Spatio-temporal development of the long and short-wave vortex-pair instabilities

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(Received 19 October 1999; accepted 25 January 2000)

We consider the spatio-temporal development of the long-wave and short-wave instabilities in a pair of counter-rotating vortices in the presence of a uniform axial advection velocity. The stability properties depend upon the aspect ratio a/b of the vortex pair, where a is the core radius of the vortices and b their separation, and upon W_0/U_0 the ratio between the self-induced velocity of the pair and the axial advection velocity. For sufficiently small W_0/U_0, the instabilities are convective, but an increase of W_0/U_0 may lead to an absolute instability. Near the absolute instability threshold, spatial growth rates are larger than those predicted by temporal stability theory. Considering aeronautical applications, it is shown that instabilities of the type considered in this communication cannot become absolute in farfield wakes of high aspect ratio wings. [S1070-6631(00)01705-0]

Reduction of aircraft separations at landing and take-off leads one to consider the dissipation mechanisms of the vortex wakes.[1] Sufficiently far from the aircraft, these wakes amount to a pair of counter-rotating vortices, and two types of instabilities are thought to participate in their dissipation; the long-wave instability first considered by Crow[2] and the short-wave instability characterized by Moore and Saffman[3] and Tsai and Widnall.[4] These instabilities have been fully described in the framework of temporal stability analysis which predicts the "natural" growth of perturbations in the reference frame of the vortex pair. Such a temporal development has been observed in laboratory experiments[5,6] and in numerical simulations.[7] Forcing of these instabilities by onboard control devices is envisaged to accelerate the wake dissipation.[6,8,9] This leads one to consider the spatio-temporal development of the perturbations in the frame of the aircraft. Two situations may occur.[10,11] If the instabilities are convective, perturbations are amplified while being advected but do not amplify in the frame of the airplane. In this case one may perform a spatial stability analysis, which describes the response of the wake to harmonic forcing. On the other hand, if the instabilities are absolute, perturbations are amplified in place, eventually leading to self-sustained oscillations. Recent analyses have revealed the presence of absolute instabilities in isolated vortices in the presence of core axial flow.[12–14] The scope of the present communication is the evaluation of the absolute or convective nature of the long and short-wave instabilities of a vortex pair in the presence of a uniform axial advection, and the computation of their spatial growth rates in the convective regime.

Base flow. We consider a pair of counter-rotating vortices, of core radius a, circulation $\pm\Gamma$, separated by a distance b. The vortex cores are assumed to have constant vorticity (Rankine-type vortices). The vortices are moving in a direction perpendicular to their axes with a self-induced velocity $W_0=\Gamma/2\pi b$ [see Fig. 1(a)], and they are advected along their axis with a velocity U_0 [Fig. 1(b)]. This base flow may then be characterized by the two dimensionless parameters a/b and W_0/U_0.

Temporal and spatio-temporal stability. In principle, an inviscid stability analysis consists of linearizing the Euler equations around a base flow such as the vortex pair described above. After decomposing the perturbations into normal modes proportional to $e^{i(kx-\omega t)}$ a dispersion relation $D(k,\omega;a/b,W_0/U_0)$ which relates the pulsation ω and wave number k may be obtained, given a/b and W_0/U_0. As it is usually very difficult to obtain explicitly the "full" dispersion relation derived from the linearized Euler equations, one is lead to consider some physically relevant asymptotic limit in which the dispersion relation can be obtained analytically. In the following two asymptotic limits will be considered: the long- and the short-wave limits obtained by assuming $(a/b)^2\ll 1$ and, respectively, $|kb|=O(1)$ and $|ka|=O(1)$.

The temporal stability analysis assumes k real and ω complex. The temporal growth rate is given by the imaginary part ω_i of the pulsation. Galilean invariance leads to independance of the temporal growth rate upon the frame of reference,[10,11] consequently $\omega_i(k;a/b)$ does not depend upon W_0/U_0. We will denote by $\omega_{i,\max}^T(a/b)$ the maximum temporal amplification over all real wave numbers k, given a/b.

As mentioned for instance in Refs. 10 and 11, the abso-

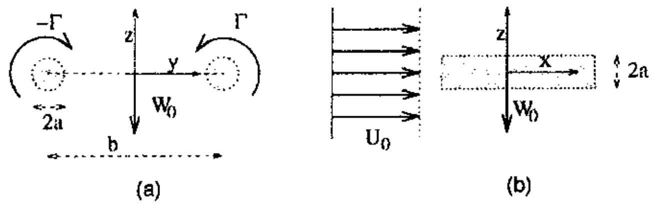

FIG. 1. Vortex pair base flow. (a) View in a plane orthogonal to the axis of the vortices, (b) side view.

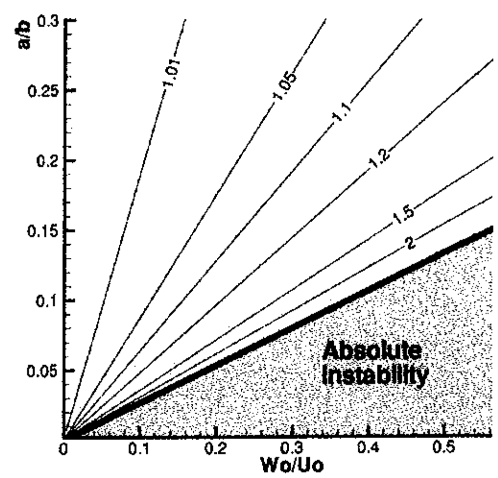

FIG. 2. Short-wave instability. Absolute and convective regions. The ratio of spatial to temporal growth rates is displayed in the convective region.

lute or convective nature of the instabilities may be determined through inspection of the saddle points (k_0, ω_0) which satisfy the system,

$$d\omega/dk(k_0,\omega_0;a/b,W_0/U_0)=0, \tag{1}$$

$$D(k_0,\omega_0;a/b,W_0/U_0)=0. \tag{2}$$

A necessary condition for absolute instability is that one can find a saddle point (k_0,ω_0) such that $\omega_{0,i}>0$. This condition is not sufficient, and the saddle point must also verify a causality condition; it must result from a "pinching" between spatial branches coming from opposite sides of the complex k-plane. These two conditions constitute the Briggs–Bers criterion (see Refs. 10,11). When this criterion is not verified the instability is convective and the spatial stability analysis may be performed by assuming ω real and k complex. The spatial growth rate is given by $-k_i$, and we denote $k^S_{i,\max}(a/b;W_0/U_0)$ as the maximum spatial growth rate over all ω, given a/b and W_0/U_0. In the limit $W_0/U_0\to 0$, spatial results can be related to temporal ones by means of an asymptotic expansion of the dispersion relation,[11]

$$k^G_{i,\max}(a/b)=-\omega^T_{i,\max}(a/b)/U_0. \tag{3}$$

Relation (3) is equivalent to Gaster's relation (see Ref. 11). In the following this relation will be used in order to compare the maximum amplification rates $k^S_{i,\max}(a/b;W_0/U_0)$ predicted by the spatial theory and the corresponding temporal results.

Short-wave instability. The short-wave instability, first considered by Moore and Saffman[3] and Tsai and Widnall,[4] develops on wavelengths of the order of the vortex core, i.e., $|ka|=O(1)$. This instability arises in the neighborhood of wave number values k_c where a resonance condition occurs between two inertial waves of one vortex and the straining field induced by the other vortex. The dispersion relation is derived in Refs. 3 and 15 by using an asymptotic method, assuming a weak strain [which is equivalent to consider $(a/b)^2\ll 1$]. The addition of a uniform advection U_0 leads to a "Doppler shift" of the pulsation from ω to $\omega-U_0k$. Considering the dimensional wave number and pulsation, the dispersion relation is

$$\left(\frac{2\pi a^2}{\Gamma}\right)^2(\omega-kU_0)^2=Q^2(ka-k_ca)^2-(a/b)^4R^2. \tag{4}$$

Temporal stability analysis[3,4] predicts the existence of a narrow band of unstable wave numbers centered on each k_c, with a width $|ka-k_ca|<(a/b)^2R/Q$ and a maximal amplification rate $\omega^T_{i,\max}=\Gamma/(2\pi b^2)R$. The coefficients Q and R have been determined as functions of k_c in Ref. 4. The first unstable band is centered at $k_ca=2.50$ and corresponds to $R=1.142$ and $Q=0.266$. Other unstable bands are centered at larger wave number values k_c, and correspond to similar values of R (and hence similar growth rates) and smaller values of Q. In Refs. 3 and 4, dispersion relation (4) is solved only for real values of k, but the asymptotic methods used to derive it remain valid for complex k. Moreover, this dispersion relation is quadratic in k and ω and its convective/absolute and spatial stabilility analyses are straightforward. Following, for instance, Ref. 16. pp. 276–279, one may verify that, applying the Briggs–Bers criterion to dispersion relation (4), the instability is absolute if

$$W_0/U_0>Q^{-1}(a/b). \tag{5}$$

In the convectively unstable regime it is found that the ratio between the spatial growths given by the spatial and the temporal theory is

$$\frac{k^S_{i,\max}}{(-\omega^T_{i,\max}/U_0)}=\left[1-Q^2\left(\frac{W_0}{U_0}\right)^2\left(\frac{a}{b}\right)^{-2}\right]^{-1/2}. \tag{6}$$

Figure 2 shows the results for the first band of unstable wave numbers. The absolute (gray shaded) and convective (white) regions are depicted in the $(W_0/U_0,\ a/b)$ plane, and the ratio given by Eq. (6) is displayed in the convective region. As expected, when $W_0/U_0\to 0$ the instability is convective and there is almost no difference between spatial and temporal results. When W_0/U_0 is increased, the spatial theory predicts growth rates which are larger than the growth rates converted from temporal theory. Near the absolute instability threshold, large differences can be attained. For other bands of unstable wave numbers, Eq. (5) indicates that absolute instabilities occurs at larger values of W_0/U_0.

Long-wave instability. The long-wave instability, first considered by Crow,[2] corresponds to symmetrical displacements of the vortex centerlines on scales such that $|kb|=O(1)$, when $a/b\ll 1$. Crow[2] modeled the vortices as two vortex filaments and computed their velocities with the Biot–Savart law and the cutoff approximation. Crow's derivation is also reproduced by Saffman (pp. 235–238 of Ref. 15). After replacing the pulsation ω by $\omega - kU_0$ to account for the uniform axial advection, the dispersion relation given in Refs. 2 and 15 takes the form,

$$\left(\frac{2\pi b^2}{\Gamma}\right)^2 (\omega - kU_0)^2 + (1-\psi+\varpi)(1+\chi-\varpi) = 0. \quad (7)$$

In this relation ψ and χ are functions characterizing the mutual straining of the vortices, and the function ϖ accounts for the self-induced rotation of the vortices. These functions were expressed by Crow as Biot–Savart integrals, with real k. We extend the validity of dispersion relation (7) for complex k as follows. First, it has been shown[8,9,15] that the function ϖ corresponds to the pulsation of a bending wave on an isolated vortex. An expression of this pulsation for $|ka| \ll 1$ and complex k is given in Ref. 13,

$$\varpi = \frac{(kb)^2}{2}\left(\ln\frac{2}{skb} - \ln\frac{a}{b} - \gamma + \frac{1}{4}\right), \quad (8)$$

where s is the sign of k_r (real part of k) and $\gamma = 0.577$ is Euler's constant. Then, functions ψ and χ can be evaluated by solving a Laplace equation in the potential flow surrounding one vortex and matching the solution with a displaced vortex line, as suggested in Ref. 9. This leads to the following expressions:

$$\psi = skbK_1(skb) + (kb)^2 K_0(skb), \quad (9)$$

$$\chi = skbK_1(skb), \quad (10)$$

where K_0, K_1 are modified second kind Bessel functions. With these expressions, the dispersion relation (7) is analytic in the complex half-plane defined by $k_r > 0$, which is sufficient for the spatio-temporal stability theory to apply.[10] Moreover, as expected, for real k it corresponds to the expressions given by Crow[2] and Saffman.[15] The temporal stability analysis[2,8,15,17] shows the existence of an instability domain for $|kb| = O(1)$. For moderate values of a/b, the maximal temporal growth rate is[2,15] $\omega^T_{i,\max}(a/b) \approx 0.8\Gamma/(2\pi b^2)$.

A combination of Eqs. (1), (2) and (7) shows that the wave numbers k_0 of the saddle-points are solutions of

$$\left(\frac{W_0}{2U_0}\right)^2 [(1-\psi+\varpi)(\chi'-\varpi') - (\psi'-\varpi')(1+\chi-\varpi)]^2$$
$$+ (1-\psi+\varpi)(1+\chi-\varpi) = 0, \quad (11)$$

where $'$ denotes the differentiation with respect to kb. The absolute/convective and spatial stability analyses of the long-wave instability were performed by numerically solving Eqs. (7) and (11) in the complex k-plane using a standard Newton–Raphson method. When applying the Briggs–Bers criterion both conditions presented above were taken into account. Figure 3 displays the absolute (gray shaded) and convective (white) regions in the plane of parameters, as well as the ratio between the spatial and temporal growth rates [see Eq. (6)] in the convective region. As found in the case of the short-wave instability, the spatial growth given by spatial theory may be larger than the one given by temporal theory.

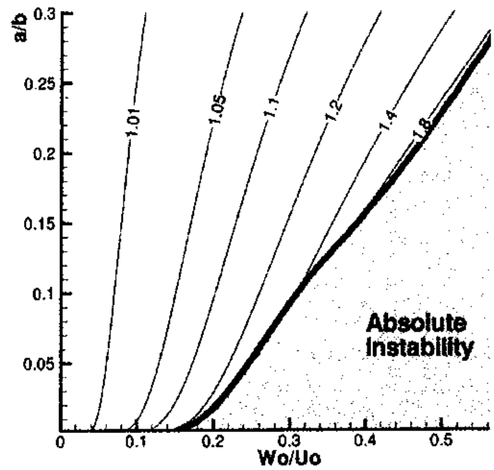

FIG. 3. Long-wave instability. Absolute and convective regions. The ratio of spatial to temporal growth rates is displayed in the convective region.

Note that, contrary to the short–wave case, a finite ratio $W_0/U_0 > 0.14$ is now needed to promote absolute instability in the limit $a/b \to 0$.

As an illustration, Fig. 4 shows the most amplified spatial mode corresponding to $a/b = 0.15$ and $W_0/U_0 = 0.166$. The displacements of the vortex centerlines in the z and y direction are such that[2,15,17]

$$\frac{\hat{z}}{\hat{y}} = \sqrt{\frac{1-\psi+\varpi}{1+\chi-\varpi}}, \quad (12)$$

where $y = \hat{y}e^{i(kx-\omega t)}$ and $z = \hat{z}e^{i(kx-\omega t)}$. In the spatial case, this ratio is complex, which leads to a slight helical shape for the vortices, as it can be observed in Fig. 4.

Application. Considering aeronautical applications, equating the induced drag and the wake kinetic energy for a pair of Rankine vortices leads to[2,8,15] $a/b \approx 0.1$. If U_∞ denotes the aircraft velocity, one has

$$W_0/U_\infty = \frac{C_L}{4\pi AR}\left(\frac{b_0}{b}\right)^2,$$

where C_L is the lift coefficient, AR the wing aspect ratio and b_0 the wing span (note that U_∞ differs from the axial veloc-

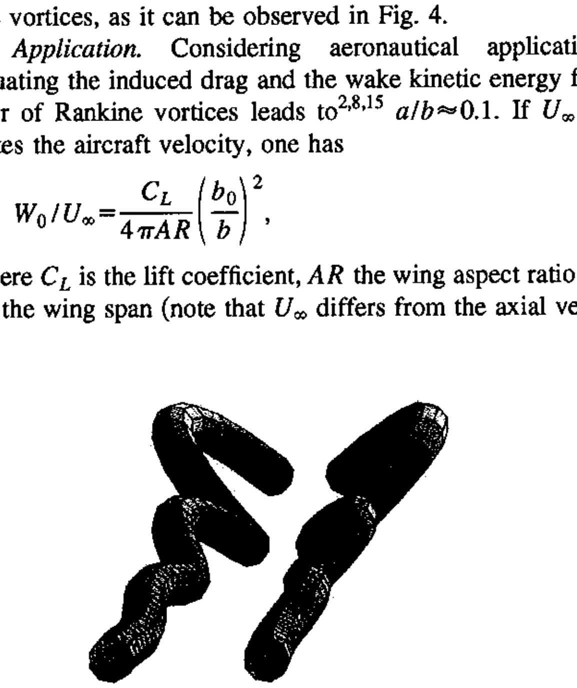

FIG. 4. Most amplified long-wave spatial mode for $a/b = 0.15$ and $W_0/U_0 = 0.166$. Perspective and top views.

ity U_0 because the plane of the wake is tilted with respect to the flight axis). Considering an elliptically loaded wing ($b/b_0 = \pi/4$) of high aspect ratio, one obtains $W_0/U_0 \approx 0.13 C_L/AR$. For a conventional transport aircraft in landing configuration, $C_L \approx 2$ and $AR \approx 10$, thus giving $W_0/U_0 \approx 0.026$. We conclude that, for high aspect ratio wings, the instabilities considered in this communication are convective and the spatial growth rates are very close to temporal ones. On the other hand, in the wake of low aspect ratio wings or high lift devices, the temporal analysis may fail to provide a good description of the development of these instabilities and absolute instabilities could take place.

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