Absolute and Convective Secondary Instabilities in Spatially Periodic Shear Flows

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The generic problem of the spatiotemporal instability of a periodic basic flow (Stuart vortices) is considered in order to interpret the sequence of bifurcations observed in open shear flows. Using a novel numerical technique, we show that the more concentrated the vortices, the smaller the backflow needed to trigger absolute instability. These results allow us to propose an alternative interpretation for the subharmonic resonance observed in forced shear flows, which is classically attributed to an acoustic feedback. [S0031-9007(96)02213-2]

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The classical description of the transition from laminar flow to turbulence, or more generally from order to disorder in extended systems, involves a sequence of primary, secondary, ... bifurcations which successively break the symmetries of the original problem [1]. In closed flows, such as Rayleigh-Bénard convection or the Taylor-Couette experiment, this sequence occurs while varying some control parameter, for example, the temperature difference or the angular velocity. In particular, for these systems the first bifurcation breaks the invariance under translation in one direction, x, and leads to a periodic solution in x, like convection rolls or Taylor's rings. At higher values of the control parameter the periodic solution will itself become unstable and will give rise to a new state eventually with less symmetry. For open flows, such as shear flows, the picture is somewhat different as these flows are strongly unstable and evolve in space [2]. From experimental observations, their dynamics may still be described by a sequence of bifurcations which now take place successively in space. For instance, the spatial evolution of a mixing layer initially involves a two-dimensional instability which saturates into a row of Kelvin-Helmholtz billows [3]. Further downstream, this row of vortices is destabilized by the *pairing* instability associated with the spatial growth of the first subharmonic [4]. Ultimately this secondary mode saturates into a new row of larger vortices with twice the initial spacing. This spatial sequence of instability and saturation may repeat itself until three-dimensional secondary instabilities induce transition to turbulence. A similar sequence is obtained for a temporally evolving shear flow, as realized in the tilting tank experiment by Thorpe [5]. Numerous numerical simulations [6] and theoretical analyses [7,8] of this temporal shear flow have shed light on the 2D and 3D instability mechanisms. In particular Pierrehumbert and Widnall [8], studying the Stuart model [9] of a row of vortices, have identified three types of secondary instability: the *helical pairing*, most unstable for 2D modes; the *translative* instability which preserves the periodicity of the Stuart row and corresponds to the elliptic instability [10] at large spanwise wave numbers and to the so-called zigzag instability [1] at short spanwise wave numbers; and finally a *core* instability associated with a varicose modulation of the core of the vortices. But, from the causality principle, results from the temporally evolving flow are transposable to the spatially evolving flow if, and only if, the considered instabilities are convective as defined by Briggs [11,12]. Physically this means that the dynamics of a temporally evolving flow, for which the future is not supposed to influence the past (causality), will be equivalent to the dynamics of a spatially evolving flow only if the downstream evolution of the flow does not influence the upstream instability. If this is not true, the spatial case will exhibit a global behavior [13,14], which is the result of the resonant loop due to the downstream part of the emerging flow structure inducing the genesis of its own upstream part [15].

If the importance of the absolute and convective instability concept is now widely recognized for the natural and the controlled dynamics of open flows, the majority of the studies have only considered its implication for the primary instability. But, as stressed by Huerre in pioneering work [16], "primary and secondary instabilities arising in fluid flows need not have the same absolute/convective character," and "absolute secondary instability" might induce energetic transition to turbulence or select a secondary mode radically different from what might be deduced from temporal studies. This idea is not restricted to open flow dynamics but applies to any pattern forming system supporting traveling waves such as chemical reactions [17], nonlinear optics [18], binary fluid convection [19], or dynamo theory of dishlike objects [20]. The route to disorder involving intrinsic absolute instability cascade or extrinsic noise induced response would have to be explored in those various fields. The novel technique we implement should be easily transposed to any of those problems as it relies on a simple, nearly naive but efficient method: absolute or convective instability being defined [11,12] by the behavior of the impulse response of the system with respect to a particular frame of reference, we numerically compute the impulse response in a single frame and then evaluate the growth rate in any

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moving frame using the Galilean invariance of the system. Application of this principle to shear flow secondary instability is detailed in the following as much as possible in a general and concise manner because we feel that the technique as well as the physical implications are equally important.

The present Letter represents a first attempt to determine the absolute/convective nature of a secondary instability of a primary saturated periodic mode. Following Pierrehumbert and Widnall [8], we consider the Stuart model and analyze the absolute/convective nature of the 2D *pairing* instability. The theoretical foundation of our study relies on a recent paper by Brevdo and Bridges [21] which extends the absolute/convective criteria to periodic base flows. Although the proof is highly technical, the final result is remarkably simple. They show, using the Floquet theory, that the homogeneous criterion [11,12] (i.e., the sign of the imaginary part of the frequency ω of the wave such that $d\omega/dk = 0$, with k its complex wave number) stays valid, with ik being now the logarithm of the Floquet multiplier. Instead of numerically determining the complex dispersion relation and its associated saddle point, we implement a direct numerical determination of the asymptotic wave packet issuing from a localized initial perturbation using the technique developed by Delbende *et al.* [22]. We obtain, at large t, the absolute or convective nature of the instability in any frame moving at velocity v compared to the frame of the simulation by measuring the growth rate $\sigma(v)$ of the wave packet on the spatiotemporal ray defined by $x = vt$. In reality the laboratory frame singled out by the boundary or entrance conditions will correspond to a particular value v_0 and the instability will be absolute if $\sigma(v_0)$ is positive. In the case of the mixing layer, a single relative velocity profile corresponds to several experimental configurations, as increasing the speed of both streams by the same amount just changes the value of v to be considered. Therefore for a given relative velocity profile, the quantity $\sigma(v)$ defines the nature of the instability for a family of experimental configurations differing only by the mean advection velocity v.

In order to compute the asymptotic linear impulse response of the periodic flow, the two-dimensional incompressible Navier-Stokes equation expressed for the vorticity Ω and the stream function Ψ (such that $\Omega = -\Delta\Psi$) is linearized around the Stuart basic state defined by the stream function $\Psi_b(x,y) = \frac{1}{2}\ln[\cosh(2y) - \rho\cos(2x)]$, and solved using a pseudospectral Fourier code validated by Vincent and Meneguzzi [23] and Brancher *et al.* [24]. The perturbation is supposed periodic in both x and y directions but the domain is extremely elongated in the longitudinal direction x (32 periods of the base flow resolved by 1024 collocation points) and large in the transverse direction y (256 collocation points representing 8 x-periods of the basic flow) in order to minimize the boundary effects. On the test case, $\rho = 0$ (hyperbolic tangent profile) doubling the size of the box or the resolution has been shown to have no significant effect. This size represents therefore an optimum for computer efficiency.

The numerical simulation is initialized by a localized perturbation with a Gaussian envelope whose size is chosen large enough to be well represented in the truncated spectral domain. Comparing results while varying the location of the initial perturbation, its actual size, and the total duration of the simulation allows us to estimate the error on all the measured quantities. The initial perturbation gives rise to a wave packet growing in time and expanding in space. In order to separate the phase and the amplitude of the signal, we construct its analytic continuation by applying a Hilbert transform [25]. Instead of computing the Floquet exponent while moving on a ray, we filter out all the wave numbers higher than the wave number of the basic flow. In this particular case, because the spectrum in x presents a suitable band structure, this filtering proves itself sufficient and the computed response exhibits no variation synchronized with the underlying basic flow. Using the Hilbert transform on the wave packet one has to be conscious that the associated convolution with $i/\pi x$ produces an algebraic spatial decay on the side of the wave packet. Therefore the reconstructed amplitude was systematically plotted with the initial signal to delineate the region where the wave packet amplitude was correctly estimated. After this control procedure, the time series of the wave packet envelope was drawn versus $v = x/t$, t (x) being the time (distance) from the initial perturbation. Figure 1, obtained for $\rho = 0$ (defining the hyperbolic tangent profile), clearly shows that the wave packet grows exponentially between two critical values of $v = x/t$, $\pm v_c$. The growth rate measured on each ray $x/t = v$ is presented in Fig. 2. Asymptotic theory for an infinite domain tells us that, when t goes to infinity, the wave which emerges on a ray corresponds to $d\omega/dk = v$ and its growth rate along the ray is such that $\sigma(v) = \mathrm{Im}\,(\omega - vk)$. As our

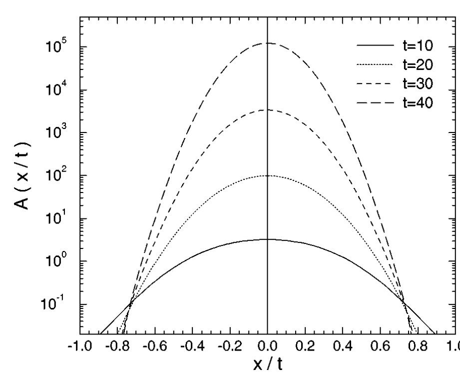

FIG. 1. Evolution of the wave packet generated by an initial perturbation localized in space. The time series is computed by a direct numerical simulation of the Navier-Stokes equations linearized around the homogeneous hyperbolic tangent profile. The amplitude of the wave packet is plotted versus $x/t = v$ in order to follow its evolution on rays radiating from the initial perturbation location.

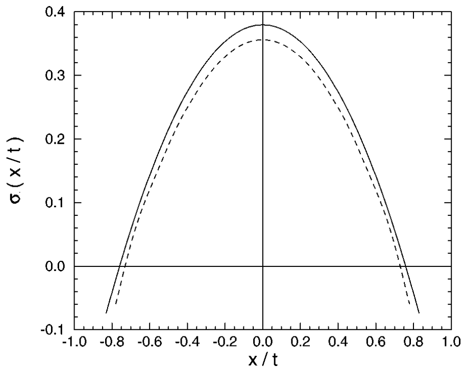

FIG. 2. Growth rate of the wave packet envelope measured at fixed $x/t = v$ on Fig. 1 (dashed line) versus the growth rate predicted by Huerre and Monkewitz [26] (solid line).

numerical domain is finite, the total time of the simulation $t_{\max}$ should remain short enough for the wave packet to be smaller than half the box in order to avoid interaction between the leading and the trailing edges of the wave packet, but large enough to correspond to the asymptotic regime. In the numerical simulation $t_{\max}$ is about 40 and the validity of this crucial approximation is verified *a posteriori* for each run by checking that the growth rate $\sigma(v)$ has indeed saturated.

Figure 2 compares the growth rate on rays $x/t = v$ obtained for the hyperbolic tangent profile ($\rho = 0$) by theoretical inviscid analysis of Huerre and Monkewitz [26] and by our numerical technique. The agreement between theoretically and numerically estimated values of $\sigma(v)$ is remarkably good, the small departure being due to viscous effects as our Reynolds number Re is finite and equal to 500. This is confirmed by the fact that the maximum spatiotemporal growth rate we have measured is extremely close to the maximum temporal growth rate reported in the literature [2] for this Reynolds number as it should. From Fig. 2 we determine the trailing edge velocity, $v_c = 0.735 \pm 0.015$, as the point such that $\sigma(v_c) = 0$. Using a Galilean transformation to return to the laboratory frame, we predict that, for the homogeneous viscous mixing layer, the velocity in the lower layer of the profile should be negative and such that the velocity ratio [26] $R \equiv 1/v$ is greater than $R_c \equiv 1/v_c = 1.36 \pm 0.03$ for the instability to be absolute, in excellent agreement with the inviscid theoretical result of Huerre and Monkewitz [26], who find the value 1.315. Having now validated the numerical procedure, we repeat this study for the family of Stuart vortices with ρ varying from 0 to 0.75 (Fig. 3). The inviscid theoretical result [27] for the absolute instability of a single row of point vortices ($\rho = 1$), which gives $\sigma(v) = 1 - v^2$ and $k = 1 + iv$, is reported on Fig. 3 together with the numerical results. Figure 3 shows that the more concentrated the Stuart vortices are, the faster the impulse wave packet spreads, reaching asymptotically the front velocity ± 1 for $\rho = 1$. Practically, this means that the back flow needed to trigger abso-

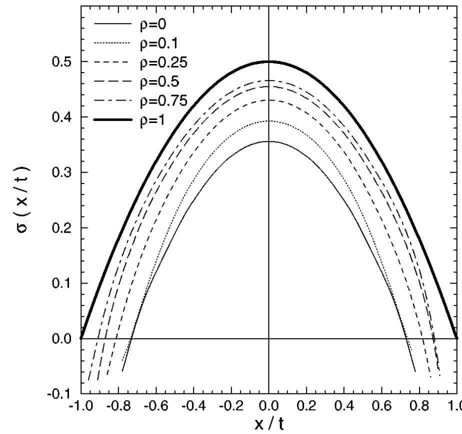

FIG. 3. Same as Fig. 2 for various values of the Stuart parameter ρ. For $\rho = 0$ to 0.75 the growth rate is computed, whereas for $\rho = 1$ (the limiting case of point vortices) the growth rate is theoretically predicted [27].

lute instability is smaller when the primary vortices are more concentrated. These results are synthesized in Fig. 4 where the threshold for the velocity ratio R, which will yield absolute instability, is plotted versus ρ. Whereas a strong backflow ($R = 1.36$) is necessary to trigger absolute primary instability (with no preexisting vortices $\rho = 0$), the secondary instability becomes absolute for a backflow almost nil ($R \sim 1$) when the saturated primary vortices are sufficiently concentrated ($\rho \sim 1$). In this latter case the selected wave number (obtained from the gradient in the x direction of the phase of the analytical signal) on the trailing edge of the wave packet tends to 1 and therefore corresponds to period doubling (pairing instability).

When considering the nonlinear evolution of a separated shear flow (a mixing layer, a jet, or a backward facing step) one may imagine that, for small or even nearly

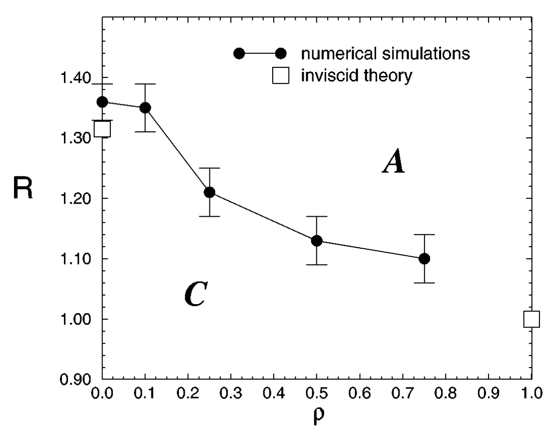

FIG. 4. Domains of absolute (A) and convective (C) instability in the (R, ρ) plane. The error bars represent the maximum variation of the critical values of R while varying the location of the initial perturbation, its actual size, and the total duration of the simulation.

zero backflow, a subharmonic resonance, due to the presence of a region of absolute secondary instability prevailing on the saturated row of Kelvin-Helmoltz billows, could occur when the flow is forced at a suitable fundamental frequency. A subharmonic cascade could occur, because the pairing instability while saturating will give rise to a periodic row of vortices with twice the original spacing, which in turn might be absolutely unstable to the pairing. If one forces the primary mode of a shear layer, one should observe a coherent sequence of pairing events, associated with a strong subharmonic component in the power spectrum of any physical signal. This cascade, although attributed to an acoustic feedback, has been observed in the recent experiments [28] where a jet or a flow over a backward facing step is forced close to its natural frequency, and in which period doubling and quadrupling are naturally strongly present and phase locked. The absolute subharmonic instability mechanism constitutes another interpretation of these observations. New experiments or numerical simulations where, for example, a coflow is added around the jet, are desirable to discriminate between the acoustic feedback and the absolute instability explanations.

A similar analysis is presently being undertaken for the three-dimensional instabilities of the same basic flow (Stuart vortices), because we believe that transition to an absolute secondary 3D instability might actually be an alternative interpretation of the mixing transition and a route to turbulence for open shear flows. The concept of secondary absolute or convective instability, already recognized and applied for amplitude equations [13,21,29], should be systematically taken into account when considering the sequence of bifurcations occurring in an extended system in which a particular frame (here the laboratory frame) is singled out by the boundary conditions or continuous forcing. The present Letter gives a practical and general tool to determine for a real flow the absolute or convective nature of the secondary instability. These results allow us to propose an alternative interpretation based on the absolute subharmonic instability for the subharmonic resonance of a forced jet or a forced backward facing step flow, attributed previously to an acoustic feedback [28].

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