY cluster expansion [5]—stopping the growth of supports and providing the closure, and an "analytical" truncation stopping the increase of the total order α of moments.

In extensively chaotic systems, spatial correlations decay fast—often exponentially—far enough from critical points [7]. In terms of moments, strict decorrelation is expressed by (possibly complicated) factorization relations involving moments of smaller geometrical support. These relations are implicitly and efficiently taken into account when *cumulants* are considered instead of moments. Decorrelation of one subset of $(i_1, \ldots, i_n)$ from the other points of the geometrical support is simply expressed by saying that cumulants with the same support vanish. To implement the approximation scheme, a criterion is needed to designate the supports for which the cumulants are to be canceled. A natural choice is to keep all cumulants such that the maximal distance between any two points of their support is less than some threshold distance $r_{\max}$. The analytical truncation is also based on cumulants. Numerical observations show that the cumulants decrease when their total order α increases (see Table I). This is related to the unimodal structure of the probability densities observed [8]. In practice, a maximum order $\alpha_{\max}$ is chosen, and cumulants with higher total order are set to zero [9].

TABLE I. Values of single-variable moments and cumulants for $\mu = 1.5725$ (period 2 collective behavior) for the lattice defined in Fig. 1. Data are given for both the original CML (numerical simulation) and the result of the approximation for $\alpha_{\max} = 2$, $r_{\max} = 3$, and $n_{\max} = 30$.

	$\langle X \rangle$	$\langle X^2 \rangle$	$\langle X^2 \rangle_c$	$\langle X^3 \rangle$	$\langle X^3 \rangle_c$
CML	0.856	0.745	0.012	0.657	−0.0012
	−0.171	0.091	0.062	−0.027	0.0097
Approx.	0.881	0.794	0.017	0.730	0
	−0.165	0.076	0.048	−0.028	0

Hierarchy (2) is given in terms of moments, whereas both truncation steps involve cumulants. We thus need to compute moments from cumulants (and vice versa). This is achieved in practice by making recursive use of the relation [10]

$$\langle X_1^{\alpha_1} \cdots X_n^{\alpha_n} \rangle = \sum_{\beta_i \leq \alpha_i} \frac{\sum_{i=1}^{n} \beta_i}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \binom{\alpha_i}{\beta_i} \times \langle X_1^{\alpha_1 - \beta_1} \cdots X_n^{\alpha_n - \beta_n} \rangle \langle X_1^{\beta_1} \cdots X_n^{\beta_n} \rangle_c \,. \tag{3}$$

Let us summarize our approach so far. The two-step truncation leaves a finite set C of nonzero cumulants. Let us call $\mathcal{D}_C$ the set of all probability densities [9] whose only nonzero cumulants are inside C. The probability density $\mathbf{f}^t$ describing the system is approximated by $\mathbf{g}^t$, the unique element of $\mathcal{D}_C$ sharing the same cumulant values $\vec{C}^t$ over C. The image of $\mathbf{g}^t$ by the (truncated) hierarchy belongs to $\mathcal{D}_C$. The truncation thus also provides a closure via relation (3).

Although the approximation is self-consistent, producing a time series of elements of $\mathcal{D}_C$, the situation is less satisfactory than it looks. Elements of $\mathcal{D}_C$ are "generalized Gaussian" distributions. In particular, they possess tails and therefore do not belong to $\mathcal{D}_I$, the set of probability densities with their support in $I = [-1, 1]^N$. Under the action of the local map S, points outside $[-1, 1]$ quickly diverge to infinity. Indeed, implementations at this stage lead to the divergence of moment values.

We now present a resummation scheme which forces the probability densities produced by the approximation to belong to $\mathcal{D}_I$. We look for $\mathbf{h}^t$ in $\mathcal{D}_I$ approximating the functions $\mathbf{g}^t$ produced by the truncation. The "natural" solution—closely linked to the definition of cumulants in terms of the characteristic function (see below)—is to periodize $\mathbf{g}^t$ on I and multiply it by the indicator function $\mathbb{1}_I$ of I [$\mathbb{1}_I(X) = 1$ if $X \in I$, $\mathbb{1}_I(X) = 0$ otherwise],

$$\mathbf{h}^t = \mathcal{P}_I(\mathbf{g}^t) \cdot \mathbb{1}_I \,,$$

where $\mathcal{P}_I$ symbolizes the operator periodizing the densities [$\mathcal{P}_I(\mathbf{g}(\vec{X})) = \sum_{\vec{q} \in \mathbb{Z}^n} \mathbf{g}(\vec{X} + \vec{q}\Delta)$ where $\Delta = 2$ is the

length of I]. This amounts to "cutting" the tails of $\mathbf{g}^t$ outside of I and "pasting" each of them at the other end of the interval ($\mathbf{h}^t$ is the aliased approximation of $\mathbf{g}^t$ appropriate to fast Fourier transforms). If $\mathbf{g}^t$ is a good approximation of $\mathbf{f}^t$ (the "true" probability density belonging to $\mathcal{D}_I$), then its tails outside of I are small, and consequently $\mathbf{h}^t$ itself is a good approximation of $\mathbf{g}^t$ and $\mathbf{f}^t$. In other words, the coefficients $c_{\vec{n}}$ of the multidimensional Fourier series of the density $\mathbf{f}^t$ are approximated by

$$c_{\vec{n}} = \left\langle \exp\left(\frac{2i\pi\vec{n}\cdot\vec{X}}{\Delta}\right)\right\rangle = \tilde{\mathbf{g}}^t\left(\frac{2\pi\vec{n}}{\Delta}\right),$$

where $\tilde{\mathbf{g}}^t$ is the characteristic function of $\mathbf{g}^t$, given by the *finite* sum

$$\tilde{\mathbf{g}}^t(\vec{k}) = \exp\left\{\sum_{\langle X_1^{\alpha_1}\cdots X_n^{\alpha_n}\rangle_c \in C}\left[\prod_{j=1,\ldots,n}\frac{(ik_j)^{\alpha_j}}{\alpha_j!}\right]\langle X_1^{\alpha_1}\cdots X_n^{\alpha_n}\rangle_c^t\right\}.$$

All the moments of $\mathbf{h}^t$ are then given by

$$\langle X_{i_1}^{\alpha_1}\cdots X_{i_n}^{\alpha_n}\rangle^t = \sum_{\vec{n}} c_{\vec{n}} \prod_{j=1}^{n} \mathcal{A}(n_j,\alpha_j), \qquad (4)$$

with

$$\mathcal{A}(n_j,\alpha_j) = \int_I X_j^{\alpha_j}\exp\left(-\frac{2i\pi n_j X}{\Delta}\right)dX\,.$$

Since $\mathcal{A}(n_j,0) = 0$, for every moment or cumulant, the only vectors $\vec{n}$ involved in (4) are those of dimension n, where n is the number of points in the geometrical support [$\vec{n} = (n_1,\ldots,n_n)$]. Moreover, the Fourier series are truncated to $n_{\max}$ coefficients in each dimension ($n_j \in \{-n_{\max},\ldots,n_{\max}\}$), so that only a finite number of $c_{\vec{n}}$ have to be calculated. We note, to conclude this presentation of the scheme, that the moments of the function $\mathbf{h}^t \in \mathcal{D}_I$ are close to those of $\mathbf{g}^t$, but $\mathbf{h}^t$ possesses an infinite number of (small) nonzero cumulants.

In realistic cases, the calculations are too tedious to be done by hand. This is all the more acute as (1) we eventually want to apply the approximation to high-dimensional CML, and (2) we know that, even though correlations decay fast, the cutoff distance $r_{\max}$ may have to be chosen large in order to ensure a satisfactory level of decorrelation. Therefore, we have written a symbolic manipulation program in C++ to implement the scheme on a computer. While it is impossible to give a detailed account of the program here [10], we now describe its general structure. A first step calculates the set C of the cumulants kept and—implicitly—the equations giving the self-consistent evolution of densities in $\mathcal{D}_C$. Except for the data defining the CML (local map, lattice, coupling), the only parameters on input are $r_{\max}$, the cutoff distance, and $\alpha_{\max}$, the maximum total order of the moments or cumulants retained. Next comes the actual numerical simulation of the evolution of an initial probability density $\mathbf{h}^0 \in \mathcal{D}_I$. First, the truncated evolution rule is applied, yielding $\mathbf{g}^1 \in \mathcal{D}_C$. The resummation is then performed, calculating the coefficients $c_{\vec{n}}$ and then the moments of $\mathbf{h}^1 \in \mathcal{D}_I$. The process is iterated,

$$\cdots \mathbf{h}^t \in \mathcal{D}_I \overset{\text{rule}}{\rightarrow} \mathbf{g}^{t+1} \in \mathcal{D}_C \overset{\text{resummation}}{\rightarrow} \mathbf{h}^{t+1} \in \mathcal{D}_I \cdots,$$

building a time series of probability densities from which the evolution of various observables can be deduced.

We now present our first results, obtained for the $d = 2$ lattice of democratically coupled logistic maps. In this case, the nontrivial collective behavior observed for $\mu > \mu_\infty$ are periodic cycles of periods 1, 2, 4, 8, and 16 (Fig. 1). For almost every initial condition, instantaneous site values eventually end up all in the same "band," leading to statistically homogeneous spatial configurations. Droplets of one such state into another shrink and disappear. The observed bands have no direct relationship with those shown by the (chaotic) logistic map at the same μ value. Since they are just one-dimensional projections of the full CML state, they can (and often do) overlap. Moreover, the global bifurcation points are shifted with respect to the band-merging points, so that the collective period observed is different from the number of bands of the local map. This is not the only effect of the coupling: the dynamics of the CML is not "explained" by simply considering that of the local map, especially in the strong-coupling case of interest here. Except at the global bifurcation points where algebraic decay is observed [7], correlations decay exponentially, instantaneous distributions are unimodal, and cumulants

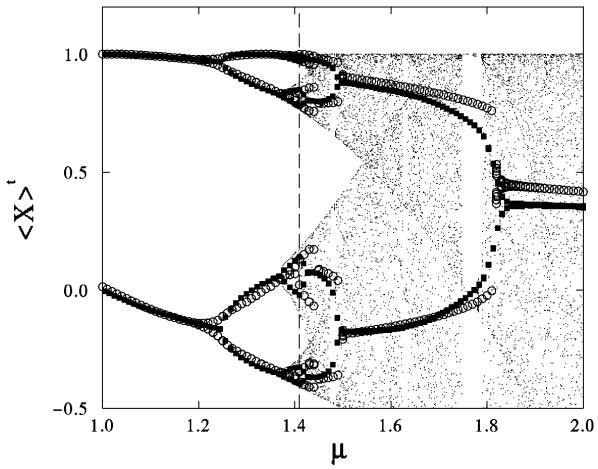

FIG. 1. Bifurcation diagram of $\langle X\rangle^t$ vs parameter μ for a $d = 2$ square lattice of democratically coupled logistic maps. Black squares: simulation of a lattice of $N = 1024^2$ sites with periodic boundary conditions and random initial conditions. Open circles: result of the approximation for $\alpha_{\max} = 2$, $r_{\max} = 3$, and $n_{\max} = 60$. The initial condition corresponds to a uniform distribution on the interval $[-0.5, 0.5]$; 10 iterations are shown after a transient of 50 time steps. Dots: single logistic map. The dashed line is at μ_∞.

decay fast with their total order (see Table I), so that the assumptions of our approach are satisfied.

We have performed our approximation scheme at maximal order $\alpha_{\max} = 2$ for various values of $r_{\max}$ and $n_{\max}$. Results are nearly identical as soon as $r_{\max} \geq 3$ and $n_{\max} \geq 20$, indicating the convergence of the scheme. The same attractor is reached for most initial conditions, and the nontrivial collective behavior observed for $\mu > \mu_\infty$ is recovered (Fig. 1), with only slight discrepancies with respect to the original CML: (1) the critical behavior near bifurcation points is not captured by the approximation, as expected from the presence of long-range correlations in these regions; (2) in the region $\mu \in [1.7; 2]$, the strong asymmetry of the distributions, not accounted for with $\alpha_{\max} = 2$, is at the origin of the small quantitative disagreement. The synchronized states observed for $\mu < \mu_\infty$ are also recovered. We note that even though the approximation at this stage might appear very simple—all probability distributions are (multidimensional) Gaussian functions for $\alpha_{\max} = 2$—the truncated hierarchy already involves 8 degrees of freedom and 25 intermediate variables.

The approximation is also very successful for the $d = 3$ case [10], but its success will have to be confirmed in the future when applied to even higher-dimensional systems. Preliminary results, to be detailed in [10], indicate that the main difficulty is controlling the *admissibility* of the approximated distributions [11] when $\alpha_{\max}$ must be chosen larger than 2. We nonetheless believe that our approach is very promising. At the very least, it sheds light on the problem of the origin of the particular type of collective dynamics arising in a given system when the local behavior is of a different type. Neglecting correlations beyond a "mesoscopic" scale of order $r_{\max}$, the collective motion is revealed. However, this does not solve the problem of the existence of nontrivial collective behavior in the infinite-size, infinite-time, "thermodynamic" limit: Even though, formally, systems of infinite size in their asymptotic state are treated, the approximation makes the assumption of spatial homogeneity and thus cannot account for (possible) fluctuations such as droplet excitations. Approximations of the type developed here do not prove the stability of the observed long-range order against intrinsic fluctuations, which should be studied in detail in another framework. Nevertheless, our results suggest an intermediate picture made of noisy mesoscopic units coupled together, a Langevin-like level of description at which the methods and results of statistical mechanics could be applied (as, e.g., in [12], in the case of cellular automata rules). Finally, we note that the approach presented here may be applied to other systems such as cellular automata [1], globally coupled maps and oscillators [13], and randomly connected networks [14], for which fascinating types of nontrivial collective behavior have also been reported.

[1] H. Chaté and P. Manneville, Europhys. Lett. **14**, 409 (1991); Prog. Theor. Phys. **87**, 1 (1992); J. A. C. Gallas *et al.*, Physica (Amsterdam) **180A**, 19 (1992).

[2] See, e.g., M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[3] P. Manneville, in *Macroscopic Modeling of Turbulent Flows*, edited by O. Pironneau, Lecture Notes in Physics Vol. 230 (Springer, New York, 1985); P. C. Hohenberg and B. I. Shraiman, Physica (Amsterdam) **37D**, 109 (1989); D. A. Egolf and H. S. Greenside, Nature (London) **369**, 129 (1994).

[4] One notable exception is the case of one-dimensional systems, for which it is possible to derive systematic improvements of the simple mean-field theory. See, e.g., H. A. Gutowitz and J. D. Victor, Physica (Amsterdam) **28D**, 18 (1987); H. A. Gutowitz, Physica (Amsterdam) **45D**, 136 (1990). See also J. M. Houlrik, I. Webman, and M. H. Jensen, Phys. Rev. A **41**, 4210 (1990).

[5] See, e.g., G. Stell, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic Press, New York, 1976), Vol. 5b.

[6] In this context, the notion of *asymptotic periodicity*, discussed in M. C. Mackey, Rev. Mod. Phys. **61**, 981 (1989), and recently extended to CML [J. Losson and M. C. Mackey, Phys. Rev. E **50**, 843 (1994); J. Losson, J. G. Milton, and M. C. Mackey, Physica (Amsterdam) **81D**, 177 (1995)], seems to be the appropriate mathematical framework to deal with nontrivial collective behavior from a probabilistic point of view [H. Chaté and J. Losson (to be published)].

[7] Ph. Marcq, H. Chaté, and P. Manneville, "Critical Properties of Lattices of Coupled Logistic Maps" (to be published).

[8] This is in agreement with the assumption of the statistical equivalence of all points in the lattice. The cumulants of a nonunimodal distribution do not decrease with order. Such a situation may be encountered when the system exhibits regular (e.g., "antiferromagnetic") patterns in space, or when macroscopic domains of different "phases" are present (e.g., near critical points), all cases where the above assumption is not legitimate.

[9] The term "probability density" is used to describe the truncated functions (later called $\mathbf{g}^t$ and $\mathbf{h}^t$) approximating the original density $\mathbf{f}^t$. This is somewhat abusive insofar as these *functional approximations* are not, strictly speaking, probability densities, since they may be nonpositive. This point will be discussed at length in [10].

[10] A. Lemaître, H. Chaté, and P. Manneville (to be published).

[11] N. I. Akhiezer, *The Classical Moment Problem* (Hafner, New York, 1965).

[12] H. Chaté and L.-H. Tang (unpublished); H. Chaté, L.-H. Tang, and G. Grinstein, Phys. Rev. Lett. **74**, 912 (1995).

[13] K. Kaneko, Physica (Amsterdam) **86D**, 158 (1995), and references therein; N. Nakagawa and Y. Kuramoto, Prog. Theor. Phys. **89**, 313 (1993); Physica (Amsterdam) **75D**, 74 (1994); V. Hakim and W. J. Rappel, Phys. Rev. A **46**, 7347 (1992).

[14] N. Mousseau, Europhys. Lett. **33**, 509 (1996).