Absolute and Convective Instabilities in Nonlinear Systems

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The concepts of absolute and convective instability are extended to nonlinear systems with broken Galilean invariance. As an illustrative model we describe the behavior of a flow, homogeneous in a semi-infinite domain, which undergoes a subcritical pitchfork bifurcation. The classical bifurcation phenomenology is shown to be nontrivially affected by the presence of a nonremovable advection term. In particular the existence of a hysteresis loop is shown to be restricted to the nonlinear absolute instability range. A qualitative description of the possible scenarios likely to arise in subcritically bifurcating open flows is outlined and a practical test is suggested to determine the nature of the bifurcation.

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Mixing layers, jets, or wakes represent examples of open flows where fluid particles, advected by the mean flow, enter and leave the experimental domain of interest. As a result, such flows are generated at a definite spatial location and input perturbations and mean advection have to be explicitly considered. The introduction of absolute and convective instability concepts [1-7] has recently provided a reasonable understanding of the linear spatiotemporal development of such open flows. Usually these notions are based on the behavior of the Green function, i.e., the linear response of the system to an initial localized impulse. If the wave packet representing the Green function decays asymptotically in any moving frame, the system is said to be linearly stable (LS). If this is not the case it will be linearly unstable. Moreover, it will be absolutely unstable (LA) if, at any fixed location, the response grows in time and convectively unstable if it decays (LC). For LC systems, the response to causal forcing fixed in space corresponds to the spatial amplification of the imposed excitation. On the contrary, in the LA case, the response to forcing cannot be defined because it is overshadowed by the exponentially growing initial transient. Therefore, the concepts of absolute and convective instability allow us, *in the linear regime*, to discriminate between open flows exhibiting *intrinsic dynamics* and open flows acting as *spatial amplifiers* of incoming turbulence.

Our theoretical understanding is much less complete for linearly stable open flows in which nonlinearity is destabilizing [8]. For example, boundary layers and Poiseuille flow belong to this class of subcritically unstable systems. As in the previous case, these flows are generated at a definite location and we need to describe the effect of advection and of incoming perturbations. Subcritically unstable *closed* flows are known to exhibit the catastrophic hysteresis phenomenon and it is of great interest to examine its possible existence in the *open* flow geometry. This Letter represents a first attempt at extending the notions of absolute and convective instability to nonlinear systems. These concepts are then illustrated on the real Ginzburg-Landau equation in a semi-infinite domain with destabilizing third-order and stabilizing fifth-order nonlinearities. The hysteresis loop is demonstrated to be restricted to the nonlinear absolute instability range of the control parameter. In the nonlinear convective instability region, strong steady forcing is able to trigger the instability but the flow ultimately returns to its basic state when forcing is turned off.

It appears natural to propose the following definitions of nonlinear absolute and convective instabilities: The basic state of a system is stable (S) if, for *all initial perturbations of finite extent and finite amplitude*, the flow relaxes to the basic state everywhere *in any moving frame.* A system is unstable if it is not stable in the above sense. The instability is nonlinearly convective (NLC) if, *for all initial perturbations of finite extent and finite amplitude*, the flow relaxes to the basic state everywhere *in the laboratory frame.* It is nonlinearly absolute (NLA) if there exists *an initial condition of finite extent and amplitude and a location* where the system does not relax to the basic state.

The physical significance of these concepts may be clarified by considering as an illustration the simple one-dimensional Ginzburg-Landau equation

$$\frac{\partial A}{\partial t}+U_0\frac{\partial A}{\partial x}=R(A)+\frac{\partial^2 A}{\partial x^2}, \qquad (1)$$

where A represents the real amplitude of a bifurcating mode that breaks a discrete symmetry of the problem. The reaction term is chosen to be of the form $R(A) = -\partial V(A)/\partial A$, the potential density $V(A)=-\mu A^2/2 - A^4/4+A^6/6$ giving rise to a subcritical pitchfork bifurcation. The operator $U_0\partial/\partial x$ represents advection at the velocity U_0 taken to be positive. The Galilean invariance is assumed to be broken by the presence of solid boundaries or by external forcing. For instance, experimentally generated open flows may be modeled by an amplitude equation which is to be solved in a semi-infinite domain $[0,+\infty[$ with a suitable boundary condition at $x=0$.

In order to fully understand the effect of the broken Galilean invariance on the behavior of nonlinear states, it is first necessary to recall a few classical results [9-12]

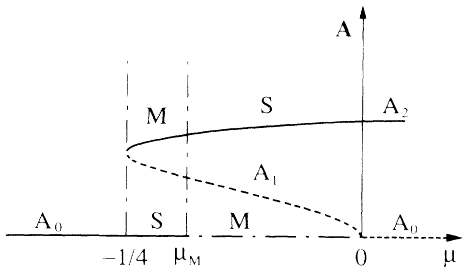

FIG. 1. Bifurcation diagram of spatially uniform states.

pertaining to the infinite domain and the parameter value $U_0=0$. Equation (1) then possesses the Lyapunov functional

$$\mathcal{L}(A)=\int [\tfrac{1}{2}(\partial A/\partial x)^2+V(A)]dx\,.$$

One notes that $\mathcal{L}$ is minimum for uniform solutions that also minimize the potential density $V(A)$. The bifurcation diagram is represented in Fig. 1. On account of the symmetry $A\to -A$, only positive values of the real amplitude A will be considered unless stated otherwise. When $\mu<-\frac{1}{4}$, $A_0=0$ is the only minimum. When $-\frac{1}{4}<\mu<0$, there exist two minima at $A_0=0$ and $A_2=(\frac{1}{2}+\sqrt{\mu+1/4})^{1/2}$. The amplitude $A_1=(\frac{1}{2}-\sqrt{\mu+1/4})^{1/2}$ corresponds to a maximum of $V(A)$ and therefore to an unstable solution. When $\mu>0$, $A_0=0$ becomes a maximum of $V(A)$, i.e., unstable, A_1 disappears, and A_2 is the only stable solution. The parameter value $\mu_M=-\frac{3}{16}$ defines the Maxwell point at which the solutions A_0 and A_2 have an equal potential density. The relative position of μ with respect to μ_M determines whether a sufficiently large "droplet" of bifurcated state A_2 surrounded by the basic state A_0 shrinks [$\mu<\mu_M$, Fig. 2(a)] or expands [$\mu>\mu_M$, Fig. 2(b)]. When $\mu_M<\mu<0$, A_0 is then said to be metastable (M) and A_2 is stable (S). When $-\frac{1}{4}<\mu<\mu_M$, A_0 and A_2 exchange roles. Note that, since $U_0=0$, the system displays $x\to -x$ symmetry. As a result, the velocity $v_f(\mu)$ of a front separating the basic state at $-\infty$ from the bifurcating state at $+\infty$ is positive (negative) when the basic state is stable (metastable). The critical value μ_M is such that $v_f(\mu_M)=0$. The front velocity $v_f(\mu)$ satisfies the nonlinear eigenvalue problem

$$-v_f(\mu)\frac{dA}{dx}=-\frac{\partial V(A)}{\partial A}+\frac{d^2A}{dx^2}\,, \tag{2}$$

with the boundary conditions $A(-\infty)=A_0$ and $A(+\infty)=A_2$. Equation (2) describes a dynamical system in x for a particle in a potential $-V(A)$ with friction coefficient v_f. When $\mu<0$ the unique value $v_f(\mu)$ corresponds to a heteroclinic orbit linking the steady state A_0 at $-\infty$ to the bifurcated state A_2 at $+\infty$. When $\mu<\mu_M$ the particle loses potential energy to travel from A_0 to A_2 and $v_f(\mu)$ has to be positive in order for the particle to stop at A_2. Finally, when $\mu>\mu_M$, the particle gains potential energy to travel from A_0 to A_2 and $v_f(\mu)$ has to correspond to a negative friction. This may also be seen in the implicit formulation

$$v_f(\mu)=\frac{V(A_2)-V(A_0)}{\int(\partial A/\partial x)^{1/2}dx}\,,$$

obtained in the case $\mu<0$ by integrating (2) along the heteroclinic orbit. Note that the velocity $v_f(\mu)$ has the same sign as the quantity $V(A_2)-V(A_0)$.

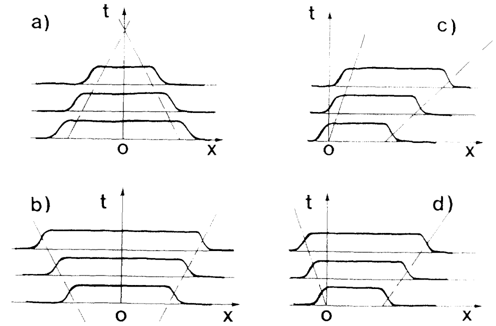

FIG. 2. Diagrams in the (x,t) plane displaying the dynamics of droplets of bifurcated state: (a) A_0 stable, A_2 metastable; (b) A_0 metastable, A_2 stable; (c) nonlinear convective instability; (d) nonlinear absolute instability.

When the advection velocity U_0 is different from zero these considerations can be extended by resorting to the definitions of nonlinear absolute (NLA) or convective (NLC) instabilities. The range of metastability of the basic state must then be separated into two regions: a nonlinearly convective range where the expanding droplet is limited by fronts moving in the same direction [Fig. 2(c)], and a nonlinearly absolute range where the droplet expands in opposite directions [Fig. 2(d)]. As U_0 is taken to be positive this distinction only relies on the sign of the asymptotic left front velocity: $v_f^0(\mu)\equiv v_f(\mu)+U_0$. We shall assume the existence of a negative critical parameter value μ_A defining the transition between these two regimes and such that $v_f(\mu_A)=-U_0$. Since $v_f(\mu_A)<0$, μ_A is necessarily larger than μ_M. The following scenario (see Fig. 3) holds for the dynamics of droplets of the A_2 state initially embedded in the A_0 basic state: For

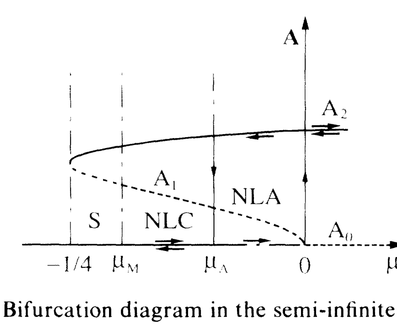

FIG. 3. Bifurcation diagram in the semi-infinite domain.

$-\frac{1}{4} < \mu < \mu_M$, all droplets shrink back to the A_0 state. For $\mu_M < \mu < \mu_A$, droplets of sufficient initial size expand but are advected away. The A_0 state is linearly stable but NLC. For $\mu_A < \mu < 0$, droplets of sufficient initial size expand and contaminate the entire medium. The A_0 state is linearly stable but NLA. For $\mu > 0$, all initial perturbations of the A_0 state grow and lead to the A_2 state. The A_0 state is linearly unstable (LC or LA) and NLA.

As a result, note that *the range of metastability in the laboratory frame is restricted to the interval* $\mu_A < \mu < 0$. An hysteresis loop is possible only between μ_A and $\mu = 0$ as indicated in Fig. 3. This phenomenon is brought out by examining the response to an excitation of constant amplitude B applied at $x=0$. The amplitude A of a steady solution must then satisfy

$$U_0 \frac{dA}{dx} + \frac{\partial V(A)}{\partial A} - \frac{d^2 A}{dx^2} = 0 . \tag{3}$$

In the metastable range considered here, three phase portraits of the dynamical system (3) in the $(A, dA/dx)$ plane (Fig. 4) can be distinguished depending on the relative position of μ with respect to μ_A. The only orbits corresponding to steady solutions must stay finite as x goes to infinity since they must have been reached from a localized finite initial state by minimizing the Lyapunov functional $\mathcal{L}$. Therefore allowable phase-space trajectories of Eq. (3) end either in $A_0 = 0$ or in A_2 as indicated by continuous heavy lines in Fig. 4. Solutions pertaining to a particular excitation amplitude B correspond to portions of these trajectories initiated at the intersection point with the vertical line $A = B$.

The marginal value $\mu = \mu_A$ corresponds to the limiting case where a heteroclinic orbit links A_0 to A_2 [Fig. 4(a)]. The orbit ending at the origin possesses a maximum A_0^* between A_1 and A_2. For the initial amplitude $B = 0$ the only solution is $A(x) = 0$. For any $B > 0$ there exists a solution asymptotic to A_2 as $x \to +\infty$. In the range $0 \le B \le A_0^*$ a second solution is possible which decays to zero as $x \to +\infty$. An *hysteresis* loop between the two asymptotic states *is therefore possible as the forcing amplitude is varied in the range* $0 < B \le A_0^*$ [shaded region in Fig. 4(a)].

For $\mu < \mu_A$ [Fig. 4(b)], the flow is NLC, the heteroclinic orbit ending at A_2 spirals around the A_1 state with a minimum at A_2^*. For any initial condition in the range $A_0 < B < A_2^*$ the unique steady solution decays to zero as $x \to +\infty$. A_2^* may be viewed as a minimum threshold value for the entrance perturbation to be "efficient." For $A_2^* \le B \le A_0^*$ two solutions are possible ending respectively at the origin and at A_2. For $B > A_0^*$ a single solution exists and it is asymptotic to A_2. Thus in the NLC case, *hysteresis is only possible for large enough forcing amplitude in the range* $A_2^* \le B \le A_0^*$ [shaded region in Fig. 4(b)]. Note that the system returns to the rest state $A = 0$ as the forcing amplitude B decreases to zero.

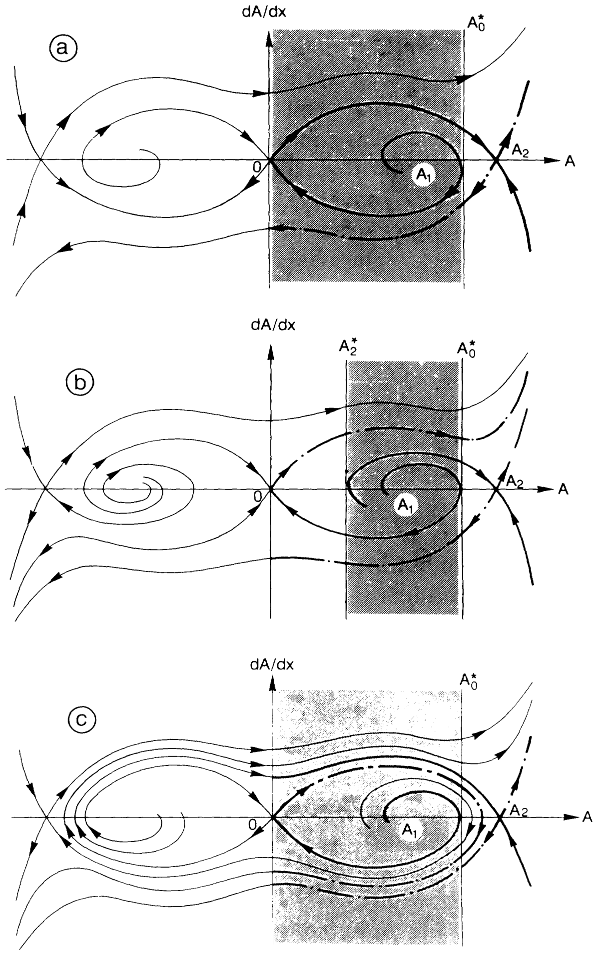

FIG. 4. Phase portraits of steady solutions: (a) $\mu = \mu_A$; (b) $\mu < \mu_A$; (c) $\mu > \mu_A$. Shaded regions indicate the coexistence of two states respectively asymptotic to A_0 and A_2 at $+\infty$. Heavy lines represent steady solutions asymptotic to A_0 or A_2 as $x \to +\infty$ (continuous lines) or $x \to -\infty$ (broken lines).

For $\mu > \mu_A$ [Fig. 4(c)] the flow is NLA. The phase diagram is similar to the first case except that the allowable orbit ending at A_2 crosses the vertical axis. Therefore a steady solution asymptotic to A_2 at $+\infty$ exists even at zero forcing. *Multiple states coexist in the entire range* $0 \le B \le A_0^*$ [shaded region in Fig. 4(c)]. Note, however, that, in contrast with the NLC case, the system does not return to the rest state $A = 0$ as the forcing amplitude is decreased to zero.

The predictions of the potential model (1) can be summarized as follows: When a flow is NLC, the only observable steady solution in the absence of forcing is the rest state $A = 0$. As the forcing amplitude is increased to A_2 and then decreased back to zero, one observes an *hysteresis loop* composed of the spatially decaying state asymptotic to $A = 0$ and a spatially growing state asymptotic to A_2. There is, however, a *reversible return* to the rest state $A=0$ when forcing is turned off. In contrast, when a flow is NLA, both the rest state $A=0$ and the spatially growing state asymptotic to A_2 are observable in the absence of forcing. Thus, starting from the rest state, similar variations of the forcing magnitude trigger an *irreversible transition* to a spatially growing state asymptotic to A_2. *The rest state $A=0$ is not recovered when the forcing is suppressed.* It is important to bear in mind that the above conclusions assume both steady solutions to be stable.

The analysis presented on this very simple model may be extended in a straightforward manner to the Ginzburg-Landau equation with complex amplitude and real coefficients. In fact the conclusions of the preceding discussion rely only on the existence of a well-defined speed for the front separating the basic state from the bifurcating state. A qualitative description of the possible scenarios likely to arise in subcritically bifurcating open flows has been outlined and a practical test has been suggested to determine the nature of the bifurcation. There is presently no detailed experimental confirmation of the validity of this analysis. However, some preliminary observations [13] of the Görtler flow on a concave wall provide evidence for the NLC nature of the instability, in qualitative agreement with the theoretical study of Park and Huerre [14]: The nonlinear impulse response displays the same features as in Fig. 2(c). Görtler vortices are not detected at zero forcing. Nonetheless, the application of a localized steady excitation close to the origin of the flow gives rise to a spatially growing perturbation for forcing amplitudes above a definite threshold.

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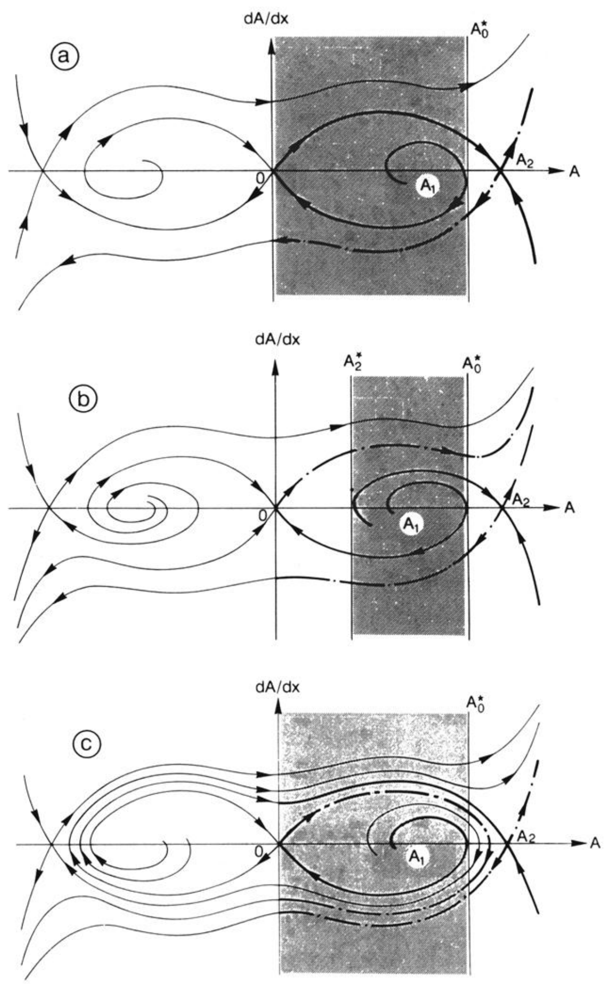

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