Viscous structure of plane waves in spatially developing shear flows

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(Received 16 November 1994; accepted 8 February 1995)

This paper is concerned with the propagation of linear plane waves in incompressible, two-dimensional weakly nonparallel shear flows for large Reynolds numbers. Waves are analyzed for arbitrary complex frequency ω and local wave number k when nonparallel effects are assumed to be due to weak viscous diffusion. The inviscid approximation is shown to correctly describe, at leading order, the cross-stream variations of local plane waves at all stations where they are locally amplified in a frame of reference moving at the local phase speed $\Re e\omega/\Re ek$, i.e., at stations where the temporal growth rate $\sigma \equiv \Im m\omega - \Im mk\ \Re e\omega/\Re ek$ remains positive. This result also holds as long as the local phase speed lies outside the range of values reached by the basic velocity profile. By contrast, the inviscid approximation fails to represent cross-stream variations in the critical layers when waves are locally neutral ($\sigma=0$), and in large viscous regions when they become damped ($\sigma<0$). Uniformly valid WKBJ approximations are derived in these regions and the results are applied to the description of forced spatial waves and self-excited global modes.

I. INTRODUCTION

Most incompressible, two-dimensional unbounded shear flows are known to exhibit under certain conditions coherent large-scale structures. The goal of this study is to describe such structures in terms of local plane waves evolving on a weakly nonparallel basic flow when the Reynolds number is large. Sufficient conditions are obtained for the validity of linearized inviscid theory pertaining to arbitrary complex values of frequency and local wave number. Spatial domains in physical space where the inviscid approximation breaks down are clearly identified. Finally, uniformly valid asymptotic approximations are obtained in these regions, which are governed by viscous diffusion.

Convectively unstable flows, for instance mixing layers and jets, are known to be very sensitive to noise excitation and numerous experiments (see Ho and Huerre[1] for a review) have been conducted to analyze the vortical structures that appear in response to a time-periodic excitation. Such vortices have been found to be well-described by linear spatially growing instability waves at their early stages of evolution. Bouthier[2] appears to have been first in applying the WKBJ approximation to account for weak nonparallel effects in boundary layers. Crighton and Gaster[3] and Gaster *et al.*[4] subsequently developed a similar formalism in the case of jets and mixing layers, respectively, and obtained reasonably good agreement with experiments. Hultgren[5] has shown that viscous and nonparallel effects come at the same order in the high-Reynolds-number limit, so that Bouthier's approach, which treats the spreading rate independently of the Reynolds number, is not strictly correct.

Locally parallel linear theory indicates that, for a given forcing frequency within the unstable range, the local spatial growth rate ultimately decreases with downstream distance, becomes zero at some locally neutral streamwise station and negative further downstream. Thus, at leading order in the linearized WKBJ approximation, the wave amplitude increases, reaches a maximum at the locally neutral point and decays thereafter. The critical point singularity (see Maslowe[6] for a review) where the inviscid Rayleigh equation breaks down happens to be located in the cross-stream direction precisely at the locally neutral station where the phase velocity is by definition real. Several effects may then be invoked to smooth the singularity: If the maximum amplitude is sufficiently large, the critical layer surrounding the critical point is necessarily dominated by nonlinearities, as demonstrated by Goldstein and Leib.[7] By contrast, if it is sufficiently small, viscous diffusion dominates the critical layer structure, as in the classical hydrodynamic stability theory of parallel flows (Lin,[8] Drazin and Reid[9]). In the present study, we are only concerned with the linear evolution of spatial waves in the *viscous critical layer régime*. In other words, local plane waves grow exponentially, reach neutral, and spatially decay as they propagate downstream, according to locally parallel linearized stability theory.

It is important to note that viscous effects are not expected to be solely confined to the critical layer in the limit of large Reynolds numbers: From strictly parallel stability theory, it is known that large $O(1)$ viscous regions appear on the real cross-stream axis as soon as waves become temporally damped (Lin[8]). We therefore anticipate inviscid WKBJ

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 1070-6631/95/7(6)/1337/11/$6.00

approximations derived in the context of weakly diverging flows to become invalid in large cross-stream intervals downstream of the locally neutral station. One of the objectives of this study is to obtain viscous WKBJ approximations in such subdomains.

Local plane waves are also essential ingredients of the global mode problem. Several shear flows, such as wakes (Karniadakis and Triantafyllou[10]), hot jets (Monkewitz *et al.*[11]), and counterflow mixing layers (Strykowski and Niccum[12]) exhibit self-excited oscillations. In such situations, the observed coherent structures are not created by external forcing but appear through a destabilization of the entire flow. Typically, these are represented in terms of global modes, i.e., linear time-harmonic perturbations of the basic flow satisfying homogeneous boundary conditions in all spatial directions. The complex global frequencies are then the result of an eigenvalue problem. If the basic flow is weakly nonparallel, global modes can be studied by the WKBJ method (Huerre and Monkewitz[13]). In this framework, a global mode theory has been developed in the context of the Ginzburg–Landau equation (Chomaz *et al.*,[14] Le Dizès *et al.*[15]) and recently applied to shear flows by Monkewitz *et al.*[16] and Pesenson and Monkewitz.[17] In particular, each global mode has been demonstrated from a purely inviscid point of view to be a local plane wave of complex global frequency and spatially varying wave number as in the forcing problem studied, for instance, by Crighton and Gaster.[3] The same questions therefore arise as to the validity of the linearized inviscid approximation:

- For a given complex frequency and local wave number, what is the cross-stream scale of the associated eigenfunction, or equivalently, in which cross-stream intervals does the inviscid analysis apply?
- What is the local plane wave approximation in regions where the inviscid approximation breaks down?

This paper is organized as follows. The basic equations are stated in Sec. II: The determination of the characteristic cross-stream scale, viscous or inviscid, is shown to be locally specified by to the associated parallel flow problem. Classical results from temporal theory (k real), concerning the validity of the inviscid approach for parallel flows, are extended to arbitrary complex wave numbers in Sec. III. They are applied to slowly varying flows in Sec. IV. In Sec. V, a uniformly valid approximation for local plane waves is constructed in the viscous regions and matched with the inviscid approximation. Application of the results to the signaling and global mode problems is briefly discussed in a concluding paragraph.

II. BASIC FORMULATION

Consider a two-dimensional, incompressible, spatially developing shear flow governed by the vorticity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi_T}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi_T}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi_T=\frac{1}{\mathrm{Re}}\nabla^2\nabla^2\Psi_T, \qquad (1)$$

for a total streamfunction Ψ_T. The streamwise and cross-stream coordinates x and y are normalized with respect to a typical instability wavelength λ or a characteristic vorticity thickness. Time t and all velocities are scaled by $\lambda/\bar{U}$ and $\bar{U}$, respectively, where $\bar{U}$ is a characteristic basic streamwise velocity.

The basic flow defined as the time-independent part $\Psi^{(0)}$ of Ψ_T satisfies:

$$\left(\frac{\partial\Psi^{(0)}}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\Psi^{(0)}}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\Psi^{(0)}=\frac{1}{\mathrm{Re}}\nabla^2\nabla^2\Psi^{(0)}. \qquad (2)$$

The weakly nonparallel assumption is equivalent to assuming that for $\mathrm{Re}\gg 1$, the leading-order approximation $\Psi_0^{(0)}$ of $\Psi^{(0)}$ varies on the slow viscous scale x/Re. The small parameter ϵ defined by λ/L, where L is the characteristic scale of the streamwise basic flow variations is another measure of the degree of nonparallelism. It is directly linked to the Reynolds number through

$$\mathrm{Re}=\frac{R}{\epsilon};\quad R=O(1). \qquad (3)$$

Variations of $\Psi_0^{(0)}$ thus occur on the slow scale $X=\epsilon x$ and Eq. (2), written to $O(\epsilon^2)$, becomes a boundary-layer equation for $\Psi_0^{(0)}$:

$$\frac{\partial\Psi_0^{(0)}}{\partial y}\frac{\partial^3\Psi_0^{(0)}}{\partial X\,\partial y^2}-\frac{\partial\Psi_0^{(0)}}{\partial X}\frac{\partial^3\Psi_0^{(0)}}{\partial y^3}=\frac{1}{R}\frac{\partial^4\Psi_0^{(0)}}{\partial y^4}. \qquad (4)$$

If the vorticity equation is linearized around the time-independent solution $\Psi^{(0)}$, the perturbation of the basic flow $\Psi\equiv\Psi_T-\Psi^{(0)}$ satisfies

$$\left[\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)\nabla^2-U_{0yy}\frac{\partial}{\partial x}+\epsilon\left(V_0\nabla^2\frac{\partial}{\partial y}-V_{0yy}\frac{\partial}{\partial y}-\frac{1}{R}\nabla^2\nabla^2\right)+O(\epsilon^2)\right]\Psi=0, \qquad (5)$$

where the functions $[U_0(y;X),\epsilon V_0(y;X)]\equiv[\partial\Psi_0^{(0)}/\partial y,-\epsilon(\partial\Psi_0^{(0)}/\partial X)]$ are the leading-order streamwise and cross-stream basic flow velocities, and $\nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$.

We only consider local plane wave solutions of (5) and assume Ψ to be a time-harmonic function of frequency ω and local wave number k defined by

$$k=-\frac{i}{\Psi}\frac{\partial\Psi}{\partial x}. \qquad (6)$$

At leading order in ϵ, the wave number k depends continuously on space variables only through the slow scale X and without ambiguity it will thereafter be denoted by $k(\omega;X)$. Furthermore, the perturbation streamfunction Ψ is subject to exponential decay boundary conditions at $y=\pm\infty$.

Upon integrating (6), plane waves of frequency ω and wave number $k(\omega;X)$ take the form

$$\Psi=\Phi(x,y;\epsilon)e^{i/\epsilon\int_{X_m}^{X}k(\omega;r)dr}e^{-i\omega t}, \qquad (7)$$

where Φ satisfies the conditions

$$\frac{1}{\Phi}\frac{\partial \Phi}{\partial x} \ll 1 \quad \text{for all } X \text{ and } y \tag{8}$$

and

$$\lim_{y\to\pm\infty} |\Phi| = 0. \tag{9}$$

The reference point X_m in the integral is chosen by imposing a local normalization condition.

Solution (7) constitutes the starting point in the WKBJ analysis. All solutions are considered as a superposition of local plane waves of the form (7) at any streamwise station X. The frequency ω is either given real as in the forced case, where the response to an oscillating source is sought, or an unknown complex quantity in the global mode problem, to be determined by imposing suitable boundary conditions in the streamwise direction. To cover both instances, ω is considered to be arbitrary and complex.

The equation for $\Phi(x,y;\epsilon)$ is immediately obtained by inserting (7) into (5). The resulting leading-order equation depends on the characteristic cross-stream scale of the wave. If this scale is the typical wavelength λ, Φ is found to satisfy the Rayleigh equation

$$\left[[-i\omega+ik(\omega;X)U_0(y;X)]\left(\frac{\partial^2}{\partial y^2}-k^2(\omega;X)\right) - iU_{0yy}(y;X)k(\omega;X)\right]\Phi=0, \tag{10}$$

and it is, therefore, a solution of the inviscid instability problem on the locally parallel basic flow $U_0(y;X)$. As stated by Crighton and Gaster,[3] the reduction to the inviscid equation (10) is indeed justified in the signaling problem (ω real) in regions where the local plane wave is spatially amplified. However, when it is spatially damped or when the frequency is complex, such a reduction is not guaranteed. There is *a priori* no reason to assume that the characteristic scale is everywhere inviscid.

If variations in the cross-stream direction are shorter, the viscous term may become dominant. By balancing viscous and inviscid effects in (5), the viscous cross-stream scale is found to be $y_v = y/\sqrt{\epsilon}$, and the leading-order equation takes the form

$$\left([-i\omega+ik(\omega;X)U_0(y;X)]\frac{\partial^2}{\partial y_v^2}-\frac{1}{R}\frac{\partial^4}{\partial y_v^4}\right)\Phi=0, \tag{11}$$

to be compared with (10). One can easily show that for scales other than y and y_v, the leading-order equation for Φ can always be obtained from either (10) or (11): if Φ varies faster than the inviscid $O(1)$ scale, the governing equation can be deduced from (11); if variations are slower, they can be deduced from the Rayleigh equation (10).

One notes that (10) and (11) are also the only two possible leading-order expressions as Re$\to\infty$, that can be deduced from the parallel Orr–Sommerfeld (OS) equation

$$\left[[-i\omega+ik(\omega;X)U_0(y;X)]\left(\frac{\partial^2}{\partial y^2}-k^2(\omega;X)\right) - iU_{0yy}(y;X)k(\omega;X) - \frac{1}{\text{Re}}\left(\frac{\partial^2}{\partial y^2}-k^2(\omega;X)\right)^2\right]\Phi=0. \tag{12}$$

This implies that, for any fixed X and Re$\to\infty$, the cross-stream variations of Φ are governed by Eq. (12). As a consequence, one immediately deduces that the problem of finding the leading-order equation satisfied by Φ can indeed be resolved in the context of parallel-flow analysis and be formulated as follows: For a given complex pair ω and $k=k(\omega;X)$ and a given streamwise velocity profile $U_0(y;X)$, what are the regions of the y plane where the eigenfunction Φ of the OS equation (12) satisfies, at leading order, the Rayleigh equation (10) when Re$\to\infty$? This issue is examined in the next section, by extending classical results pertaining to the large-Reynolds-number asymptotics of the OS equation.

III. LARGE-REYNOLDS-NUMBER ASYMPTOTICS OF THE ORR–SOMMERFELD EQUATION

The justification of the inviscid approximation essentially relies on a careful study of the singular behavior of eigenfunctions in the vicinity of so-called critical points y_c defined in the complex y plane by

$$-i\omega+ikU_0(y_c)=0. \tag{13}$$

Major results have been obtained during the 1940's and 1950's in the temporal framework where k is assumed real. An extension is given here for arbitrary complex ω and k. Let σ and δ be defined by

$$\sigma \equiv \omega_i - \frac{\omega_r}{k_r}k_i, \tag{14a}$$

$$\delta \equiv \left(\frac{\omega_r}{k_r}-U_{0,\min}\right)\left(\frac{\omega_r}{k_r}-U_{0,\max}\right), \tag{14b}$$

where the subscripts r and i denote the real and imaginary part, respectively. The parameter σ represents the temporal growth rate in a frame of reference moving at the phase speed ω_r/k_r while δ characterizes the value of the phase speed relative to the range of basic flow velocities: δ is negative (resp. positive) when there exists (resp. does not exist) a location on the real y axis where $U_0(y_*)=\omega_r/k_r$.

The following statements are now proven:

(i) If $\delta>0$ **or** $\sigma>0$, Φ satisfies the Rayleigh equation (10) at leading order for all real y; the inviscid eigenfunction is the asymptotic limit of the OS eigenfunction as Re$\to\infty$.

(ii) If $\delta\leq 0$ **and** $\sigma=0$, points y_* on the real y axis such that $U_0(y_*)=\omega_r/k_r$ are critical points y_c in the sense of Eq. (13) around which the Rayleigh equation (10) is not valid at leading order.

(iii) If $\delta\leq 0$ **and** $\sigma<0$, there may exist large $O(1)$ viscous intervals on the y axis where the OS eigenfunction does not reduce to a solution of the Rayleigh equation (10) when Re$\to\infty$.

Note that when k is real, each statement corresponds to a classical result of temporal theory, as discussed in Lin[8] and Drazin and Reid.[9]

Statement (a) is proven using the following standard result: there exist four formal independent solutions ϕ_j, $j=1,2,3,4$ of the OS equation that admit the following expansions:

$$\phi_1 \sim \phi_1^{(0)} + \frac{1}{\text{Re}}\,\phi_1^{(1)} + \cdots\,, \tag{15a}$$

$$\phi_2 \sim \phi_2^{(0)} + \frac{1}{\text{Re}}\,\phi_2^{(1)} + \cdots\,, \tag{15b}$$

$$\phi_3 \sim e^{\int^y \sqrt{\text{Re}(ikU_0 - i\omega)}dy}\left[\phi_3^{(0)} + \frac{1}{\sqrt{\text{Re}}}\,\phi_3^{(1)} + \cdots\right], \tag{15c}$$

$$\phi_4 \sim e^{-\int^y \sqrt{\text{Re}(ikU_0 - i\omega)}dy}\left[\phi_4^{(0)} + \frac{1}{\sqrt{\text{Re}}}\,\phi_4^{(1)} + \cdots\right]. \tag{15d}$$

The two functions ϕ_1 and ϕ_2 are commonly called the "inviscid" solutions of the OS equation. Their leading-order approximations $\phi_1^{(0)}$ and $\phi_2^{(0)}$ satisfy the Rayleigh equation (10) and viscous effects only appear as $O(1/\text{Re})$ correction terms. The two other functions ϕ_3 and ϕ_4 are purely "viscous" solutions. Their leading-order approximations satisfy Eq. (11). According to Wasow[18] and Lin,[8] the four *formal* functions (15a)–(15d) indeed represent *uniformly valid asymptotic* solutions of the OS equation as $\text{Re}\to\infty$, in any domain D where any two points can be connected by a curve along which $\Re e[\int^y \sqrt{\text{Re}(ikU_0 - i\omega)}]$ varies monotonically.

Under assumption (a), the square root $\sqrt{\text{Re}(ikU_0 - i\omega)}$ can be chosen so that its real part remains strictly positive on the entire real y axis. The quantity $\Re e[\int^y \sqrt{\text{Re}(ikU_0 - i\omega)}]$ is then a strictly increasing function on the entire real y axis and the above theorem holds. Application of the boundary conditions to a continuous solution of the form

$$\Phi = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + a_4\phi_4 \tag{16}$$

implies that $a_3 = a_4 = 0$, since ϕ_3 and ϕ_4 are exponentially large at $+\infty$ and $-\infty$, respectively. Thus the leading-order approximation for ϕ reduces to a linear combination $a_1\phi_1^{(0)} + a_2\phi_2^{(0)}$ of inviscid solutions governed by Rayleigh equation (10). This proves statement (a). (Note that no assumptions have to be made regarding the number of critical points. In a certain sense, this statement also generalizes the results of temporal theory found in textbooks which usually treat problems involving only one or two critical points.)

Note that when Eq. (10) is asymptotically valid on the entire real axis and k is real, Howard's semicircle theorem guarantees that ω_r/k_r is always in the range of U_0, i.e., $\delta \leqslant 0$. No such theorem can be derived when k is complex; therefore, configurations for which ω_r/k_r is outside the range of U_0, and σ is positive, may *a priori* exist.

The proof of statements (b) and (c) is similar to their equivalent in temporal theory. It is based on the analysis of OS solutions near critical points y_c in the limit $\text{Re}\to\infty$. Some of the OS solutions are known to have no uniform decomposition into formal solutions (15a)–(15d) in a full complex neighborhood of critical points. The vicinity of y_c is in fact partitioned into Stokes sectors delimited by Stokes lines defined by $\Re e[\int_{y_c}^y \sqrt{\text{Re}(ikU_0 - i\omega)}] = 0$. Close to a first-order critical point y_c where $U_{0y}(y_c) \neq 0$, the orientation of Stokes lines is given by

$$\arg(y - y_c) = \pi/6 - \frac{\arg(k)}{3} - \frac{\arg[U_{0y}(y_c)]}{3} + 2l\pi/3, \quad l = 0,1,2, \tag{17}$$

which means that three Stokes sectors of equal angle $2\pi/3$ emanate from y_c. An important result, first proven by Wasow[18] is the following: If in one sector a solution is approximated by an "inviscid" solution, say ϕ_2, that becomes singular at y_c, then it is necessarily asymptotic to a dominant viscous solution ϕ_3 or ϕ_4 in at least one of the two remaining Stokes sectors. [At least one of the two independent "inviscid" solutions has a singular expansion at y_c. If $U_{0yy}(y_c) \neq 0$, a logarithmic singularity appears at leading order in the expansion in powers of 1/Re.] It follows that, if the real axis crosses all three sectors, a region of the physical domain may be viscous.

This is indeed what happens when σ evolves from positive to slightly negative values with ω_r/k_r in the range of U_0 ($\delta \leqslant 0$). In such a case, at least one critical point y_c crosses the real axis through a point y_* such that $U_0(y_*) = \omega_r/k_r$. [Strict crossing takes place if $U_{0y}(y_*) \neq 0$. When $U_{0y}(y_*) = 0$ for $\sigma = 0$, y_* is a double critical point: two critical points converging from opposite sides of the real axis then collide to give the configuration illustrated in Fig. 4(a).] The evolution of the Stokes line network in the neighborhood of such a critical point is illustrated in Figs. 1(a)–1(c). When $\sigma > 0$ [Fig. 1(a)], the real axis cuts only one Stokes line: Both Stokes sectors containing the real axis are inviscid at leading order and the OS eigenfunction can be expanded into $a_1\phi_1 + a_2\phi_2$ as established above. If one excludes the exceptional case where the coefficient a_2 of the singular "inviscid" solution ϕ_2 is zero, the OS eigenfunction becomes viscous in the third pale grey sector of Fig. 1(a) and in the shaded neighborhood of the critical point y_c. When σ evolves continuously towards negative values [Figs. 1(b) and 1(c)], the Stokes sectors as well as the asymptotic behavior of the OS eigenfunction are expected to vary in a continuous manner. The decomposition of Φ into $a_1\phi_1 + a_2\phi_2$ should still be possible in the two Stokes sectors which contained the real axis when σ was positive. Provided the coefficient of the singular solution ϕ_2 is not identically zero during the process, the OS eigenfunction Φ is viscous in the third pale grey sector and in the shaded critical layer, as illustrated in Figs. 1(b) and 1(c). In Fig. 1(b), the real axis cuts through the critical layer and the viscous region is localized in a neighborhood of the critical point. In Fig. 1(c), the real axis crosses the viscous Stokes sector and a large interval of the real axis is dominated by viscous diffusion. Statements (b) and (c) have therefore been established.

In Fig. 1(b) the Stokes line network pertaining to a real critical point of temporal theory (k real) is shown in dashed lines (see figure 8.2 of Lin[8] for instance). Note that in more general situations the Stokes lines bounding the viscous sector do not necessarily correspond to the angles $+\pi/6$ and

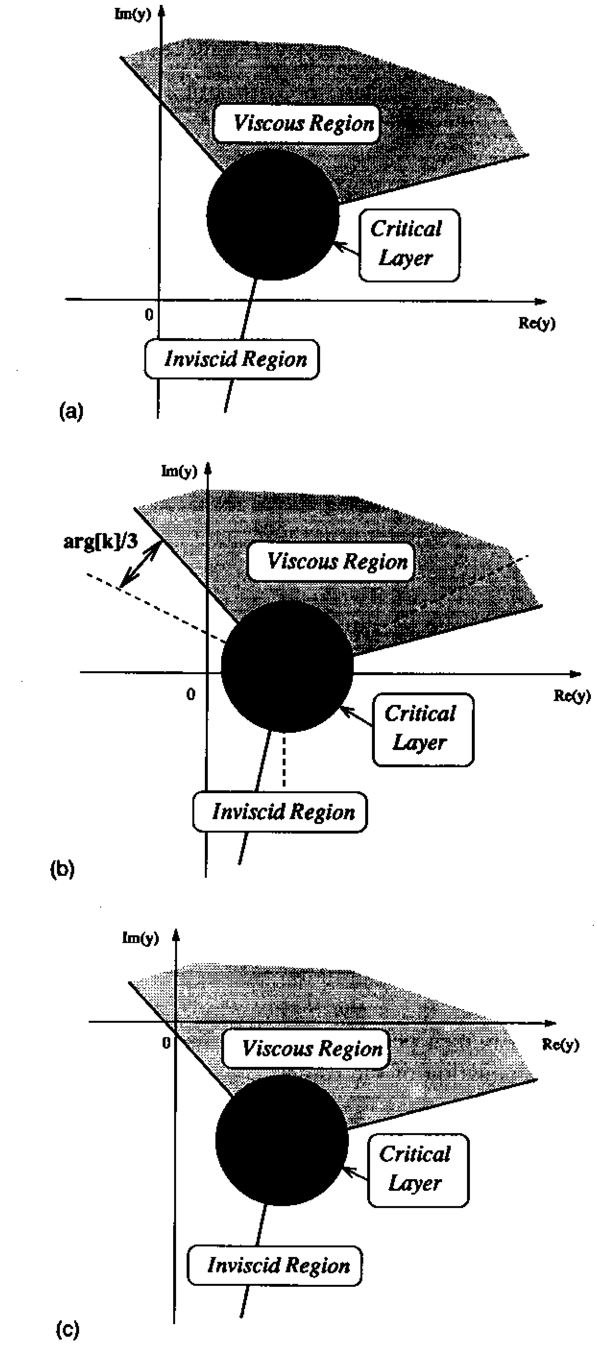

FIG. 1. Viscous and inviscid behavior of Orr–Sommerfeld eigenfunctions in the complex y plane in the neighborhood of a first-order critical point y_c; (a): $\sigma>0$, (b): $\sigma=0$, (c): $\sigma<0$.

$+5\pi/6$ but are subject to an additional rotation of angle $\arg(k)/3$ as specified by Eq. (17). If $\arg(k)=\pm\pi/2$, one of the Stokes lines delimiting the viscous region is asymptotic to the real axis near y_c. In this case, the viscous region is not confined to the critical layer but spreads along the real axis as far as the Stokes line is within a distance of order ϵ from the real axis.

We have shown that the existence of viscous regions on the real axis is guaranteed if damped modes ($\sigma<0$) can be obtained from amplified modes ($\sigma>0$) by a continuous process with ω_r/k_r remaining within the range of U_0. As illustrated in Fig. 1, the viscous regions are then determined by following the evolution of complex critical points and Stokes lines in the complex y plane as a function of σ. If such a continuous process cannot be identified, viscous regions do not necessarily exist for modes satisfying assumptions (c).

IV. VISCOUS REGIONS IN WEAKLY NONPARALLEL FLOWS

We now apply the preceding ideas to weakly nonparallel flow situations. By analogy with (14a) and (14b), the following quantities are then defined:

$$\sigma(X)\equiv\omega_i-\frac{\omega_r}{k_r(\omega;X)}\,k_i(\omega;X), \tag{18a}$$

$$\delta(X)\equiv\left(\frac{\omega_r}{k_r(\omega;X)}-U_{0,\min}\right)\left(\frac{\omega_r}{k_r(\omega;X)}-U_{0,\max}\right). \tag{18b}$$

From the conclusions of Sec. II and statement (a) of the last section, we know that at any point X where $\sigma(X)>0$ or $\delta(X)>0$, the characteristic cross-stream scale of local plane waves is everywhere inviscid and the approach of Crighton and Gaster[3] is justified. By contrast, if there is no such X station, nothing can be said regarding the characteristic scale of local plane waves at any point in the flow. (Except perhaps near $y=\pm\infty$ where local plane waves are inviscid in order to satisfy exponential decay boundary conditions.) Furthermore, if there is no region where $\sigma(X)$ evolves from positive to negative values with $\delta(X)<0$, nothing can be said concerning the viscous regions since the analysis of Sec. III cannot be applied. We therefore exclude these situations and assume that *there exists a point X_p in the flow where $\delta(X)$ is negative and $\sigma(X)$ changes sign.* The analysis of Sec. III used to prove statements (b) and (c) can then be directly applied. It follows that there exists a critical point $y_c(X)$ satisfying

$$-i\omega+ik(\omega;X)U_0[y_c(X);X]=0, \tag{19}$$

that crosses the real y axis for $X=X_p$ provided that $U_{0y}[y_c(X_p);X_p]\neq 0$. Figures 1(a)–1(c) then correctly describe the evolution with respect to X of the critical point and the viscous Stokes sector, Fig. 1(b) representing the situation prevailing at X_p. Different domains can then be distinguished in the real X-y plane, as sketched in Fig. 2. On the locally amplified side $[\sigma(X)>0]$, the characteristic cross-stream scale is inviscid as argued above. As X increases through X_p, viscous effects first appear in a critical layer surrounding $[X_p,y_c(X_p)]$. Finally, on the side where $\sigma(X)<0$, a large viscous region develops in a sector originating from the real critical point $[X_p,y_c(X_p)]$.

The angle of the viscous sector and its position with respect to the X and y axes near $[X_p,y_c(X_p)]$ can directly be obtained from Eq. (17) and definition (19) for $y_c(X)$. Note that when $\arg[k(\omega;X_p)]=\pm\pi/2$, one of the boundary lines of the viscous sector leaves the critical point $[X_p,y_c(X_p)]$ in a direction parallel to the y axis.

The particular case of spatial waves with real frequency ($\omega_i=0$) is now addressed. Any maximum of the wave amplitude $|\Psi|$ then occurs at a point (X_m,y_m) satisfying $k_i(\omega;X_m)=0$ and the spatial growth rate $-k_i$ changes sign as X goes through X_m. At X_m, the wave is therefore locally neutral and a classical result of temporal theory[19] applies:

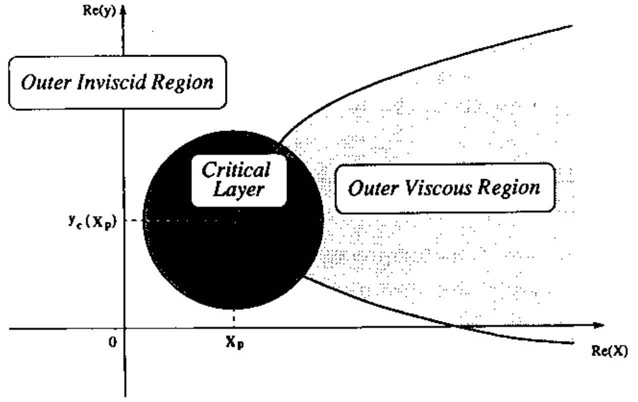

FIG. 2. Viscous structure of local plane waves near a first-order critical point $[X_p, y_c(X_p)]$ in the real (X,y) plane. The viscous region corresponds to locations such that $\sigma(X) \leq 0$.

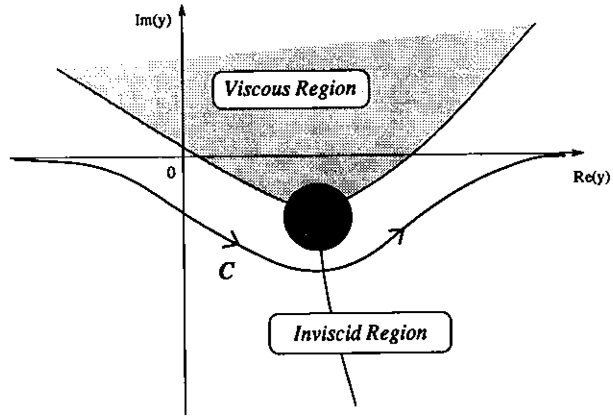

FIG. 3. Sketch of contour C avoiding viscous regions in the complex y plane.

$\omega/k(\omega;X_m)$ is within the range of $U_0(y;X_m)$. According to definition (18a), $\sigma(X)$ changes sign at X_m and the previous analysis can readily be applied: the X-y plane exhibits the same configuration as in Fig. 2 with $X_p=X_m$.

In closing, we emphasize the fact that the continuity of the local wave number $k(X)$ is fundamental to the present analysis. If this assumption is violated (one could think of a flow with a step change in stability properties induced by heat addition, for instance), the evolution of critical points and Stokes lines cannot be "followed," and the appearance of viscous regions cannot be predicted. However, the local wave number does not have to be differentiable and the results remain valid at branch points of the dispersion relation where $k(\omega;X)$ exhibits square-root-type singularities. The reader is however reminded that the local plane-wave approximation is generally different at such points (Monkewitz *et al.*[16]).

V. SPATIAL STRUCTURE OF PLANE WAVES IN WEAKLY NONPARALLEL FLOWS

In this section, the leading-order approximation for the evolution of local plane waves is given in the different regions of the X-y plane displayed in Fig. 2. At each X station the matching procedure between inviscid and viscous approximations is carried out in the complex y plane through the critical layer around $y_c(X)$. We restrict the study to streamwise locations that are not branch points of the dispersion relation, thereby allowing us to generate uniform approximations for all X. The implications of this assumption are discussed in Sec. VI.

A. Outer inviscid region

The characteristic cross-stream scale of plane waves is then the inviscid scale y. The function $\Phi(x,y;\epsilon)$ defined in (7) satisfies, at leading order, Eq. (10) and may be written as

$$\Phi(x,y;\epsilon) \sim \phi(y;X)A(x;\epsilon), \tag{20}$$

where $\phi(y;X)$ is a suitably normalized solution of Eq. (10) and $A(x;\epsilon)$ is an arbitrary amplitude. In order to match with the viscous solution, it is convenient to impose for all X the following normalization condition at the complex critical point $y_c(X)$:

$$\phi[y_c(X);X]=1. \tag{21}$$

The characteristic scale for the variations of $A(x;\epsilon)$ is at this stage unknown but it is slower than x, as required by condition (8).

If there exists in the complex y plane a path C going from $-\infty$ to $+\infty$ that remains in inviscid regions, the amplitude $A(x;\epsilon)$ can be obtained at higher order through a solvability condition involving only inviscid approximations. As shown by Crighton and Gaster,[3] this yields, at a station X which is not a branch point of the local dispersion relation, an amplitude equation of the form

$$b(X)\,\frac{\partial A}{\partial X}+p(X)A=0. \tag{22}$$

In such a case, the amplitude A is only dependent on X and is written as $A(X)$. The first-order correction to approximation (20) is easily evaluated as being $O(\epsilon)$, and the inviscid approximation to the perturbation field Ψ therefore reduces to

$$\Psi=[\phi(y;X)A(X)+O(\epsilon)]e^{(i/\epsilon)\int_{X_m}^{X}k(\omega;r)dr}e^{-i\omega t}. \tag{23}$$

The path C coincides with the real y axis at streamwise locations where $\sigma(X)>0$. At X stations where $\sigma(X)\leq 0$, it is necessarily a complex contour that avoids the viscous regions, as shown in Fig. 3. It should be emphasized that a solvability condition exclusively involving the inviscid approximation is possible only if such a path can be found. As soon as $\sigma(X)$ becomes zero, a situation may arise when two critical points [Fig. 4(b)] or a double critical points [Fig. 4(a)] lie on the real y axis; it will then be impossible to find a path C completely avoiding the viscous domains. The viscous approximation then intervenes explicitly in the solvability condition leading to the amplitude equation governing $A(x;\epsilon)$. The existence of a path C that stays in inviscid regions is however guaranteed for locations close to X_p in Fig. 2, if the real critical point is simple at X_p to avoid the

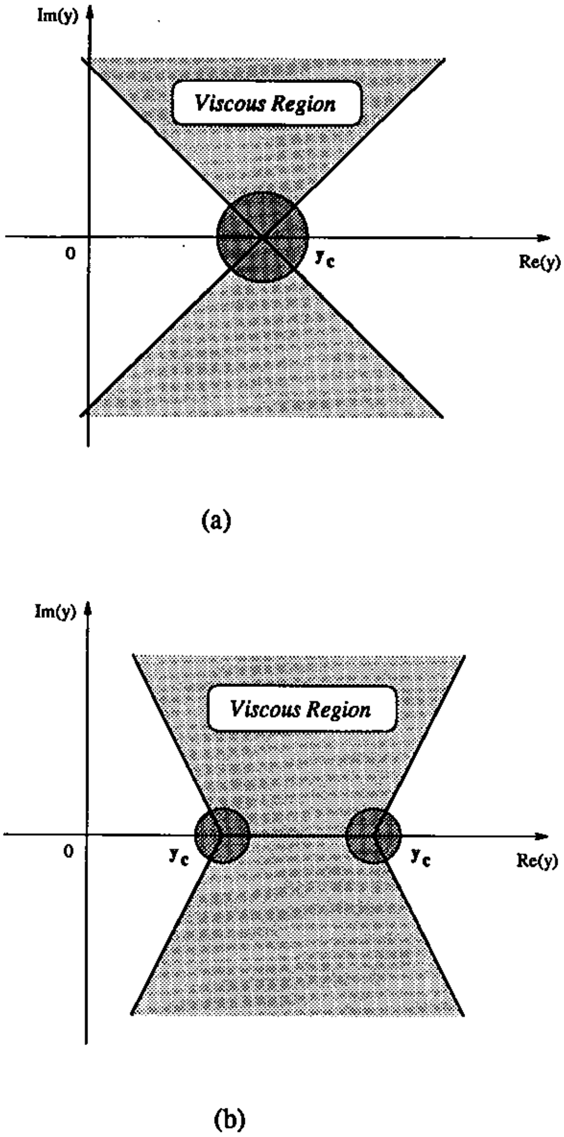

FIG. 4. Particular situations at $\sigma=0$ in which no contour C avoiding viscous regions exists; (a) two critical points merge at a real location; (b) two Stokes lines limiting viscous sectors associated to two distinct critical points coincide.

situation of Fig. 4(a), and if $\arg[k(\omega;X_p)]\neq\pm\pi/2$ to avoid the situation displayed in Fig. 4(b). In the following, these conditions are assumed to be satisfied.

B. Complex critical layer

The study of this region in the complex plane is necessary in order to match viscous and inviscid approximations. The use of WKBJ expression (7) leads to a uniform approximation with respect to X in the X-dependent complex critical layer, as shown below.

At any station X, the inviscid approximation (20) becomes singular at the complex critical point $y_c(X)$. Its expansion near $y_c(X)$ is commonly obtained by representing the Rayleigh eigenfunction $\phi(y;X)$ in terms of Frobenius series. Using the normalization condition (21) for $\phi(y;X)$, one obtains

$$\phi(y;X)=1+\{b_\phi(X)-\alpha_\phi(X)\ln[y-y_c(X)]\}[y-y_c(X)]+O\{[y-y_c(X)]^2\}, \tag{24}$$

where $\alpha_\phi(X)$ is defined by

$$\alpha_\phi(X)\equiv\frac{U^c_{0yy}(X)}{U^c_{0y}(X)}. \tag{25}$$

If $\alpha_\phi(X)$ is zero at the station under consideration, i.e., if the critical point is regular, the singularity appears at higher order but it does not fundamentally alter the ensuing analysis.

Following Lin[8] and Drazin and Reid,[9] one introduces the cross-stream scale

$$\check{y}\equiv\frac{y-y_c(X)}{\epsilon^{1/3}}.$$

In view of Eq. (24), the inviscid approximation (20) expressed in terms of the inner variable $\check{y}$, reads

$$\Phi(x,y;\epsilon)=\{1+\epsilon^{1/3}\ln(\epsilon^{1/3})\alpha_\phi(X)\check{y}+\epsilon^{1/3}[b_\phi(X)-\alpha_\phi(X)\ln\check{y}]\check{y}+O(\epsilon^{2/3})\}A(X). \tag{26}$$

Accordingly, a critical layer expansion is sought in the form

$$\check{\Phi}\sim[\check{\Phi}_0+\epsilon^{1/3}\ln(\epsilon^{1/3})\check{\Phi}_1+\epsilon^{1/3}\check{\Phi}_2+\cdots]. \tag{27}$$

The trivial solutions $\check{\Phi}_0$ and $\check{\Phi}_1$ can directly be deduced from expression (26) as

$$\begin{cases}\check{\Phi}_0(\check{y};X)=A(X),\\ \check{\Phi}_1(\check{y};X)=A(X)\alpha_\phi(X)\check{y}.\end{cases} \tag{28}$$

The third-order term in (26) is singular and is not directly a solution of the critical layer equation. The function $\check{\Phi}_2$ is obtained by solving

$$\left(\lambda(X)\frac{\partial^4}{\partial\check{y}^4}-i\check{y}\frac{\partial^2}{\partial\check{y}^2}\right)\check{\Phi}_2=i\alpha_\phi(X)A(X), \tag{29}$$

where

$$\lambda(X)=\frac{1}{k(\omega;X)RU_0[y_c(X);X]},$$

with the matching condition $\check{\Phi}_2\sim A(X)[b_\phi(X)-\alpha_\phi(X)\ln\check{y}]\check{y}$ as $|\check{y}|\to+\infty$ in the inviscid sector. This equation can be integrated in terms of generalized Airy functions as

$$\check{\Phi}_2(\check{y};X)=A(X)(b_\phi(X)\check{y}+\alpha_\phi(X)\times[\lambda(X)/i]^{1/3}\mathbf{B}_{j_0}\{[i/\lambda(X)]^{1/3}\check{y},2,1\}). \tag{30}$$

The function $\mathbf{B}_j(z,2,1)$ for any $j=1,2,3$ has been defined by Reid[20] and in the appendix of Drazin and Reid[9] as the solution of

$$\left(\frac{\partial^4}{\partial z^4}-z\frac{\partial^2}{\partial z^2}\right)f=1,$$

which is balanced in the sector S_j of the complex z plane (Fig. 5). As demonstrated by Reid,[20] the balanced functions $\mathbf{B}_j(z,p,q)$ appear systematically in the critical layer to represent solutions which are "inviscid" in two Stokes sectors and admit a logarithmic singularity at the critical point [func-

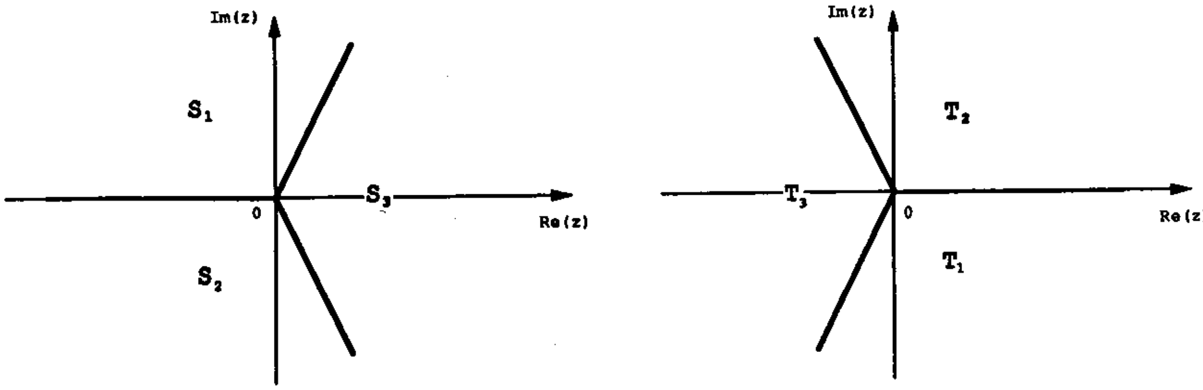

FIG. 5. Sectors S_j and T_j, $j=1,2,3$, pertaining to generalized Airy functions, in the complex z plane.

tion ϕ_2 defined by (15b)]. More explicitly, the approximation in the critical layer obtained in terms of the function $\mathbf{B}_j$ can be matched to the singular "inviscid" solution ϕ_2 of the OS equation in the outer continuation of both sectors T_k where $k=1,2,3$ and $k\neq j$. (One easily shows that the boundaries of the sector T_k are given by the critical layer expression of the Stokes lines defined in Sec. III.) By contrast, in the third sector T_j, the function $\mathbf{B}_j$ is dominant, i.e., it is exponentially large when $|z|\to\infty$ and can only be matched to a dominant outer viscous solution. It is through this kind of reasoning that the existence of viscous regions can in fact be proven. A more comprehensive account for strictly parallel flows can be found in Drazin and Reid.[9]

In expression (30), j_0 is chosen such that the sector T_{j_0}, after a rotation of angle $\arg\{[i/\lambda(X)]^{1/3}\}$, corresponds to the inner expression of the viscous sector drawn in Figs. 1(a)–1(c).

Using (7), (27), (28), and (30), the approximation in the complex critical layer is finally found to be

$$\check{\Psi}=A(X)e^{(i/\epsilon)\int_{X_m}^{X}k(\omega;r)dr}e^{-i\omega t}\{1+\alpha_\phi(X)\check{y}\epsilon^{1/3}\ln\epsilon^{1/3}+\epsilon^{1/3}(b_\phi(X)\check{y}+\alpha_\phi(X)[\lambda(X)/i]^{1/3}\times\mathbf{B}_{j_0}\{[i/\lambda(X)]^{1/3}\check{y},2,1\})+O(\epsilon^{2/3})\}. \tag{31}$$

In order to obtain the approximation pertinent to the real critical layer in the X-y plane sketched in Fig. 2, expression (31) has to be expanded near X_p with respect to the inner streamwise variable

$$\tilde{X}=\frac{X-X_p}{\epsilon^{1/3}}.$$

This task presents no difficulty and is left to the reader.

C. Outer viscous region

In this section the outer viscous approximation is obtained in the pale grey region of Fig. 2. In view of relation (3) between the Reynolds number Re and ϵ, the cross-stream characteristic variable is then viscous, and given by $y_v=y/\sqrt{\epsilon}$.

The leading-order equation governing the amplitude $\Phi^{(v)}$ in the outer viscous region is given by (11). It admits four independent solutions which correspond to the leading-order approximations of (15a)–(15d). The solution is dominated by viscous diffusion and can be written, after application of the normalization condition (21), as

$$\Phi^{(v)}=\phi^{(v)}(x,y;\epsilon)e^{(1/\sqrt{\epsilon})\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\,dy}, \tag{32}$$

where the square root is defined so that $\Re e[\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\,dy]>0$ for any y in the viscous sector. The characteristic scales for the variations of the function $\phi^{(v)}$ in the x and y directions are *a priori* unknown but from condition (8) and the integration of (11) one immediately obtains the restrictions $O(\epsilon)\leq(1/\phi^{(v)})\times(\partial\phi^{(v)}/\partial x)\ll 1$ and $O(1)\leq(1/\phi^{(v)})(\partial\phi^{(v)}/\partial y)\ll 1/\sqrt{\epsilon}$.

The equation for $\phi^{(v)}$ can be derived by inserting expressions (7) and (32) into Eq. (5), thereby leading to

$$-\frac{RU_0}{\sqrt{\epsilon}}\frac{\partial\phi^{(v)}}{\partial x}-2\sqrt{(-i\omega+ikU_0)R}\,\frac{\partial\phi^{(v)}}{\partial y}-5\partial_y[\sqrt{(-i\omega+ikU_0)R}]\phi^{(v)}+RV_0\sqrt{(-i\omega+ikU_0)R}\,\phi^{(v)}+RU_0\,\partial_X\left[\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\right]\phi^{(v)}=0. \tag{33}$$

Since matching between the complex critical layer and the inviscid region has already been accomplished in Sec. V B, it is convenient to match through the complex critical layer.

Considering the following expansion of $\mathbf{B}_{j_0}(z,2,1)$ (Drazin and Reid[9])

$$\mathbf{B}_{j_0}(z,2,1)\sim i\pi^{1/2}z^{-5/4}e^{(2/3)z^{3/2}},\quad z\in T_{j_0}\ \text{and}\ |z|\to\infty, \tag{34}$$

the behavior in the outer viscous region of the critical-layer amplitude $\check{\Phi}$ given by (27)–(30) is easily obtained as

$$\check{\Phi}\sim i\epsilon^{3/4}A(X)\alpha_\phi(X)\pi^{1/2}[\lambda(X)/i]^{3/4}[y-y_c(X)]^{-5/4}\times e^{2/(3\sqrt{\epsilon})[i/\lambda(X)]^{1/2}[y-y_c(X)]^{3/2}}[1+O(\epsilon^{1/3})]. \tag{35}$$

Since

$$e^{(1/\sqrt{\epsilon})\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\,dy}$$

$$\sim e^{(2/3\sqrt{\epsilon})[i/\lambda(X)]^{1/2}[y-y_c(X)]^{3/2}} \quad \text{when } y\to y_c(X),$$

matching between expressions (32) and (35) implies that

$$\phi^{(v)}\sim \epsilon^{3/4} iA(X)\alpha_{\phi}(X)\pi^{1/2}[\lambda(X)/i]^{3/4} \times[y-y_c(X)]^{-5/4} \quad \text{as } y\to y_c(X). \tag{36}$$

The function $\phi^{(v)}$ must therefore be sought in the form:

$$\phi^{(v)}(x,y;\epsilon)=\epsilon^{3/4}\xi(y;X). \tag{37}$$

By inserting this "ansatz" in Eq. (33), one immediately finds that the first term in (33) is negligible. Hence the leading-order equation for $\xi(y;X)$ reads

$$\frac{1}{\xi}\frac{\partial\xi}{\partial y}=-\frac{5}{2}\frac{\partial_y[\sqrt{(-i\omega+ikU_0)R}]}{\sqrt{(-i\omega+ikU_0)R}}+\frac{RV_0}{2}+\frac{RU_0}{2\sqrt{(-i\omega+ikU_0)R}}\times\partial_X\left[\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\right]. \tag{38}$$

Its solution is

$$\xi\sim B(X)[-i\omega+ik(X)U_0(X,y)]^{-5/4}e^{(R/2)\int_{y_c(X)}^{y}[V_0+(U_0/\sqrt{(-i\omega+ikU_0)R})\partial_X\{\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\}]dy}, \tag{39}$$

with $B(X)$ determined from condition (36) and expression (37) as

$$B(X)=i\pi^{1/2}A(X)\alpha_{\phi}(X)[\lambda(X)/i]^{1/3}. \tag{40}$$

By collecting the results (32), (37), (39), (40), the leading-order approximation of the local plane wave amplitude $\Phi^{(v)}$ in the viscous region is finally obtained as

$$\Phi^{(v)}\sim\epsilon^{3/4}i\pi^{1/2}A(X)\alpha_{\phi}(X)[\lambda(X)/i]^{1/3}[-i\omega+ik(X)U_0(X,y)]^{-5/4} \times e^{(R/2)\int_{y_c(X)}^{y}[V_0+(U_0/\sqrt{(-i\omega+ikU_0)R})\partial_X\{\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\}]dy}e^{(1/\epsilon)\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\,dy}. \tag{41}$$

The term

$$[-i\omega+ik(X)U_0(X,y)]^{-5/4}e^{(1/\sqrt{\epsilon})\int_{y_c(X)}^{y}\sqrt{(-i\omega+ikU_0)R}\,dy}$$

in the above expression is the WKBJ approximation of the viscous solution in parallel flows as obtained by Tatsumi and Gotoh.[21] The additional terms are due to nonparallel effects: The exponential factor $e^{(R/2)\int_{y_c(X)}^{y}V_0\,dy}$ is an amplitude correction induced by the $O(\sqrt{\epsilon})$ correction of the leading-order operator. The second exponential factor

$$e^{(R/2)\int_{y_c(X)}^{y}[(U_0/\sqrt{(-i\omega+ikU_0)R})\partial_X\{\int_{y_c(X)}^{y}(\sqrt{(-i\omega+ikU_0)R})\}]dy}$$

is due to the X dependence of the local cross-stream wave number $l=\sqrt{(-i\omega+ikU_0)R}$. It is important to notice here that this dependence has induced an $O(\sqrt{\epsilon})$ correction to the streamwise wave number which is not present in the outer inviscid domain.

Upon inspection of (7) and (41) at a given station X, one immediately sees that the amplitude in the viscous region exhibits fast oscillations and is exponentially larger than its counterpart in the inviscid region (Fig. 6). One notes however that, in the limit $\epsilon\to 0$, the strength of the damping factor $e^{(i/\epsilon)\int_{X_m}^{X}k(\omega;r)dr}$ in the streamwise direction remains much more important than the amplifying viscous factor in the cross-stream direction. The point of maximum amplitude therefore remains at the streamwise station where $e^{(i/\epsilon)\int_{X_m}^{X}k(\omega;r)dr}$ displays a maximum and the wave is locally spatially neutral.

VI. DISCUSSION

Approximations for linear local plane waves in two-dimensional weakly nonparallel shear flows have been obtained. The characteristic cross-stream scale has been determined for arbitrary complex frequency ω and local wave number $k(\omega;X)$ by extending classical parallel-flow results for real k. The inviscid approximation at a given station X is

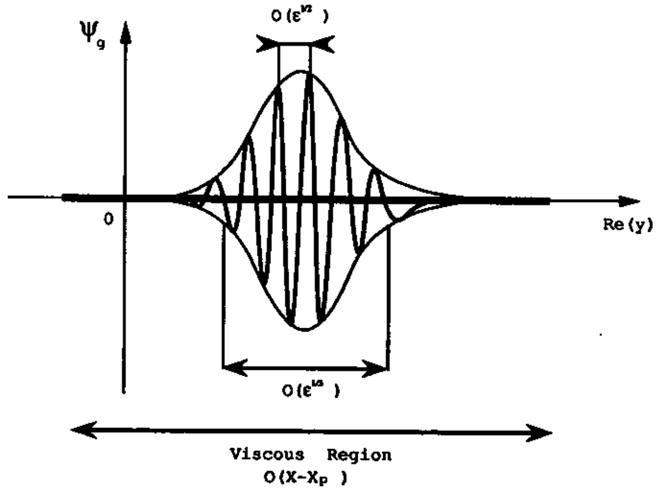

FIG. 6. Cross-stream variations of local plane waves in the viscous sector at a streamwise station where $\sigma<0$.

asymptotically valid for any y if the local plane wave is amplified in a frame of reference moving at the local phase speed ω_r/k_r, i.e., if $\sigma=\omega_i-k_i(\omega;X)\omega_r/k_r(\omega;X)>0$, or if the phase speed is not within the range of the streamwise velocity $U_0(y;X)$. As soon as the wave is locally damped ($\sigma<0$) with ω_r/k_r in the range of $U_0(y;X)$, the existence of viscous regions has been demonstrated.

The development of the viscous region in the neighborhood of a critical point $[X_p, y_c(X_p)]$ satisfying

$$-i\omega+ik(\omega;X_p)U_0[y_c(X_p);X_p]=0 \tag{42}$$

has been analyzed and proven to generically display the structure sketched in Fig. 2. In the process, leading-order approximations of local plane waves have been calculated everywhere.

These results are readily applicable to both the signaling and global mode problems. In the signaling case, local plane waves represent the flow response to an oscillating source at a given real frequency. As outlined in Huerre and Monkewitz,[13] for instance, this approach is physically relevant only in convectively unstable systems. In free shear flows the signaling problem has so far only been studied in regions where the wave is spatially amplified, i.e., for $\sigma>0$, thereby allowing the use of an inviscid approach. As shown by Crighton and Gaster[3] for jets and by Gaster *et al.*[4] for mixing layers, the outer inviscid expression (20) for the perturbation streamfunction is adequate to explain the observed variations of individual physical quantities, such as local growth rate and wavelength. A consistent description of local plane wave evolution has now been obtained around and beyond the point of maximum amplitude.

It must be said that, to our knowledge, no linear viscous waves have so far been detected experimentally or numerically in free shear flows because, in most cases, nonlinear effects become important well before the wave reaches its maximum amplitude. In fact, one has to keep in mind that theoretically the condition $|\Psi|_{\max}\ll 1$ is not sufficient to neglect nonlinear effects. As soon as $|\Psi_{\max}|$ is $O(1/\mathrm{Re}^{2/3})=O(\epsilon^{2/3})$, nonlinear terms come into play in the neighborhood of the critical point at the streamwise location of maximum amplitude and one has to proceed to a nonlinear-critical-layer analysis (Goldstein and Leib[7]).

The present results are also applicable to the description of linear global modes [Huerre and Monkewitz,[13] Monkewitz *et al.*[16]]. In the limit of slowly evolving basic flows, global modes can be decomposed locally into plane waves oscillating at the global frequency ω_g, and the analysis of Secs. IV and V directly applies.

The real issue is in fact to determine which local plane waves constitute a valid approximation of global modes at a given streamwise station. By ignoring the cross-stream structure and using a Ginzburg–Landau evolution model with two turning points in the streamwise direction, Chomaz *et al.*[14] and Le Dizès *et al.*[15] have succeeded in identifying two types of global modes: those with distant turning points (type 1) are approximated by a single local plane wave everywhere except in the neighborhood of the Stokes line connecting both turning points; global modes with a double turning point (type 2) are approximated everywhere by a single local plane wave.

Double-turning-point global modes have also been analyzed in shear flows in the context of an inviscid approach (Monkewitz *et al.*[16]). The global frequency is determined in that case by matching two subdominant WKBJ waves at the double turning point X_s. Inviscid approximations to the global modes in both the outer and double-turning-point regions have been calculated. The results obtained in the present article guarantee that at any location where $\sigma(X)$ is positive or $\omega_{g,r}/k_r(\omega_g;X)$ is not in the range of $U_0(y;X)$, such approximations are asymptotically valid for all y. We have also proven that they break down in the neighborhood of any critical point $[X_p, y_c(X_p)]$ satisfying (42) and in the viscous sector displayed on Fig. 2. In such regions, the critical layer and viscous approximations of Sec. V are then applicable, provided that they do not contain the double turning point X_s.

ACKNOWLEDGMENTS

This work is supported by DRET under Grant No. 92-098 (SLD,PH), AFOSR under Grant No. F49620-92-J-0471 (PAM) and SNECMA (SLD).

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