ARTICLE IN PRESS EJMFLU:2208

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European Journal of Mechanics B/Fluids ••• (••••) •••–•••

EUROPEAN JOURNAL OF MECHANICS B/FLUIDS

The role of boundary conditions in the instability of one-dimensional systems

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Received 2 June 2005; received in revised form 6 December 2005; accepted 20 January 2006

Abstract

We investigate the instability properties of one-dimensional systems of finite length that can be described by a local wave equation and a set of boundary conditions. A method to quantify the respective contributions of the local instability and of wave reflections in the global instability is proposed. This allows to differentiate instabilities that emanate from wave propagation from instabilities due to wave reflections. This is illustrated on three different systems, that exhibit three different behaviors. The first one is a model system in fluid mechanics (Ginzburg–Landau equation), the second one is the fluid-conveying pipe (Bourrières equation), the third one is the fluid-conveying pipe resting on an elastic foundation (Roth equation).

Keywords: Local/global instabilities; Absolute/convective; Wave reflections

1. Introduction

Numerous systems in mechanics can be addressed using one-dimensional wave equations, at which a set of boundary conditions are to be added when dealing with finite length systems. Many studies have sought to establish the instability conditions for systems of finite length in various geometrical and physical conditions (see for instance [1] in fluid mechanics, or [2] in fluid–structure interaction). Such systems, with the same local properties but with infinite length can also experience wave instability, i.e. *local* instability [1,3]. These unstable waves are often the main reason of the instability of finite length systems, i.e. *global* instability. But recently it has been shown that boundary conditions, through wave reflections, may induce instability of finite length systems despite local stability (see [4] in the case of a cantilevered fluid conveying pipe on an elastic foundation and [5] in the case of a swirling jet). This phenomenon has been attributed to the existence of over-reflecting boundaries. The reflection of waves on such boundaries results in an energy gain. On the other hand, over reflecting boundaries have been addressed in various fields, such as astrophysics [6], fluid mechanics [7] and fluid structure interaction [8,9]. In [10], the effect of asymmetric boundary conditions has been investigated in the Ginzburg–Landau equation. It has been found that the criterion for global insta-

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0997-7546/$ – see front matter
doi:10.1016/j.euromechflu.2006.01.001

bility of long systems can be that of local convective instability or that of absolute instability, depending on the degree of asymmetry and absorption of the boundaries. The aim of this paper is to provide a theoretical framework that can unify these results when a global instability is found, by quantifying the respective roles of boundary conditions and local waves in the unstable behavior of these finite length systems. The method presented here has been first detailed in [11,12]. It has been used to emphasize the destabilizing role of wave reflections in a locally stable system using a simplified model of vortex breakdown [5]. We extend here the approach to local instability.

1.1. Local stability

In this paper, *local* stability refers to the stability properties of waves borne by the medium. It is analyzed in terms of the local dispersion relation, which has the general form,

$$D(k,\omega;\mathbf{p})y(x,t)=0, \tag{1}$$

where $\mathbf{p}$ is a set of parameters, $y(x,t)=y_0\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ is the amplitude variable and x is the spatial coordinate. This links the wavenumbers k to the frequencies ω. Local stability is ensured if for any wave of infinite extent, the associated temporal evolution remains finite with time, so that for any real wavenumber k, $\mathrm{Im}(\omega)<0$. Convective and absolute instabilities may also be distinguished [13,1,3]. The convective or absolute nature of a local instability depends on the long time behavior of the impulse response $G(x,t)$ of the system. The system is linearly stable if G decreases to zero for long times along all rays $x/t=\mathrm{const}$, and unstable otherwise. If the unstable wavepacket is convected away from the source, the instability is said convective. Conversely, if the unstable wavepacket grows in place, the system is said to be absolutely unstable. The criterion for transition between convective and absolute instabilities is found in practice in terms of the wavenumber and the frequency by a residue analysis of the complex k-branches, see [13].

1.2. Global stability

When considering finite length systems with particular boundary conditions, the stability analysis is referred to as *global*. A particular value of the frequency ω is a global eigenfrequency of the finite length system if the associated spatial waves, with the wavenumbers given by the dispersion relation, can be combined to satisfy the boundary conditions at any time. This selects a discrete set of global eigenfrequencies, say ω_j, associated to global eigenmodes, ϕ_j. If one of these frequencies has a positive imaginary part, the system is said to be globally unstable, as the amplitude of one of the modes grows exponentially with time.

1.3. Model equations

1.3.1. Ginzburg–Landau model

The Ginzburg–Landau equation is a simple amplitude equation describing the dynamics of systems near the onset of linear instability. It can be seen as a model equation governing the disturbances of the velocity around an equilibrium state in the Navier–Stokes equations [1]. Since we are only interested in the onset of the instability, a linearized Ginzburg–Landau equation is considered here, as in [14]. The coefficients of the equation, although complex in the general case, will be taken real. This does not change the qualitative results [10]. The dispersion relation then reads,

$$D(\omega,k;\mu)=\mathrm{i}\left(\mu-k^2\right)+k-\omega=0. \tag{2}$$

This readily gives the local instability condition: the medium is locally unstable if $\mu>0$. The critical value of μ for transition between convective and absolute instability is also easily found as $\mu_a=1/4$ [1]. The particular boundary conditions considered in this paper are

$$y(0,t)=y(l,t)=0. \tag{3}$$

The global stability analysis is then straightforward: the system is globally unstable if $\mu>\mu_a+\pi^2/l^2$ (see for instance [15]). This critical value of μ for global instability is plotted as function of the length l in Fig. 1(a). The criterion for transition between absolute and convective instabilities $\mu=\mu_a$ is also shown.

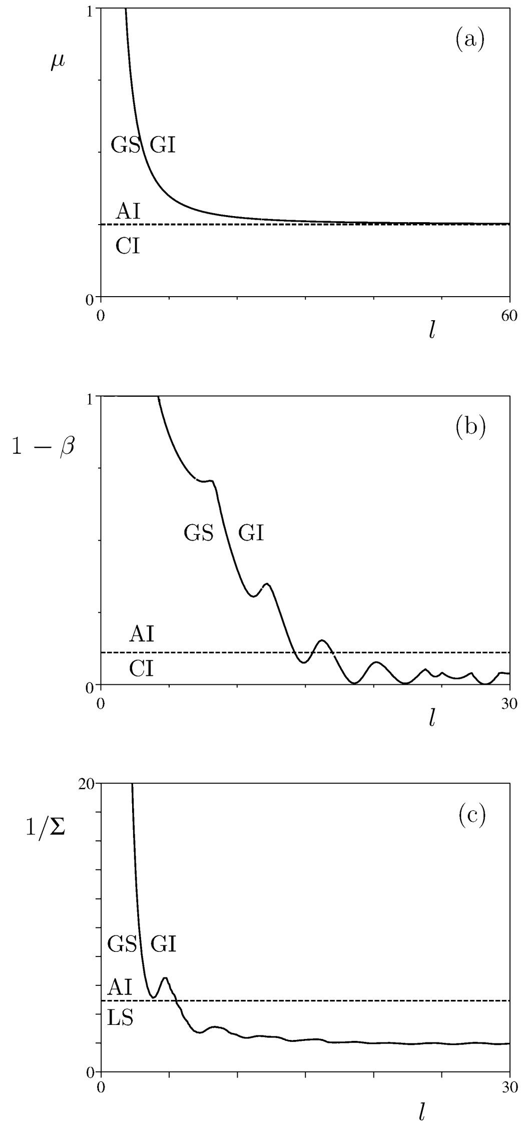

Fig. 1. Local versus global instability properties in the space of parameters; (GS), global stability; (GI), global instability; (AI), absolute instability; (CI), convective instability; (LS), local stability; (a) Ginzburg–Landau equation; (b) Bourrières equation; (c), Roth equation with $\beta = 0.1$.

1.3.2. Bourrières model

The fluid-conveying pipe is a dynamical system that attracted a lot of investigators and is recognized as a model for numerous fluid–structure interaction problems (see [2] for a detailed review). We consider here a pipe of mass per unit length m with flexural rigidity EI, no external tension or dissipation, conveying a fluid of mass per unit length M. The flow velocity is U. Using the reference length $\eta = \sqrt{EI/MU^2}$ and time $\tau = U^{-2}\sqrt{EI/\beta M}$, the non-dimensional dispersion relation reads [2],

$$D(\omega, k; \beta) = k^4 - k^2 + 2\sqrt{\beta}\, k\omega - \omega^2 = 0, \tag{4}$$

where $\beta = M/(M+m)$ is the mass ratio. In the following, Eq. (4) will be referred to as the Bourrières equation [16]. The system is known to bear unstable waves for any $\beta < 1$, instability being absolute if $\beta < 8/9$ and convective otherwise [3,17]. The stability of a cantilevered pipe of length l can be assessed considering the boundary conditions

$$y(0) = y'(0) = y''(l) = y'''(l) = 0, \tag{5}$$

where $l = L/\eta$ is the dimensionless length. Fig. 1(b) shows the critical value of β for global instability as function of l. Here, a Galerkin method is needed to determine the instability threshold, as in [18]. The limit between local absolute instability and convective instability is also shown. Note that the dimensionless length parameter l used here is actually identical to the dimensionless velocity parameter u commonly used to analyze the effect of the flow velocity U on the stability of a pipe of given length L (see [2]).

1.3.3. Roth model

When the fluid-conveying pipe lies on an elastic foundation the dispersion relation becomes [3,19],

$$D(\omega, k; \beta, \Sigma) = k^4 - k^2 + 2\sqrt{\beta}\, k\omega - \omega^2 + \Sigma = 0, \tag{6}$$

where $\Sigma = S\eta^4/EI$ is the non-dimensional stiffness of the elastic foundation, S being the dimensional stiffness. The local instability criterion is $\Sigma < (1-\beta)^2/4$ [19]. When locally unstable, the medium is absolutely unstable if $\Sigma < (8/9 - \beta)/(12\beta)$ [3]. A cantilevered pipe is again considered, the boundary conditions are thus given by Eq. (5). The global stability analysis of the fluid-conveying pipe on an elastic foundation may be also studied using a Galerkin method [20,4]. At a given value of $\beta = 0.1$, the critical value of Σ for global instability varies with length, as shown on Fig. 1(c). Also shown is the limit between local stability and instability, here absolute, at $1/\Sigma = 4/(1-\beta)^2 = 4.94$. This unusual transition from local stability to absolute instability has been discussed in a previous paper [4].

1.4. Relation between local and global stability

The three systems presented here exhibit three different behaviors, as it appears on Fig. 1. In case (a), the system cannot be globally unstable when the local instability is of the convective type, while it is allowed in case (b). And in case (c), the system can be globally unstable when locally stable. Concerning the long systems behavior, the three cases exhibited here are again very different. In case (a), the marginal stability criterion the long system is the convective to absolute instability transition criterion, while in case (b), it is the local stability to instability transition criterion. In case (c), this transition arises in the domain of local stability.

We propose to address these major differences using a new method to identify the global eigenmodes and eigenfrequencies of the system. The method is detailed in Section 2. It is then applied to the three above cited models in Section 3. Finally we draw several conclusions in Section 4.

2. From waves to modes

2.1. Local response to harmonic forcing

We consider in this section a system defined by the dispersion relation (1) subjected to an harmonic forcing localized in space,

$$F(x,t) = \delta(x)H(t)\,\mathrm{e}^{\mathrm{i}\omega_f t}. \tag{7}$$

As a global complex eigenfrequency of the system is searched, the circular frequency ω_f is considered complex. Such a forcing has hence an increasing or decreasing amplitude, depending on the imaginary part of ω_f. By an analysis of the residues in the complex ω- and k-planes, it is found [13] that the waves generated by this forcing are defined by the frequency ω_f and their associated wavenumbers through the dispersion relation. See for instance [1] for a complete discussion about the waves generated by this particular forcing.

The method for identifying the side $+x$ or $-x$ of the propagation of the waves is the following: Let $\tau_{\max} = \max_{k \text{ real}} \mathrm{Im}[\omega(k)]$ be the maximum growth rate of the system. If $\mathrm{Im}(\omega_f) > \tau_{\max}$, wavenumbers with a positive imaginary part define waves propagating downstream of the point of excitation while wavenumbers with a negative imaginary part define waves propagating upstream. By reducing $\mathrm{Im}(\omega)$ to zero and following the evolution of k-branches, upstream- and downstream-propagating waves are identified for values of ω_f with imaginary part smaller than $\tau_{\max}$.

A downstream wavenumber with a positive imaginary part corresponds to a spatially evanescent wave and with a negative imaginary part to a spatially amplified wave. A wavenumber that is real refers to a spatially neutral wave. As in [1], k^+ wavenumbers refer to waves propagating in the $x > 0$ space while k^- wavenumbers refer to waves

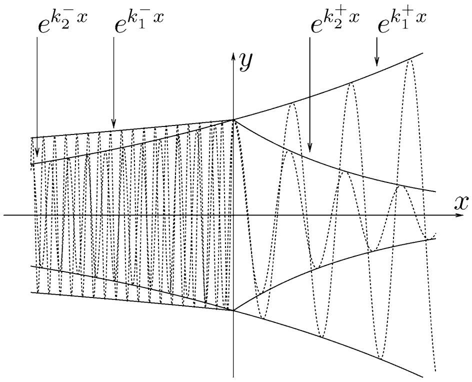

Fig. 2. Illustration of the waves generated by a forcing at a complex frequency, fluid-conveying pipe case at $\omega = 0.21 + 0.003\mathrm{i}$ for $\beta = 0.95$.

propagating in the $x < 0$ space. As an illustration, Fig. 2 represents the waves generated by a forcing at the frequency $\omega = 0.21 + 0.003\mathrm{i}$ on an infinite fluid-conveying pipe modelized by the Bourrières equation for $\beta = 0.95$. On this figure, and throughout the end of the paper, the subscript are chosen such that k_1^+ and k_1^- refer to the most amplified (or less evanescent) downstream and upstream waves respectively. One may keep in mind that the case of Fig. 2, the forcing and the envelope of the waves have an exponentially amplifying amplitude associated to a growth rate $\mathrm{Im}(\omega) = 0.003$.

2.2. The gain matrix

A system of finite length is considered now. It is forced at its downstream end. From the previous section, we know that several upstream waves, say n, are generated, with corresponding wavenumbers k_j^-, $j = 1$ to n. Their amplitudes at the forcing point are denoted a_j^- and they combine to form the displacement,

$$y^-(x,t) = \sum_{j=1}^{n} a_j^- \, \mathrm{e}^{\mathrm{i}[k_j^-(x-l)-\omega t]}. \tag{8}$$

The amplitudes of the waves at the upstream boundary b_j^- are therefore given by setting $x = 0$ so that $b_j^- = a_j^- \, \mathrm{e}^{-\mathrm{i}k_j^- l}$. In a matrix form it reads

$$\mathbf{b}^- = P^- \mathbf{a}^-, \tag{9}$$

where P^- is the diagonal matrix such that,

$$p_{jm}^- = \mathrm{e}^{-\mathrm{i}k_j^- l} \delta_{jm}. \tag{10}$$

The reflection of the n waves induces n downstream waves, with amplitudes at the upstream end b_j^+. The more general case where the number of upstream and downstream waves differ is discussed in the last section of the paper. At any time, the n waves must satisfy the n upstream boundary conditions, which are expressed as n linear conditions on the n unknowns b_j^+, $j = 1..n$. This can be also expressed in a matrix form,

$$\mathbf{b}^+ = R^- \mathbf{b}^-. \tag{11}$$

Here, R^-, referred to as a reflection matrix, is a full $n \times n$ matrix, with coefficients depending only on the boundary conditions. In the following examples, the reflection matrices will only be functions of the wavenumbers. In the same manner, the amplitudes at the downstream end of the downstream propagating waves are,

$$\mathbf{a}^+ = P^+ \mathbf{b}^+. \tag{12}$$

And finally, the downstream waves, incident on the downstream end, generate n upstream waves of amplitudes $a_j'^-$, which are related to the amplitudes a_j^+ of the impinging downstream waves by,

$$\mathbf{a}'^- = R^+ \mathbf{a}^+. \tag{13}$$

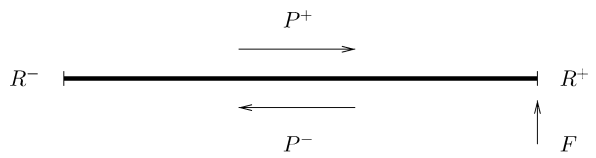

Fig. 3. Schematic view of the back and forth travel.

We may define a gain matrix G between the original upstream waves amplitudes a_j^- and those obtained after a back and forth travel $a_j'^-$,

$$\mathbf{a}'^- = G\mathbf{a}^-, \quad G = R^+P^+R^-P^-. \tag{14}$$

This matrix is a function of the parameters $\mathbf{p}$ of the dispersion relation, of the length l and of ω through the wavenumbers. The method described here is illustrated in Fig. 3.

2.3. From the gain matrix to the global modes

If for a given set of parameters $(\mathbf{p}, l)$ and a given circular frequency ω we have $\mathbf{a}'^- = \mathbf{a}^-$, no external forcing is necessary to sustain the waves. In that case $\mathbf{a}^-$ is an eigenvector of G with the corresponding eigenvalue being

$$\lambda = 1. \tag{15}$$

In this case the upstream waves obtained after a back and forth travel have the same amplitude and phase at the forcing point than the initial upstream waves. Moreover, at this particular complex frequency, it is possible to combine the four corresponding waves to satisfy the boundary conditions. This is the condition for ω to be an eigenfrequency of the system. An eigenfrequency has thus been identified. The corresponding eigenmode is obtained by combining the $2n$ waves,

$$\phi(x) = \left\{e^{ik_1^-(x-l)}, \ldots, e^{ik_n^-(x-l)}\right\} \cdot \mathbf{a}^- + \left\{e^{ik_1^+(x-l)}, \ldots, e^{ik_n^+(x-l)}\right\} \cdot \mathbf{a}^+ \tag{16}$$

with, after Eqs. (9), (11) and (12), $\mathbf{a}^+ = P^+R^-P^-\mathbf{a}^-$.

2.4. Respective roles of waves and boundary conditions

In the form of Eq. (14), the gain matrix G does not allow us to separate the contribution of propagation and the contribution of the reflections. However, only the eigenvalues of G, referred to as λ, are relevant here, as they allow to identify the eigenfrequencies through the criterion $\lambda = 1$, Eq. (15). These eigenvalues all satisfy

$$|\lambda| < |G|, \tag{17}$$

where $|\cdot|$ is here the 2-norm of a matrix or a vector, defined by

$$|M| = \max_{\mathbf{v}} \frac{|M\mathbf{v}|}{|\mathbf{v}|}, \qquad |\mathbf{v}| = \sqrt{{}^t\mathbf{v}\mathbf{v}}, \tag{18}$$

where the superscript t refers to the conjugate transpose. The 2-norm of a matrix M is found in practice by calculating the square root of the maximum eigenvalue of ${}^tMM$. The 2-norm of G satisfies the following property,

$$|G| \leqslant \left|R^+\right|\left|P^+\right|\left|R^-\right|\left|P^-\right|. \tag{19}$$

By denoting $\mathcal{P} = |P^+||P^-|$ and $\mathcal{R} = |R^+||R^-|$, we finally have, combining (17) and (19),

$$|\lambda| \leqslant \mathcal{P}\mathcal{R}. \tag{20}$$

The upper bound for the eigenvalues of G is now expressed in terms of a product of two scalars $\mathcal{P}$ and $\mathcal{R}$, exclusively depending on the properties of the waves and their reflections respectively. As P^+ and P^- are diagonal matrices, we simply have

$$\mathcal{P} = \max_j\left(e^{ik_j^+l}\right) \times \max_j\left(e^{-ik_j^-l}\right). \tag{21}$$

If $\mathcal{P} > 1$, the most amplified (or less evanescent) waves upstream and downstream have together an amplifying role. Conversely, if $\mathcal{P} < 1$, these waves have a damping role. Then, to allow the system to exhibit a global unstable frequency, that is to have $\lambda = 1$ for some frequencies with positive imaginary part, the boundaries must compensate with amplifying wave reflections, i.e. $\mathcal{R} > 1$. This allows to identify the situations where the global instability originates from a gain at boundary conditions. This is now applied to the particular systems described in Section 1.3.

3. Application

3.1. Ginzburg–Landau equation

This medium, defined by Eq. (2) bears two waves, one travelling upstream and one travelling downstream. The matrices of propagation and reflection and the gain matrix are hence only scalars. The reflection matrices are such that the boundary conditions (3) are satisfied, $R^+ = R^- = -1$. The gain matrix is therefore,

$$G(\omega) = \mathrm{e}^{\mathrm{i}[k^+(\omega) - k^-(\omega)]l}. \tag{22}$$

The condition for ω to be an eigenfrequency of the finite length system is $G(\omega) = 1$, which implies

$$\mathrm{Im}\big(k^+ - k^-\big) = 0, \qquad \mathrm{Re}\big(k^+ - k^-\big) = \frac{2n\pi}{l}. \tag{23}$$

The first condition is exactly the Kulikovskii criterion [15], see the discussion in the last section.

For this system, $\mathcal{R} = 1$ and boundaries are neutral for all frequencies. When the medium is convectively unstable, it is found using the dispersion relation (2) that $\mathrm{Im}(k^+) < -\mathrm{Im}(k^-)$ for every frequencies with positive imaginary part. Hence, $\mathcal{P} < 1$ and $|G| \leqslant \mathcal{P}\mathcal{R} < 1$ for any unstable frequency. Global instability is therefore impossible in the case of local convective instability, as it appears in Fig. 1(a).

3.2. Bourrières equation

The dispersion relation (4) is now used with the boundary conditions (5). The matrices of propagation read,

$$P^+ = \begin{bmatrix} \mathrm{e}^{\mathrm{i}k_1^+ l} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}k_2^+ l} \end{bmatrix}, \qquad P^- = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}k_1^- l} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_2^- l} \end{bmatrix}. \tag{24}$$

Hence,

$$\mathcal{P} = \big|\mathrm{e}^{\mathrm{i}(k_1^+ - k_1^-)l}\big| = \mathrm{e}^{\mathrm{Im}(k_1^+ - k_1^-)}. \tag{25}$$

See Appendix A for the calculation of the reflection matrices.

In Fig. 4, two sets of parameters (β, l) such that a global instability arises are considered. The corresponding unstable eigenfrequency is shown in the complex ω-plane. Here, the frequency is computed using a standard Galerkin method, see [2]. On the same figures are shown the regions where $\mathcal{P} > 1$ and $\mathcal{P} < 1$, derived from the dispersion relation (4) and using (21) and (24). For $\beta = 0.95$ and $l = 17.7$, Fig. 4(a), the global unstable eigenfrequency falls in the region where $\mathcal{P} < 1$. Wave propagations have hence a damping behavior and $\mathcal{R} > 1$ is necessary for the existence of this unstable eigenfrequency. Here, the global instability is due to wave reflections. Conversely, for $\beta = 0.912$ and $l = 15$, Fig. 4(b), the global unstable eigenfrequency appears in the region where $\mathcal{P} > 1$. In this case, $\mathcal{R}$ must be computed to conclude on the driver of the instability.

On Fig. 5, the evolutions of $\mathcal{P}$ and $\mathcal{R}$ are plotted just after the onset of global instability as function of β, in the range of convective instability. The length is equal to $l_c(\beta) + \epsilon$, $\epsilon \ll 1$, where l_c is the critical length of global instability (see Fig. 1(b)). For $\beta > 0.94$, we have $\mathcal{P} < 1$ and $\mathcal{R} > 1$, and the global instability is therefore due to wave reflections. For $\beta < 0.94$ $\mathcal{P} > 1$ and $\mathcal{R} > 1$, hence wave propagations and wave reflections may both contribute to the global instability.

We have thus clearly shown that for $\beta > 0.94$, despite local instability, the global instability emanates from a gain by wave reflections. On the other hand, for $\beta < 0.94$, both wave propagations and wave reflections may contribute to the global instability.

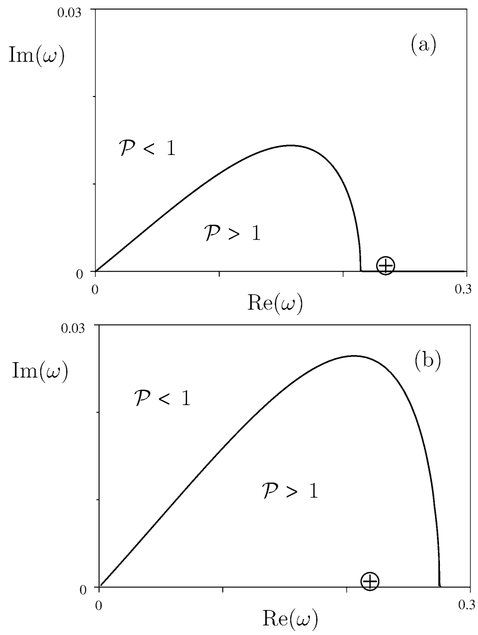

Fig. 4. Contour lines of $\mathcal{P} = 1$ in the upper complex ω-plane (——): $\oplus$, global eigenfrequencies; (a), $\beta = 0.95$, $l = 17.7$; (b), $\beta = 0.912$, $l = 15$.

3.3. Roth equation

The analysis is here straightforward. Using Eq. (6) in the range of local stability, it is found that all waves associated to frequencies with positive imaginary part are evanescent. Hence $\mathcal{P} < 1$ for any unstable global frequency that might exist. This ensures that global instability is always due to wave reflections in the range of local stability of the parameter Σ, see Fig. 1(c).

4. Concluding remarks

4.1. The role of wave propagation

For unidimensional systems described by a linear dispersion relation, four typical cases for the role of waves in a global instability have bee distinguished,

(a) Absolute instability: As previously mentioned, this case is not described by the present approach but provided the system is sufficiently long, global instability arises [15].
(b) Convective instability with $\mathcal{P} > 1$ for some unstable frequencies: In this case, wave propagations can drive a global instability.
(c) Convective instability with $\mathcal{P} < 1$ for all unstable frequencies: Wave propagations cannot drive a global instability.
(d) Stability: Wave propagations cannot drive a global instability.

4.2. The particular role of neutral and evanescent waves

In each system studied in this paper, global instabilities due to wave reflections have been observed only when there exists a range of real frequencies for which all corresponding wavenumbers are real, independently of the existence of a local instability. This range has been referred to as neutral in a previous work, [4]. At these real frequencies, $\mathcal{P} = 1$. But in the evanescent case, due to the possible existence of at least one upstream and one downstream neutral wave for some real frequencies, we may also have $\mathcal{P} = 1$ for these frequencies. The possibility for a global instability in neutral media is due to the fact that any combination of the waves is formed of only neutral waves and no energy is dissipated in the medium. Hence, a gain by wave reflections is not dissipated in the medium and a global instability may arise.

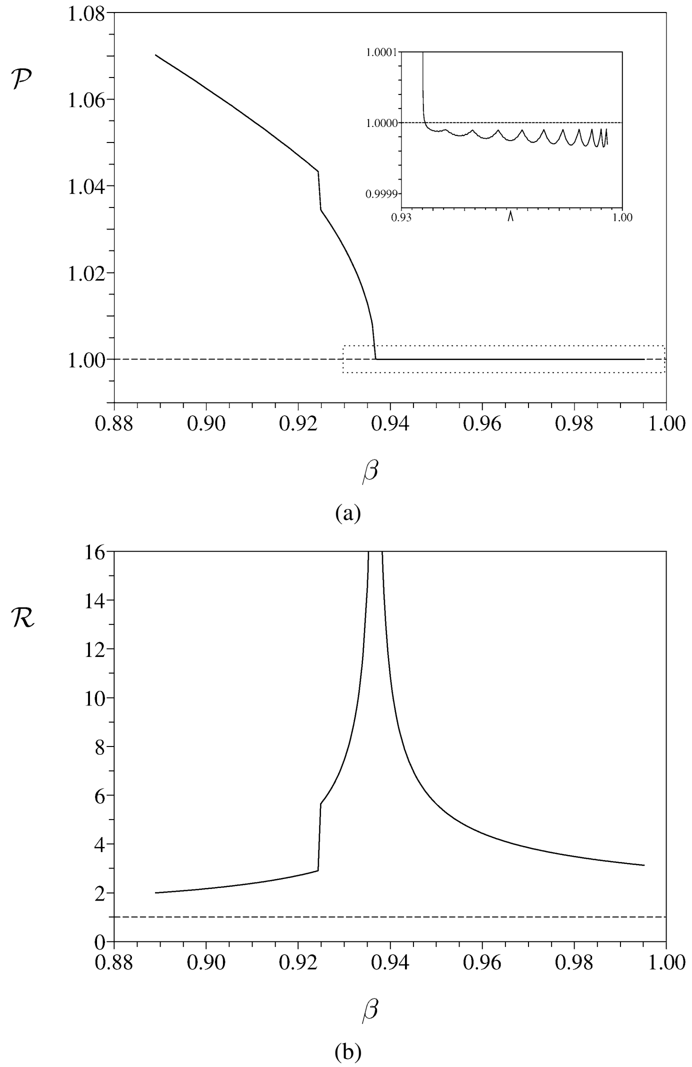

Fig. 5. (a) Value of $\mathcal{P}$ above the instability threshold as function of β in the range of convective instability; (b) value of $\mathcal{R}$. The jump appearing for both $\mathcal{P}$ and $\mathcal{R}$ at $\beta = 0.925$ corresponds to an abrupt change of l_c, see Fig. 1(b).

Conversely, in the evanescent case, any combination of the waves forming a global mode contains evanescent waves. These waves dissipate energy in the medium that cannot be compensated by a gain by wave reflections. Neutral and evanescent media can be distinguished by the value of $\mathcal{P}_{\min} = \min_j(\mathrm{e}^{\mathrm{i}k_j^+ l}) \times \min_j(\mathrm{e}^{-\mathrm{i}k_j^- l})$. For neutral media $\mathcal{P} = \mathcal{P}_{\min} = 1$ for some real frequencies, for evanescent media $\mathcal{P} = 1$ for some real frequencies but $\mathcal{P}_{\min} < 1$ for these frequencies.

4.3. Comparison with previous work

The Kulikovskii criterion [15] determines the global eigenfrequencies of the system when its length tends to infinity. It is strictly equivalent to state that $\mathcal{P} = 1$. For the Ginzburg–Landau equation, convective instability exists with $\mathcal{P} < 1$ in the upper half-plane of complex angular frequencies. This signifies that the Kulikovskii criterion is satisfied only by frequencies with negative imaginary part, predicting global stability for long systems. Conversely, for the Bourrières equation, convective instability exists with $\mathcal{P} > 1$ in the upper ω-plane. We have hence global instability for long systems. This criterion is able to explain the different behaviors of the marginal stability criteria for long systems on Figs. 1(a) and 1(b). In the case of the Roth equation, instability always arises at a neutral frequency. The

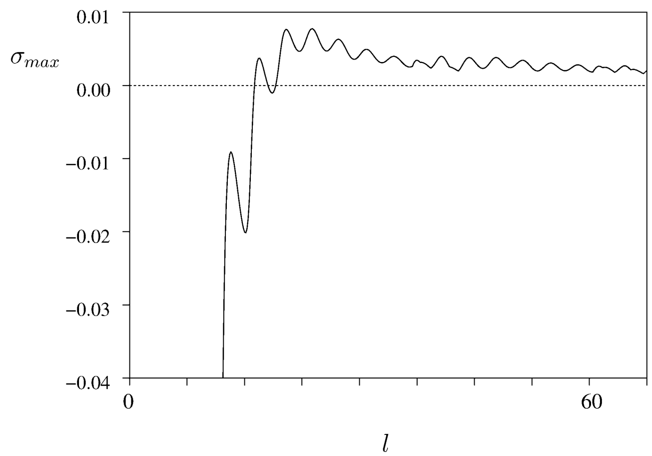

Fig. 6. Growth rate of the most unstable mode of a cantilevered fluid-conveying pipe on elastic foundation (Roth equation) as function of length for $\beta = 0.2$ and $\Sigma = 0.198$; (·····) maximum growth rate for $l \to \infty$ predicted by the Kulikovskii criterion.

criterion for global instability when the length grows to infinity has been found to be the criterion for transition from evanescence to neutrality. But as neutrality means that there exist real frequencies with associated real wavenumbers, the Kulikovskii criterion predicts global stability when $l \to \infty$. In Fig. 6, the maximum growth rate of global stability is compared to the maximum growth rate predicted by the Kulikovskii criterion when the length is increased. The former becomes positive at $l \simeq 20$ and tends to zero when $l \to \infty$ while the latter is zero. Hence our results and that of Kulikovskii are consistent; the maximum global growth rate tends asymptotically to zero, i.e. marginal stability, but has positive values for any length $l > 20$. This shows that although wave reflections may have an amplifying role that can lead to global instability, the exponential nature of wave propagations dominates when the length tends to infinity.

In the work of Lee and Mote [9] on the stability of a fluid conveying pipe, using dispersion relation (4), it is assumed that the bulk of the medium cannot contribute to the instability. More precisely, in their case which is in the range of absolute instability, two real wavenumbers and two complex wavenumbers are found at a given real frequency. It is argued in this paper that complex wavenumbers are associated to evanescent waves, which determines their direction of propagation. The group velocity is then used to determine the direction of propagation of the two other waves and the velocity at which energy is advected. Stating that only purely propagating waves (k real) convey energy, they calculate the energy gain by a wave reflection. Using an energy balance, predictions are given on the unstable behavior of the systems in presence of boundary conditions. This approach is erroneous for two distinct reasons: (a) when the system is unstable, the direction of propagation of waves can only be given by a branch analysis in the complex k- and ω-plane, [1], (b) the concept of group velocity as the velocity of energy transport is not valid when there is local instability [21]. Indeed, in the range of parameters where absolute instability arises the absolute frequency dominates the local response of the system and approaches based on the study of the waves generated by harmonic forcing, like ours or that of Lee and Mote, are not applicable.

In the work of Fox and Proctor [10], two Ginzburg–Landau equations are coupled by the boundary conditions. Unstable waves propagate in both directions, it is hence expected that $\mathcal{P} > 1$ for this system and global instability can exist even when the local instability is of the convective type. This is what was indeed observed.

4.4. *Further applications*

This general methodology can be useful for other systems that can be described by amplitude equations. For instance when the number of downstream and upstream waves differ, the matrices used in Eq. (14) are not square, but the main result, Eq. (20) remains. In the particular application of our method proposed by [5] for the stability of swirling jets, one upstream and three downstream waves exist. The instability of this system, similarly to the instability of the Roth equation presented here, is due to wave reflections, under the condition that there exist a neutral range.

These results can also be applied to active or passive control, since it makes possible to define the optimal design of boundaries that prevents such instabilities. The transfer function of a system is the response to forcing ratio, it is hence expected that the matrix gain can provide this global information on the system. Similarly, pseudospectra and transient growth properties help understand the instability transition, as it has been recently emphasized [22,23]. These properties should be contained in the gain matrix. These aspects are under current investigation.

Appendix A. Calculation of the reflection matrices for the Bourrières equation

Let us consider first the upstream boundary condition. Incident waves are the two upstream waves with local amplitudes b_1^- and b_2^-. Reflected waves are the two downstream waves with local amplitudes b_1^+ and b_2^+. The amplitudes of the reflected waves at the upstream end must be such that the total deformation,

$$y(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}\left(b_1^+ \mathrm{e}^{\mathrm{i}k_1^+ x} + b_2^+ \mathrm{e}^{\mathrm{i}k_2^+ x} + b_1^- \mathrm{e}^{\mathrm{i}k_1^- x} + b_2^- \mathrm{e}^{\mathrm{i}k_2^- x}\right), \tag{A.1}$$

satisfies the boundary condition, for all t, that is, for a clamped end, $y = \partial y/\partial x = 0$. There are two conditions for this boundary, leading to two algebraic equations with two unknowns b_1^+ and b_2^+. This can be expressed in a matrix form,

$$\mathbf{b}^+ = R^- \mathbf{b}^-, \tag{A.2}$$

with,

$$R^- = \frac{1}{k_2^+ - k_1^+}\begin{bmatrix} k_1^- - k_2^+ & k_2^- - k_2^+ \\ k_1^+ - k_1^- & k_1^+ - k_2^- \end{bmatrix}. \tag{A.3}$$

A similar calculation can be done for a free end and general form of a reflection matrix is finally obtained,

$$\begin{Bmatrix} a_1^r \\ a_2^r \end{Bmatrix} = \begin{bmatrix} r(k_1^i, k_2^r, k_1^r) & r(k_2^i, k_2^r, k_1^r) \\ r(k_1^i, k_1^r, k_2^r) & r(k_2^i, k_1^r, k_2^r) \end{bmatrix} \begin{Bmatrix} a_1^i \\ a_2^i \end{Bmatrix},$$

where the i and r indices refer to incident and reflected waves respectively. For a clamped end we have,

$$r(x,y,z) = (x-y)/(y-z), \tag{A.4}$$

and for a free end,

$$r(x,y,z) = \left(x^3 - x^2 y\right)/\left(z^2 y - z^3\right). \tag{A.5}$$

For the Roth equation described in Section 1.3.3, the boundary conditions are the same, so are the reflection matrices.

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